EECS20n, Solution to Mock Midterm 2, 11/17/00

1. **15 points** Write the following in Cartesian coordinates (i.e. in the form $x + jy$)

   (a) **1 point** $j^3 - j^2 + j - 1 = 0$

   (b) **2 points** $\sum_{k=0}^{11} e^{jk\pi/6} = \frac{1-e^{j12\pi/6}}{1-e^{j\pi/6}} = 0$

   (c) **2 points**$(1 + j1)/(1 - j1) = j$

   (d) **2 points** $\sqrt{\cos\pi/4 + j\sin\pi/4} = \sqrt{e^{j\pi/4}} = \pm e^{j\pi/8} = \pm(\cos\pi/8 + j\sin\pi/8)$.

   Write the following in polar coordinates (i.e. in the form $re^{j\theta}$)

   (a) **2 points** $1 + j1 = \sqrt{2}e^{j\pi/4}$

   (b) **2 points** $(1 + j1)3 = 3\sqrt{2}e^{j\pi/4}$

   (c) **2 points** $[\cos\pi/4 + j\sin\pi/4]^{1/2} = \pm 1.e^{j\pi/8}$

   (d) **2 points** $(1 + j1)/(1 - j1) = 1.e^{j\pi/2}$

2. **15 points** Which of the following discrete-time or continuous-time signals is periodic. Answer yes or no. If the signal is periodic, give its fundamental period and state the units. Suppose that for a discrete-time signal, n denotes **seconds**, and for a continuous-time signal, t denotes **minutes**.

   (a) **2 points** $\forall n \in$ *Ints*, $\quad x(n) = e^{j\sqrt{2}n}$ Periodic NO

   (b) **2 points** $\forall t \in$ *Reals*, $\quad x(t) = e^{j\sqrt{2}t}$ Periodic YES, Period = $2\pi/\sqrt{2}$ min.

   (c) **2 points** $\forall n \in$ *Ints*, $\quad x(n) = \cos 3\pi n + \sin(3\pi n + \pi/7)$ Periodic YES, Period = 2 sec.

   (d) **2 points** $\forall t \in$ *Reals*, $\quad x(t) = \cos 3t + |\sin 3t|$ Periodic YES, Period = $2\pi/3$ min.

   (e) **2 points** $\forall n \in$ *Ints*, $\quad x(n) = |\cos 3\pi n| + \sin(3\pi n + \pi/7)$ Periodic YES, Period = 2 sec.

   5 points Find A, θ, ω in the following expression:

$$\begin{aligned} A\cos(\omega t + \theta) &= \cos(2\pi \times 10,000t + \frac{\pi}{4}) + \sin(2\pi \times 10,000t + \frac{\pi}{4}) \\ &= \sqrt{2}\cos(2\pi \times 10,000t) \end{aligned}$$

   So, $A = \sqrt{2}, \omega = 20,000\pi, \theta = 0$.

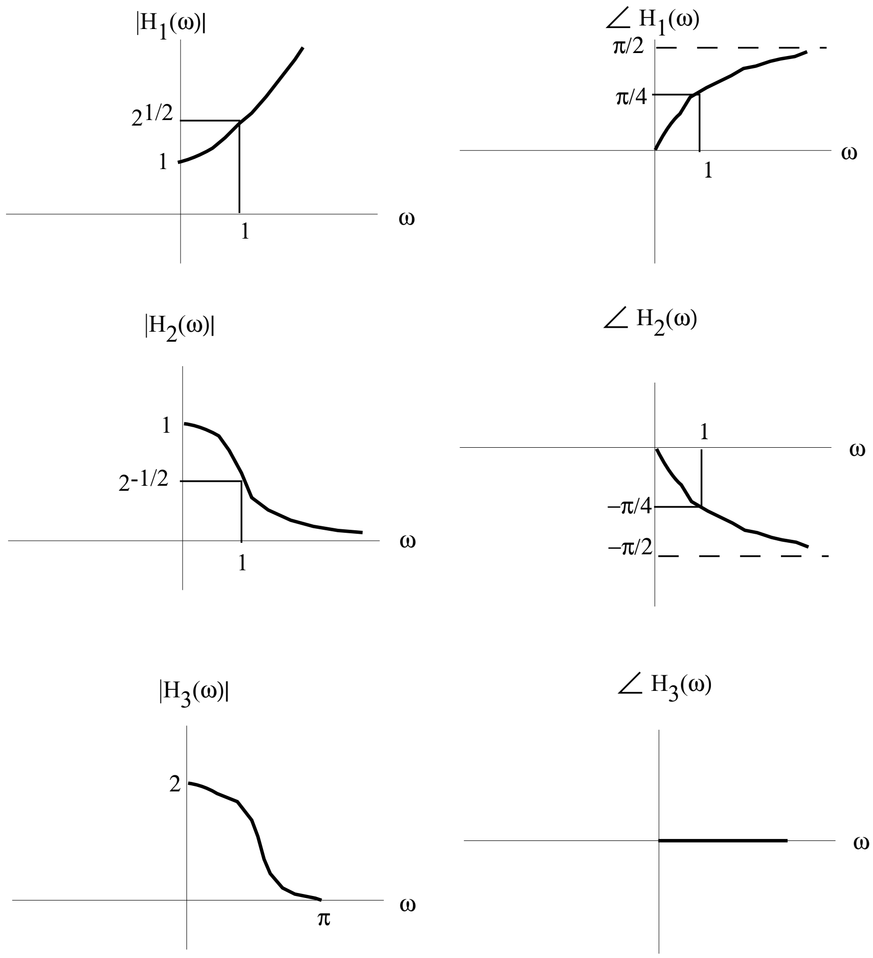

Figure 1: Plots for Problem 3

3. **15 points** On Figure 1 plot the amplitude and phase response of the following frequency responses. On your plots carefully mark the values for $\omega = 0$ and for one other non-zero value of ω.

   (a) **4 points** $\forall \omega \in$ *Reals*, $\quad H_1(\omega) = 1 + j\omega$.
   So, $|H_1(\omega)| = [1+\omega^2]^{1/2}$, $\angle H_1(\omega) = \tan^{-1}\omega$. $|H_1(0)| = 1, |H_1(1)| = \sqrt{2}$, $\angle H_1(0) = 0, \angle H_1(1) = \pi/4$.

   (b) **4 points** $\forall \omega \in$ *Reals*, $\quad H_2(\omega) = \frac{1}{1+j\omega}$. So, $|H_2(\omega)| = [1+\omega^2]^{-1/2}$, $\angle H_2(\omega) = -\tan^{-1}\omega$. $|H_2(0)| = 1, |H_2(1)| = 1/\sqrt{2}$, $\angle H_2(0) = 0, \angle H_1(1) = -\pi/4$.

   (c) **4 points** $\forall \omega \in$ *Reals*, $\quad H_3(\omega) = 1 + \cos\omega$.
   So $|H_3(\omega)| = 1 + \cos\omega, \angle H_3(\omega) = 0$. $H_3(0) = 2, H_3(\pi) = 0$.

   3 points Which of H_1, H_2, H_3 can be the frequency response of a discrete-time system?

   H_3. Since it is periodic with period 2π.

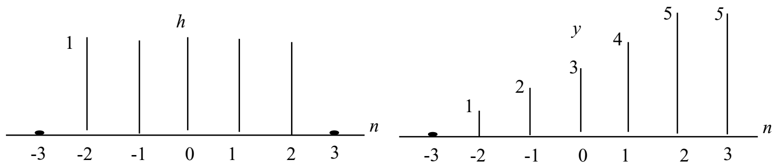

Figure 2: Impulse and step response for Problem 4

4. **15 points** A discrete-time system H has impulse response $h : Ints \to Reals$ given by

$$h(n) = \begin{cases} 1, & n = -2, -1, 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) **2 points** Sketch h.

(b) **5 points** What is the step response of H, i.e. the output signal when the input signal is *step*, where $step(n) = 1, n \geq 0$, and $step(n) = 0, n < 0$? You can give your answer as a plot or as an expression.

(c) **5 points** What is the frequency response of H?

(d) **3 points** What is the output signal of H for the following input signals?

i. $\forall n, \quad x(n) = \cos n$

ii. $\forall n, \quad x(n) = \cos(n + \pi/6)$

iii. $\forall n, \quad x(n) = \sin 100n$

(a,b)Figure 2 shows h and the step response y. An alternative way to calculate y is

$$\begin{aligned} \forall n \in \textit{Ints}, \quad y(n) &= \sum_{m=-\infty}^{\infty} h(m)x(n-m) \\ &= x(n+2) + x(n+1) + x(n) + x(n-1) + x(n-2) \\ &= 0, n \leq -3; 1, n = -2; 2, n = -1; 3, n = 0; 4, n = 1; 5, n \geq 2. \end{aligned}$$

(c) The frequency response is

$$\begin{aligned} \hat{H}(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= \sum_{n=-2}^{2} e^{-j\omega n} \\ &= 1 + 2\cos\omega + 2\cos 2\omega \end{aligned}$$

(d) Use the fact that the response to a signal $x(n) = \cos(\omega n + \theta)$ is $y(n) = |\hat{H}(\omega)|\cos(\omega n + \theta + \angle\hat{H}(\omega))$.

i. $\forall n, \quad y(n) = \hat{H}(1)x(n) = [1 + 2\cos 1 + 2\cos 2]\cos n$

ii. $\forall n, \quad y(n) = \hat{H}(1)x(n) = [1 + 2\cos 1 + 2\cos 2]\cos(n + \pi/6)$

iii. $\forall n, \quad y(n) = \hat{H}(100)x(n) = [1 + 2\cos 100 + 2\cos 200]\sin 100n$

5. **15 points**

   (a) **5 points** Find the frequency response for the LTI systems described by these differential equations (input is x, output is y)

      i. $\dot{y}(t) - 0.5y(t) = x(t)$

      ii. $\ddot{y}(t) - 0.5\dot{y}(t) + 0.25y(t) = \dot{x}(t) + x(t)$

   (b) **5 points** What is the response of the second system above for the input $\forall t, \quad x(t) = e^{j(100t+\pi/4)}$?

   (c) **5 points** Find the frequency response for the LTI systems described by these difference equations (input is x, output is y)

      i. $y(n) - 0.5y(n-1) = x(n)$

      ii. $y(n) - y(n-1) + 0.25y(n-2) = x(n) + x(n-1)$

(a) Using $y(t) = \hat{H}(\omega)e^{j\omega t}$ is the response to $x(t) = e^{j\omega t}$, we get

i. $\frac{1}{j\omega - 0.5}$

ii $\hat{H}(\omega) = \frac{j\omega+1}{-\omega^2 - 0.5j\omega + 0.25} = \frac{1+j\omega}{0.25-\omega^2-0.5j\omega}$

(b) The response is $\forall t, \quad \hat{H}(100)e^{j(100t+\pi/4)}$.

(c) Using $y(n) = \hat{H}(\omega)e^{j\omega n}$ is the response to $x(n) = e^{j\omega n}$, we get

i. $\frac{1}{1-0.5e^{-j\omega}}$

ii. $\frac{1+e^{-j\omega}}{1-e^{-j\omega}+0.25e^{-2j\omega}}$.

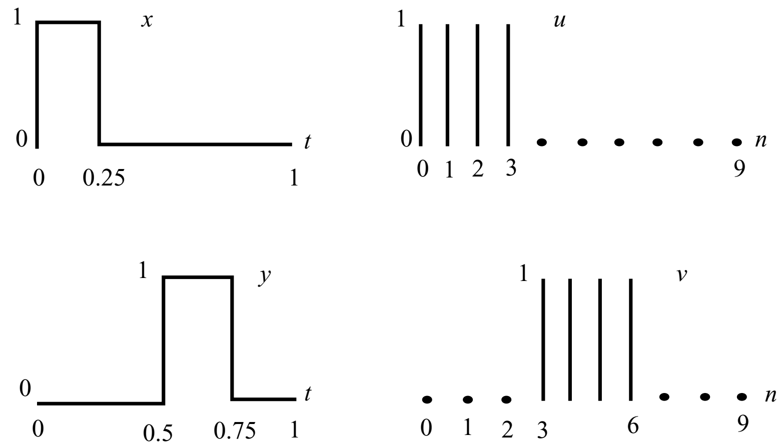

Figure 3: Periodic signals for Problem 6

6. **15 points** Figure 3 plots two continuous-time periodic signals x and y both with period 1 second, and two discrete-time signals u and v both with period 10 samples. The plots are given only for one period. Suppose the exponential Fouriers Series representations of these signals are given as:

$$\begin{aligned}
\forall t \in \textit{Reals}, \quad x(t) = &= \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_x t} \\
\forall t \in \textit{Reals}, \quad y(t) = &= \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_y t} \\
\forall n \in \textit{Ints}, \quad u(n) &= \sum_{k=0}^{9} U_k e^{jk\omega_u n} \\
\forall n \in \textit{Ints}, \quad v(n) &= \sum_{k=0}^{9} V_k e^{jk\omega_v n}
\end{aligned}$$

(a) **3 points** Give the values of $\omega_x = 2\pi$ rad/sec, $\omega_y = 2\pi$ rad/sec, $\omega_u = \pi/5$ rad/sample, $\omega_v = \pi/5$ rad/sample.

(b) **4 points** Calculate the values of the coefficients $X_0 = 0.25$, $Y_0 = 0.25$, $U_0 = 0.4$, $V_0 = 0.4$.

These are just the average values of the signal over one period.

(c) **2 points** Suppose the signals x is measured in Volts. What is the unit of X_0? Volts.

(d) **4 points** Calculate the values of the coefficients $X_1 = \int_{t=0}^{0.25} e^{-j2\pi t} dt = \frac{1-j}{2\pi}$

$Y_1 = \int_{t=0.5}^{0.75} e^{-j2\pi t} dt = \frac{-1+j}{2\pi}$

$U_1 = \frac{1}{10} \sum_{n=0}^{9} x(n) e^{-j\pi/5n} = \frac{1}{10} \sum_{n=0}^{3} e^{-j\pi/5n} = \frac{1-e^{-j4\pi/5}}{10(1-e^{-j\pi/5})}$

$V_1 = \frac{1}{10} \sum_{n=3}^{6} e^{-j\pi/5n} = e^{-j3\pi/5} \frac{1-e^{-j4\pi/5}}{10(1-e^{-j\pi/5})}$

(e) **3 points** Express y as a delayed version of x and v as a delayed version of u.

$\forall t, y(t) = x(t-0.5)$, $\forall n, v(n) = x(n-3)$.

(f) **4 points** Express the FS coefficients $\{Y_k\}$ in terms of $\{X_k\}$ and $\{V_k\}$ in terms of $\{U_k\}$.
$\forall k, \quad Y_k = X_k e^{-jk\pi}$, $\forall k, \quad V_k = U_k e^{-jk3\pi/5}$.

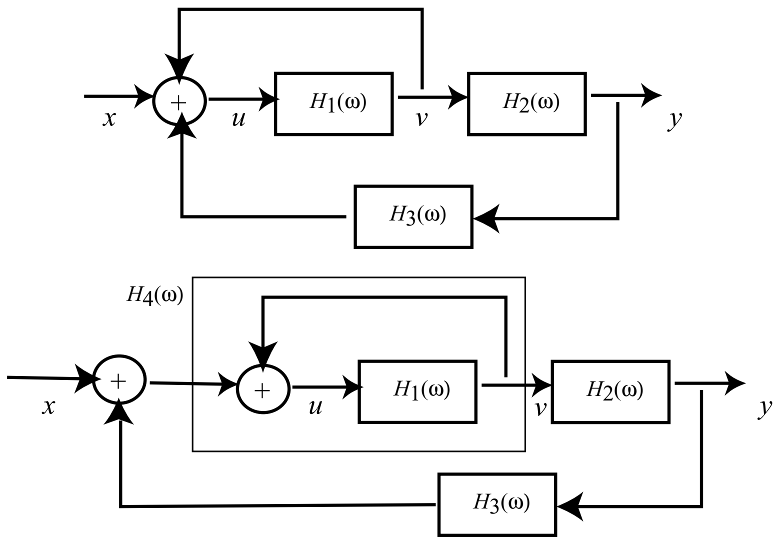

Figure 4: Feedback systems for Problem 7

7. **15 points** Figure 4 shows two feedback systems. In these figures, $H_k(\omega), k = 1, 2, 3$ is the frequency response of the three LTI systems.

   (a) **9 points** Calculate the closed-loop frequency response $H(\omega)$ of the first feedback system in terms of the H_k. Hint: Use the fact that the two systems are the same.
   By the hint, $H_4 = H_1/(1 - H_1)$. So,

$$H = \frac{H_4 H_2}{1 - H_4 H_2 H_3} = \frac{H_1 H_2}{1 - H_1 - H_1 H_2 H_3} \tag{1}$$

   (b) **6 points** Suppose $H_k(\omega) = 1/(1 + j2\omega)$ for all $k = 1, 2, 3$. Calculate $H(0)$, $H(1)$ and $\lim_{\omega\to\infty} H(\omega)$.
   We have $H_k(0) = 1, H_k(1) = 1/(1 + j2)$, $\lim_{\omega\to\infty} H_k(\omega) = 0$. Substitution in (1) gives,

$$H(0) = -1, H(1) = \frac{\frac{1}{(1+2j)^2}}{1 - \frac{1}{1+2j} - \frac{1}{(1+2j)^3}} = \frac{1 + 2j}{(1 + 2j)^3 - (1 + 2j)^2 - 1}$$

   $lim_{\omega\to\infty} H(\omega) = 0.$

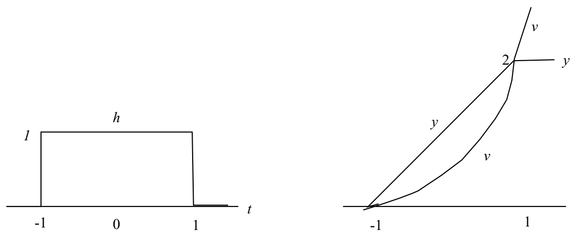

Figure 5: Impulse response for Problem 8

8. **15 points** A continuous-time LTI system has the impulse response

$$\forall t \in \textit{Reals}, \quad h(t) = \begin{cases} 1, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) **3 points** Sketch the impulse response, and mark carefully the relevant points on your plot.

This is shown in Figure 5

(b) **3 points** Is this system causal? Answer yes or no. NOT CAUSAL.

(c) **3 points** What is the step response of this system, i.e. the response to *step*$(t) = 1, t \geq 0$ and $= 0, t < 0$?

The step response is the integral of the impulse response,

$$y(t) = \int_{s=-\infty}^{t} h(s)ds = \begin{cases} 0, & t \leq -1 \\ t+1, & -1 \leq t \leq 1 \\ 2, & t \geq 1 \end{cases}$$

A sketch of y is in the figure.

(d) **3 points** What is the ramp response of this system, i.e. the response to *ramp*$(t) = t, t \geq 0$, and $= 0, t < 0$?

The ramp response v is the integral of the step response,

$$v(t) = \int_{s=-\infty}^{t} y(s)ds = \begin{cases} 0, & t \leq -1 \\ \frac{1}{2}(t+1)^2, -1 \leq t \leq 1, & \\ 2 + 2(t-1), & t \geq 1 \end{cases}$$

(e) **3 points** What is the respone of this system to the input signal *impulsetrain*, where

$$\forall t \in \textit{Reals}, \quad \textit{impulsetrain}(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k).$$

The response is

$$\begin{aligned} \forall t, \quad w(t) &= h * \textit{impulsetrain}(t) \\ &= \int_{\tau=-\infty}^{\infty} h(t-\tau)\textit{impulsetrain}(\tau)d\tau \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} h(t-\tau)\delta(\tau-2k) \\
&= \sum_{k=-\infty}^{\infty} h(t-2k) = 1.
\end{aligned}
$$