Mid-term 2 practice problems, Nov 6, 2000.

1. Represent the following numbers in Cartesian coordinates (i.e. in the form $x + jy$) and in polar coordinates (i.e. in the form $re^{j\theta}$).

   (a) $\frac{1+j1}{1-j1}$

   (b) $(1 + j1)(1 - j1)$

   (c) $\frac{-1-j1}{2+j3}$

   (d) $\frac{-1-j1}{2+j3} + \frac{1+j1}{1-j1}$

2. Find A, θ, ω in the following expression:

$$A\cos(\omega t + \theta) = \cos(2\pi \times 10,000t + \frac{\pi}{4}) + \sin(2\pi \times 10,000t + \frac{\pi}{4}).$$

3. Which of the following signals are periodic, what is their period, what is their fundamental frequency? Give the units of the period and frequency. Justify your answer in each case.

   (a) $\forall t \in \textit{Reals}, \quad x(t) = \cos(\sqrt{3}t) + \sin(2\sqrt{3}t) + \cos(3\sqrt{3}t + \frac{5\pi}{6})$.

   (b) $\forall n \in \textit{Ints}, \quad y(n) = \cos(3\pi n) + \sin(12\pi n)$

4. (a) Show that for any frequency $|\omega| > \pi$ rads/sample, there is a frequency ω_0 with $|\omega_0| \leq \pi$ rads/sample such that

$$\forall n \in \textit{Ints}, \quad \cos(\omega n + \theta) = \cos(\omega_0 n + \theta).$$

   (b) Find ω_0 for $\omega = \pm 5\pi/4, \pm 3\pi/2, \pm 7\pi/4, \pm 5\pi/2, \pm 7\pi/2$.

   (c) Plot ω_0 as a function of ω.

5. By contrast with Problem 4a suppose

$$\forall t \in \textit{Reals}, \quad \cos(\omega t + \theta) = \cos(\omega_0 t + \theta).$$

   Show that $|\omega| = |\omega_0|$.

6. This problem asks you to determine various properties of Fourier series, some of which are used in later problems. Let $x \in \textit{ContPeriodic}_p$ be a signal with exponential Fourier series coefficients $X_k, k \in \textit{Ints}$.

   (a) (Time-shift) Suppose $y = D_\tau(x)$. Obtain the FS coefficients Y_k in terms of X_k.

   (b) (Time-scale) Suppose $\forall t, y(t) = x(at)$ where $a > 0$. Show that y is periodic with period p/a. Obtain the FS coefficients Y_k in terms of X_k.

   (c) (Time-scale) Repeat the previous problem with $a < 0$.

7. Repeat problem 6 for $x : \textit{Ints} \to \textit{Reals}$, taking τ and p/a to be integers.

8. Figure 1 (a) is a portion of the graph of a periodic signal x with period p seconds.

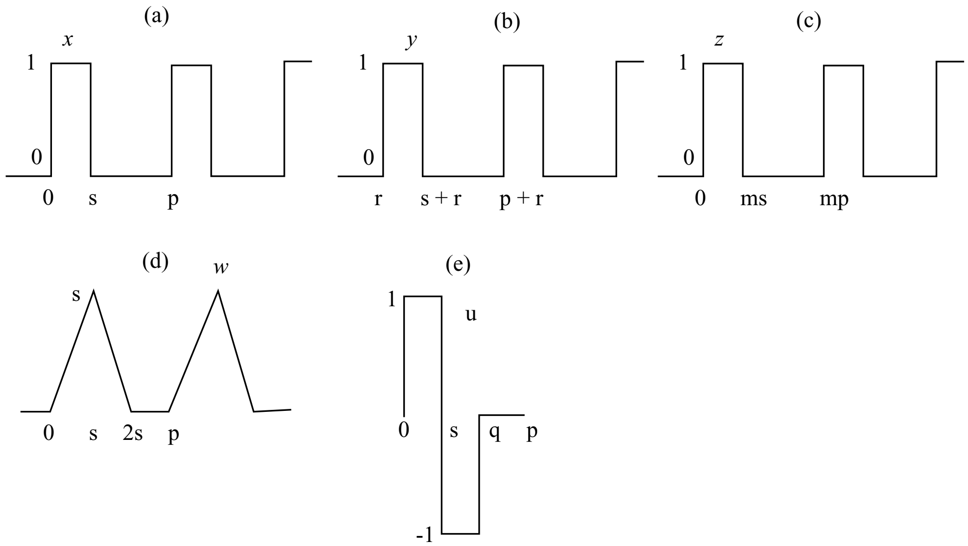


Figure 1: Figure for Problem 8

(a) Express x in the following form:

$$\forall t \in \textit{Reals}, \quad x(t) = \begin{cases} 1, & \text{if } t \in ???? \\ 0, & \text{otherwise} \end{cases}$$

(b) Find the exponential Fourier series of x.

(c) The signal y in figure 1 (b) is the signal x delayed by r, i.e. $\forall t, y(t) = x(t - r)$. Relate the exponential Fourier series coefficients of y to those of x. Then calculate the Fourier series of y.

(d) The signal z in figure 1 (c) is obtained from x by a time-scale change, since $\forall t, z(t) = x(\frac{t}{m})$ where $m > 0$ is a constant. Relate the exponential Fourier series coefficients of z to those of x. Then calculate the Fourier series of z.

9. Suppose $x, y \in \textit{ContPeriodic}_p$ are such that $\forall t, x(t) = \dot{y}(t)$. Let X_k and Y_k be the corresponding exponential FS coefficients. Show that

$$X_0 = 0, X_k = jk\frac{2\pi}{p}Y_k, \ \ k \neq 0.$$

Use this relation to obtain the FS of the signal w in Figure 1 (d) in terms of the FS of the signal x in Figure 1 (a). [Hint: $x(t) = \dot{w}(t)$, for all t.]

10. Suppose $u, v \in \textit{ContPeriodic}_p$ with FS coefficients U_k, V_k. Let $x = \alpha u + \beta v$ where α, β are real numbers.

(a) Show that $x \in \textit{ContPeriodic}_p$. Find the FS coefficients of x in terms of U_k, V_k.

(b) Use this result to find the FS of the signal u in Figure 1(e) in terms of the FS of x in Figure 1(a).

11. Let $x : \{0, 1, \cdots, p-1\} \to$ *Reals* be any finite, discrete-time signal. Then x can be uniquely represented as

$$\forall 0 \leq n \leq p-1, \quad x(n) = \sum_{k=0}^{p-1} X_k e^{j\omega_0 kn},$$

where $\omega_0 = 2\pi/p$. Find this representation for the following 4-point sequences:

   (a) $x(0) = 1, x(1) = x(2) = x(3) = 0.$

   (b) $x(0) = x(2) = 1, x(1) = x(3) = -1.$

   (c) $x(0) = x(2) = 2, x(1) = x(3) = 0.$

12. This is sometimes called upsampling. Let $x \in$ *DiscPeriodic*$_p$. Obtain $y \in$ *DiscPeriodic*$_{2p}$ by $y(n) = x(n/2)$ if n is even, and $y(n) = 0$ if n is odd.

   (a) Sketch x for $p = 3$ and then sketch y.

   (b) Find the exponential FS coefficients of y in terms of those of x.

13. Find the frequency response $\hat{H}(\omega)$ of the LTI system given by the differential equation

$$\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t) + 2x(t),$$

and draw the magnitude response $|\hat{H}(\omega)|$ and phase response $\angle\hat{H}(\omega)$.

14. Repeat Problem 13 for the difference equation

$$y(n-2) + y(n-1) + y(n) = x(n) + 2x(n-1).$$

15. The RL circuit of Figure 2 has input voltage signal x and the output is the voltage y across the resistor. The two signals are related by the differential equation

$$\frac{L}{R}\dot{y}(t) + y(t) = x(t).$$

   (a) Show that the frequency response is $\hat{H}(\omega) = \frac{1}{1+j\omega\tau}$ where $\tau = L/R$ is called the time constant of the circuit.

   (b) Plot the magnitude and phase response. Carefully mark the values for $\omega = 1/\tau$.

   (c) Suppose $\tau = 1$ ms (millisecond). What is the response y of the system for the input signal

$$\forall t, \quad \sin(100t) + \sin(1000t) + \sin(10000t). \tag{1}$$

   (d) Suppose $\hat{H}(\omega)$ is an ideal low-pass filter with cutoff 1000 rad/sec, i.e. $\hat{H}(\omega) = 1$ for $|\omega| \leq 1000$ rads/sec and $\hat{H}(\omega) = 0$, otherwise. What is the response of this system to the signal x in (1).

   (e) Why is the circuit called a first-order low-pass RL filter?

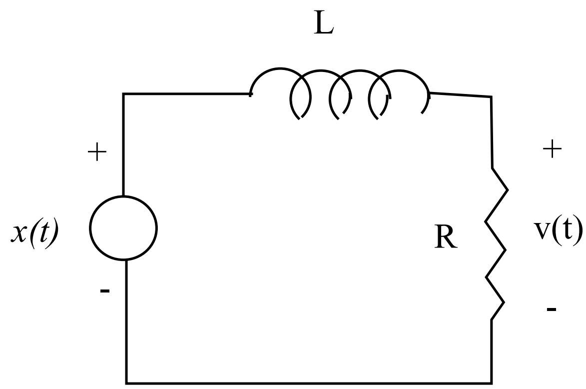


Figure 2: Circuit of Problem 15

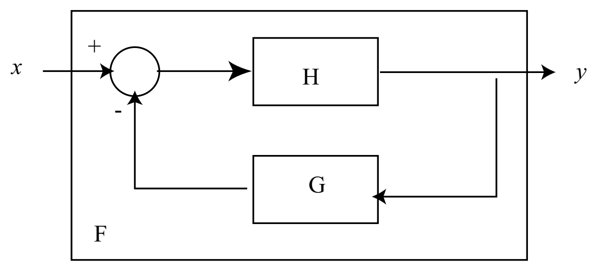


Figure 3: Feedback interconnection for Problem 16

16. In the block diagram of Figure 3, the LTI systems H, G have frequency response $\hat{H}, \hat{G}$. Find the frequency response of the close-loop system F in terms of those of H, G.

   (a) Suppose $\hat{G}(\omega) \equiv 1$ and $\hat{H}(\omega) = 1/(1 + j\omega\tau)$. What is $\hat{F}$?

   (b) Supopse $\hat{H}(\omega) \equiv 1$ and $\hat{G}(\omega) = 1(1 + j\omega\tau)$. What is $\hat{F}$?

EECS 20N Fall 2000
Practice Problem Set 1

November 7, 2000

1. Consider the impulse response $h[n]$ shown in figure 1. Find it's $H(\omega)$. Plot between $0 < \omega < \pi$. Plot between $-\pi < \omega < \pi$. Plot between $0 < \omega < 2\pi$.

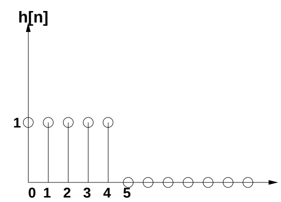


Figure 1: h[n] for Problem 1.

2. Consider the following second order difference equation:

$$y[n-1] = x[n] + x[n-2].$$

Find $H(\omega)$.

3. Find the System response for the following systems with feedback:

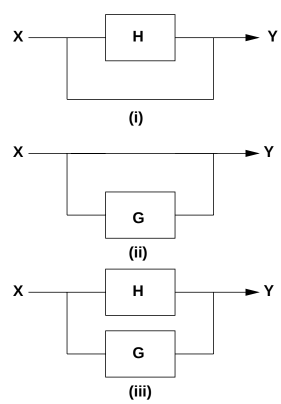


Figure 2: System diagrams for Problem 3.

EECS 20 Practice Problems

1. Express in Cartesian coordinates, a+bi) :

a. $i^3 - i^2 + i - 1$

b. $(5+3i)/(7-3i)$

c. $7e^{i\pi/6} + 3e^{-i\pi/6}$

d. $$\sum_{k=0}^{17} e^{ik\pi/6}$$

3. Given:

$$X(w) = 1/(1+iw)$$

plot $|X(w)|$ and $\angle X(w)$ for $w \in [-2\pi, 2\pi]$

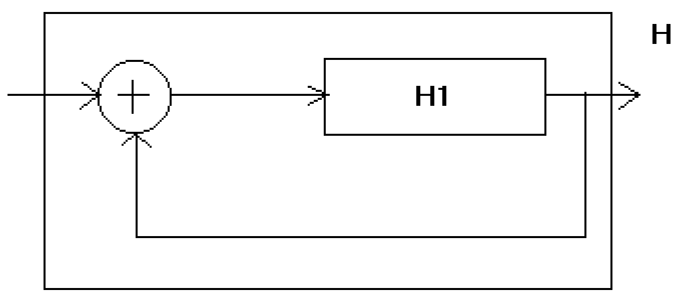
4. Prove:

$$z^n = (re^{i\theta})^n = r^n(\cos(n\theta) + i\sin(n\theta)) \quad \text{[DeMoivre's Theorem]}$$

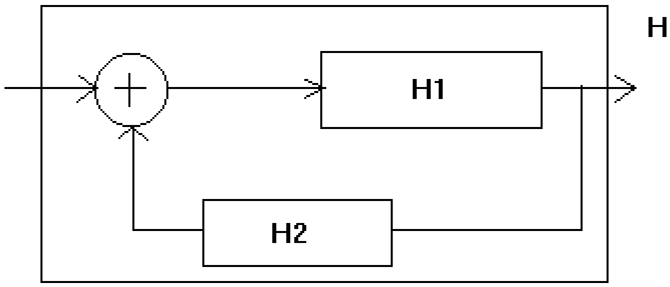
[Hint: Use Induction]

5. Find the frequency response H in terms of the frequency response H_1, H_2 and H_3 for:

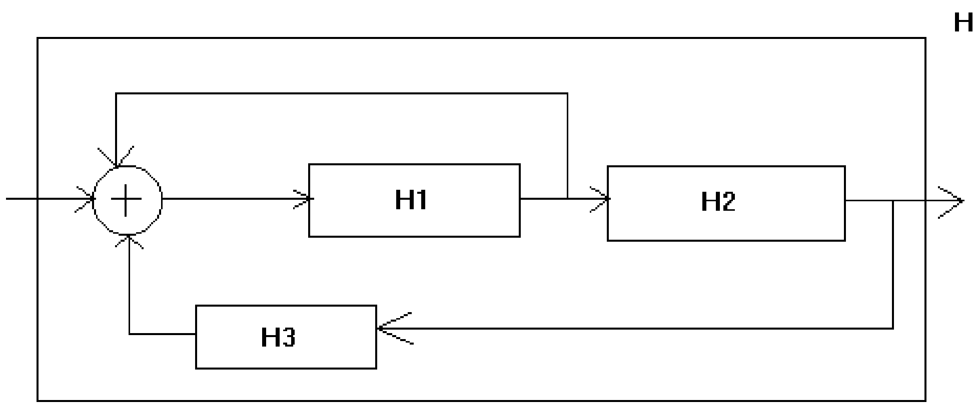
a)



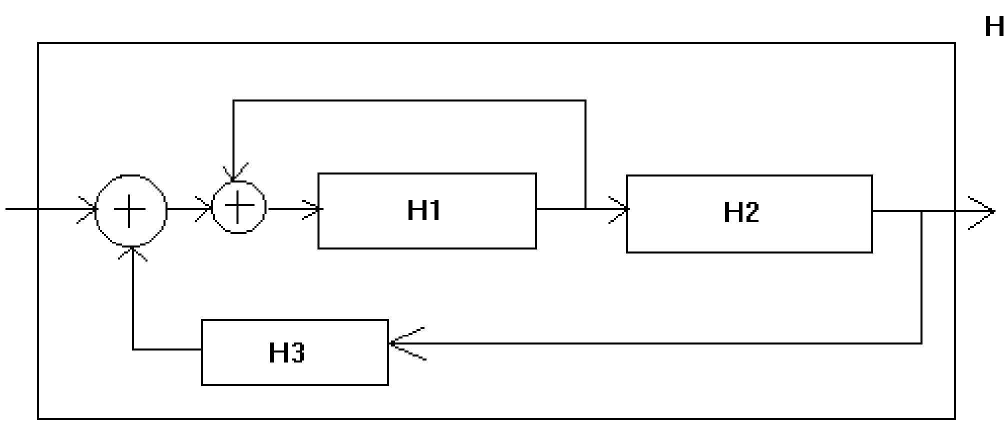
b)



c)



Hint: Redraw the above system as:



EECS 20N Fall 2000
Practice Problem Set 2

November 9, 2000

1. Here is a typical RC circuit that behaves as a LTI system.

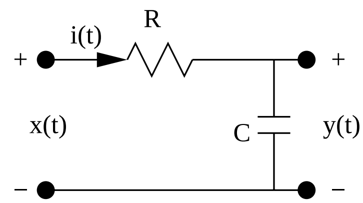


Figure 1: RC circuit for Problem 1.

Here $x(t)$ and $y(t)$ are the voltage for the input and the voltage across the capacitor. Using basic circiut analysis, we get $i(t) = C\dot{y}(t)$ from the voltage and current relation across the capacitor. And using KVL on the closed loop, we get $x(t) = Ri(t) + y(t)$.

(i) Write down the differential equation for this circuits in term of $y(t)$, $x(t)$ and $\dot{y}(t)$.

(ii) Let $X(\omega) = FT(x)$, $Y(\omega) = FT(y)$, and $FT(\dot{y}) = j\omega Y(\omega)$, where $FT(x)$ denotes the Fourier transform of $x(t)$. Write down the Fourier domain equation for the equation in (i) in term of $Y(\omega)$ and $X(\omega)$.

(iii) What is the frequency response of the system?
Hint: $H(\omega) = Y(\omega)/X(\omega)$.

(iv) Let $RC = 1/(2\pi 1000)$. Plot $|H(w)|$, $\angle(H(\omega))$ for ω between $0, 2\pi 10000$ in Matlab.

(v) Plot $|H(\omega)|_{dB}$ vs. $\log_{10}(\omega)$ for the same range of ω in Matlab.
Hint: Use semilogx instead of plot. This plot is called the bode plot of the frequency response. $|H(\omega)|_{dB} = 20 \log_{10} |H(\omega)|$.

(vi) Find $H(0), H(2\pi 1000), H(2\pi 10000)$.

(vii) Let $x(t) = sin(2\pi 1000t)$, what is $y(t)$?

(viii) Let $x(t) = sin(2\pi 1000t + \pi/4)$, what is $y(t)$?

(ix) Let $x(t) = sin(2\pi 10000t)$, what is $y(t)$? How does the magnitude of $y(t)$ compare with the $y(t)$ you obtained in (vii)? How do you explain this magnitude difference?

2. Now consider a more complex circuit with $i_C(t) = C dv_C/dt$, where $i_C(t)$ is the current through the capacitor when the voltage across it is given by $v_C(t)$. And $v_L(t) = L di_L/dt$, where $i_L(t)$ is the current through the inductor when the voltage across it is given by $v_L(t)$.

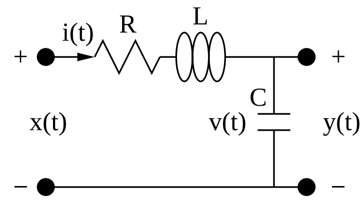


Figure 2: RLC circuit for Problem 2.

(i) Give a differential equation on time domain for this system similar to the differential equation in problem 1. What is the order of differential equation in this problem?

(ii) Given $FT(\ddot{y}) = -\omega^2 Y(\omega)$. Find $H(\omega)$.

(iii) Let $RC = 1/(1000\pi)$, $LC = (1/(2\pi 1000))^2$. Plot magnitude and phase of $H(\omega)$. What is the relationship between $H(\omega)$ in this problem with the $H(\omega)$ in problem 1?

(iv) Plot $|H(w)|_{dB}$ vs. $\log_{10}(\omega)$. How does this plot compare with the plot in problem 1? Notice the change in slope.