1 LTI system realization

1. The input x and output y of an LTI system are related by the difference equation

$$y(n) + 0.9y(n-1) + 0.2y(n-2) = x(n) + x(n-1). \tag{1}$$

(a) Implement (1) in Direct Form I. (This will have three delays.)

(b) Take this direct form. Construct a vector $s(n) \in \textit{Reals}^3$ whose three components are the outputs of the four delays at time n. Find A, b, c, d such that

$$\begin{aligned} s(n+1) &= As(n) + bx(n) \\ y(n) &= c^T s(n) + dx(n) \end{aligned}$$

(c) Implement (1) in Direct Form II. (This will have two delays.)

(d) Take this direct form. Construct a vector $s(n) \in \textit{Reals}^2$ whose two components are the outputs of the two delays at time n. Find A, b, c, d such that

$$\begin{aligned} s(n+1) &= As(n) + bx(n) \\ y(n) &= c^T s(n) + dx(n) \end{aligned} \tag{2}$$

(e) The two state machines constructed in part (1b) and part (1d) are different. What is the relation between them in terms of simulation relation?

(f) Take $x(n) = x(0)e^{j\omega n}, s(n) = s(0)e^{j\omega n}, y(n) = y(0)e^{j\omega n}$ in (2). Show that

$$H(\omega) = \frac{y(0)}{x(0)} = c^T[e^{j\omega}I - A]^{-1}b + d$$

is the frequency response of the LTI system. Calculate $H(\omega)$ using the specific values of A, b, c you obtained in part (1d).

(g) Calculate the frequency response directly from (2) and verify that it is the same as that calculated above.

(h) The frequency response is

$$H(\omega) = \frac{1 + e^{-j\omega}}{1 + 0.9e^{-j\omega} + 0.2e^{-2j\omega}}.$$

Use the fact that its denominator can be factored as $(1 + 0.5e^{-j\omega})(1 + 0.4e^{-j\omega})$ to express H as

$$H(\omega) = \frac{\alpha}{1 + 0.5e^{-j\omega}} + \frac{\beta}{1 + 0.4e^{j\omega}} \tag{3}$$

and calculate α and β.

(i) Show that the frequency response of an LTI system with impulse response $h(n) = a^{-n}, n \geq 0; = 0, n < 0$ is $[1 - ae^{-j\omega}]^{-1}$. Use this fact to obtain the impulse response of the difference equation (1) from its frequency response (3)

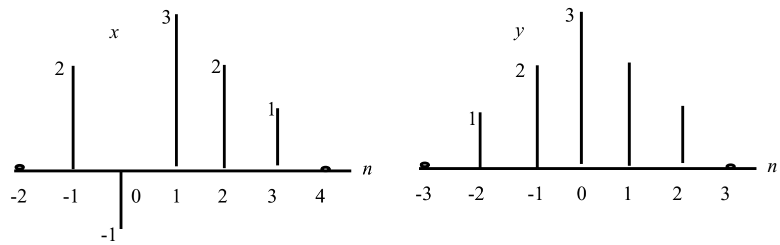

Figure 1: Signals for problem 1

(j) From (2) we also know that the (zero-state) impulse response is given by:

$$h(n) = \begin{cases} 0, & n < 0 \\ d, & n = 0 \\ c^T A^{n-1} b, & n \geq 1 \end{cases} \tag{4}$$

Verify that the impulse response calculated using (4) is the same as what you obtained above for $n = 0, 1, 2, 3$.

2 Convolution

1. Study the discrete-time signals x, y shown in figure 1. Assume that $x(n)$ and $y(n)$ equal 0 for values of n that are not shown.

   (a) For $n = 0, 4, -4$, sketch the signals x_n, y_n given by

   $$\forall m \in \textit{Ints}, \quad x_n(m) = x(n-m), y_n(m) = y(n-m).$$

   (b) Calculate $x * y(-4), x * y(0), x * y(4), x * y(16)$.

2. Sketch the continuous-time signals v, w constructed from x, y of problem 1 by

   $$v(t) = x(n), \;\; w(t) = y(n), \text{ for } n \leq t < n+1.$$

   (a) For $t = 0, 3.5, -3.5$, sketch the signals v_t, w_t defined by

   $$\forall s \in \textit{Reals}, \quad v_t(s) = v(s-t), w_t(s) = w(t-s).$$

   (b) Calculate $v * w(-3.5), v * w(0), v * w(3.5), v * w(16)$.

3. Let x, y, z be continuous-time signals as shown in the figure 2. For each of the convolutions listed in table 1 determine (1) the set of times t at which the convolution is not equal to zero, (2) the times t at which the convolution achieves its maximum value, and (3) the times at which the maximum value is achieved. The table includes the answer for the first convolution $x * x$.

4.

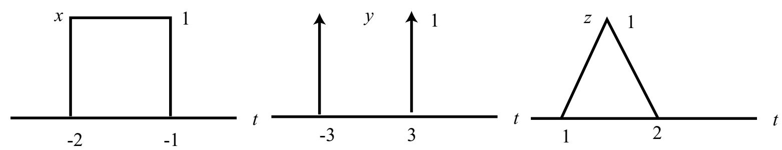

Figure 2: Signals for problem 3

signal	$\{t \mid \text{signal is non-zero}\}$	maximum value of signal
$x * x$	$(-4, -2)$	1
$x * y$		
$x * z$		
$y * y$		
$y * z$		
$z * z$		

Table 1: Table for Problem (3)

3 Fourier Transform

1. Find the CTFT X of the continuous time signal x given below and in each case plot the function: $\omega \mapsto |X(\omega)|$.

   (a) $\forall t, \quad x(t) = \cos 20t + \cos 30t$

   (b) $\forall t, \quad x(t) = \delta(t - 20) + \delta(t + 20)$

   (c) $\forall t, \quad x(t) = 1, t \in [-1, 1]; = 0,$ otherwise

   (d) $\forall t, \quad x(t) = 1, t \in [2, 4]; = 0,$ otherwise

   (e) $\forall t, \quad x(t) = (\sin t)/t$

2. Use the fact that the CTFT of the product $x \times y$ is given by the convolution $2\pi X * Y(-\omega)$ to obtain the CTFT $Z(\omega)$ of the signal

$$z(t) = \cos 20t + \cos 30t, t \in [-T, T]; = 0, \text{ otherwise}$$

   Sketch $Z(\omega)$ and explain what happens to Z as $T \to \infty$.