10/12/99 Solutions to sample problems for MT 1. Prof. Varaiya

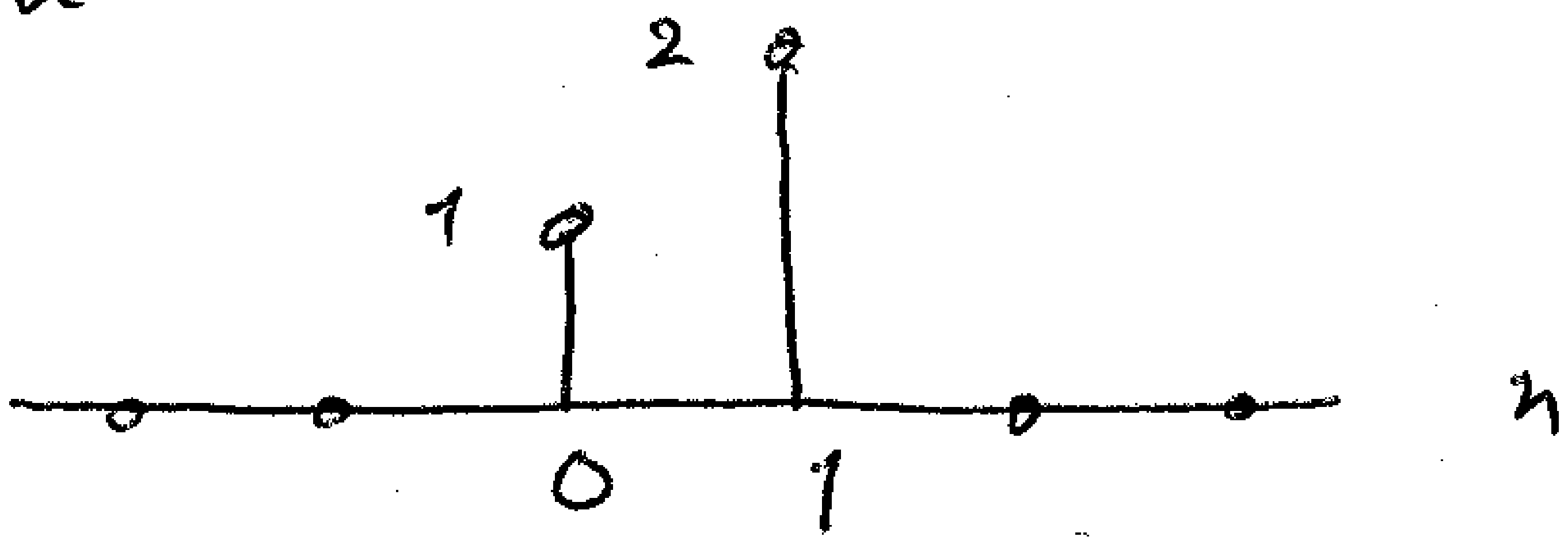
1.

System	L	TI	Memless	Causal
$H_1(x)(t) = x(t-1)$	Y	Y	N	Y
$H_2(x)(t) = x(-t)$	Y	N	N	N
$H_3(x)(t) = x(2t)$	Y	N	N	N
$H_4(x)(t) = x(t)+1$	N	Y	Y	Y

2. $\forall n, \quad h(n) = \delta(n) + 2\delta(n-1)$

So,
$$h(n) = \begin{cases} 1, & n=0 \\ 2, & n=1 \\ 0, & \text{else} \end{cases} \qquad (1)$$

a) Plot of h

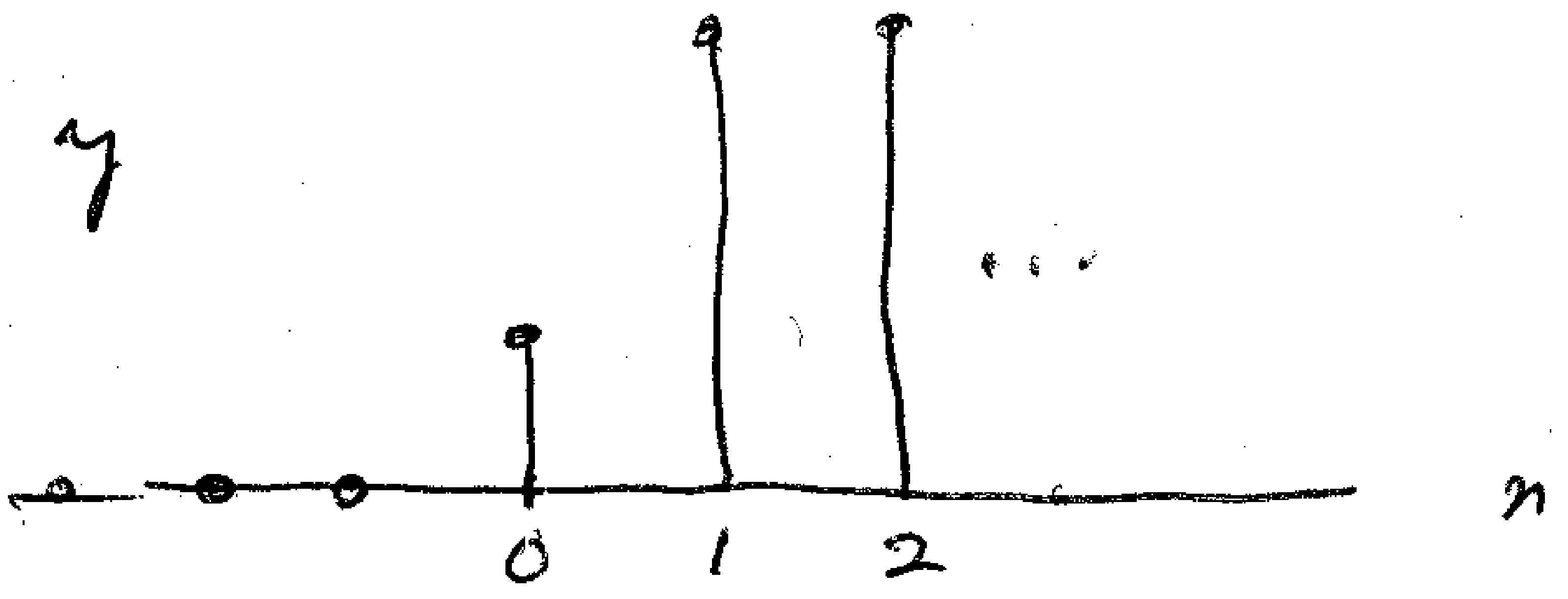


b) In general
$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= 1\,x(n) + 2\,x(n-1) \qquad \text{from } (1)$$

$$= \begin{cases} 0, & n<0 \\ 1, & n=0 \\ 3, & n \geq 1 \end{cases}$$ using the fact that x is unit step

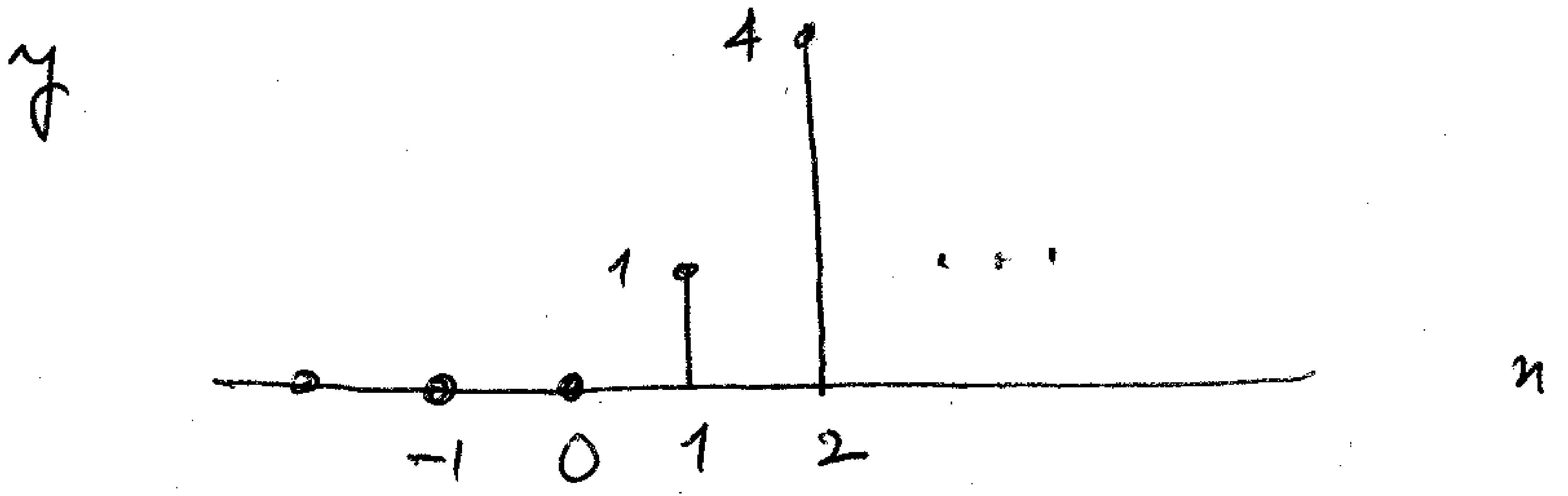
Plot of y



c) Using the form of x

$$y(n) = 1 \cdot x(n) + 2x(n-1)$$

$$= \begin{cases} 0 , & n < 0 \\ 0 , & n = 0 \\ n + 2(n-1) = 3n-2 , & n \geq 1 \end{cases}$$



d) In general $\hat{H}(\omega) = \mathrm{DTFT}(h)$,

$$\hat{H}(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-i\omega k}$$

$$= 1 + 2e^{-i\omega} , \quad \text{from} \quad (1)$$

e) For any ω

$$\hat{H}(\omega+2\pi) = 1 + 2e^{-i(\omega+2\pi)}$$

$$= 1 + 2e^{-i\omega} \cdot e^{-i2\pi}$$

$$= 1 + 2e^{-i\omega} , \quad \text{since} \quad e^{i2\pi} = 1$$

$$= \hat{H}(\omega)$$

f) $\hat{H}(\omega) = 1 + 2e^{-i\omega} = 1 + 2\cos\omega - 2i\sin\omega$

$\hat{H}(-\omega) = 1 + 2\cos(-\omega) - 2i\sin(-\omega)$

$$= 1 + 2\cos\omega + 2i\sin\omega$$

$$= [1 + 2\cos\omega - 2i\sin\omega]^* = [H(\omega)]^*$$

g) $$\hat{H}(\omega) = [(1+2\cos\omega)^2 + 4\sin^2(\omega)]^{1/2}$$

$$= [1 + 4\cos\omega + 4\cos^2\omega + 4\sin^2\omega]^{1/2}$$

$$= [5 + 4\cos\omega]^{1/2}$$

$$\measuredangle \hat{H}(\omega) = \tan^{-1}\left(-2\sin\omega / (1+2\cos\omega)\right)$$

$$= -\tan^{-1}\left(\frac{2\sin\omega}{1+2\cos\omega}\right)$$

h) $$|\hat{H}(-\omega)| = [5 + 4\cos(-\omega)]^{1/2}$$

$$= [5 + 4\cos\omega]^{1/2}, \quad \text{since } \cos(-\omega) = \cos\omega$$

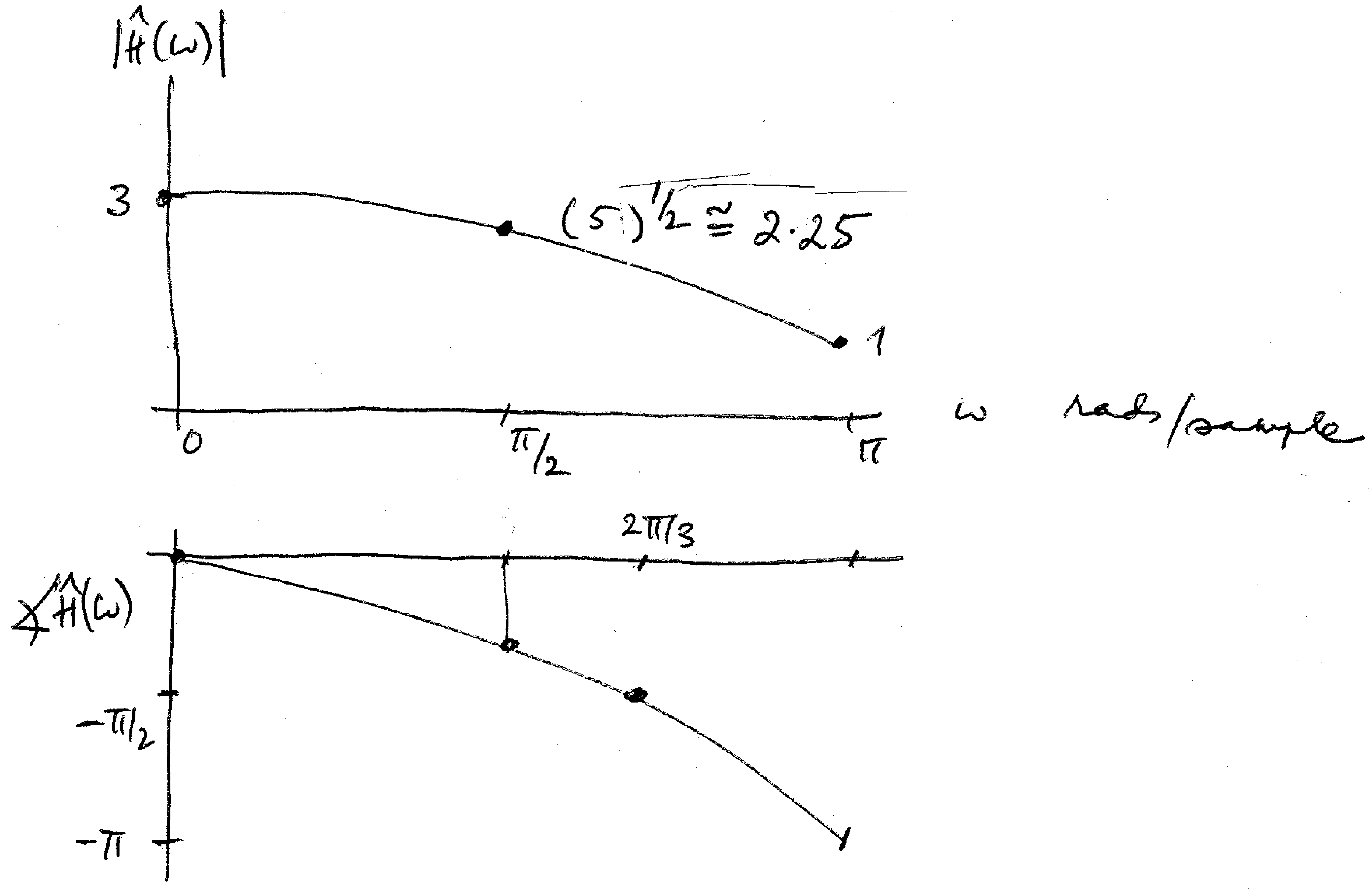
$$= |\hat{H}(\omega)|$$

$$\measuredangle \hat{H}(-\omega) = -\tan^{-1}\left(\frac{2\sin(-\omega)}{1+2\cos(\omega)}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\omega}{1+2\cos\omega}\right), \quad \text{since } \tan^{-1}(-\alpha) = -\tan^{-1}\alpha$$

$$= -\measuredangle \hat{H}(\omega)$$

(i)



To obtain these plots I use the following observations

(1) $|\hat{H}(\omega)| = [5 + 4\cos\omega]^{1/2}$

decreases as ω goes from 0 to π because $\cos\omega$ decreases. Then I use $\cos 0 = 1$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{2} = 0$, $\cos\pi = -1$ to get

$$\hat{H}(0) = 3, \quad \hat{H}\left(\frac{\pi}{4}\right) = (5 + 2\sqrt{2})^{1/2}, \quad \hat{H}\left(\frac{\pi}{2}\right) = 5^{1/2}, \quad \hat{H}(\pi) = 1$$

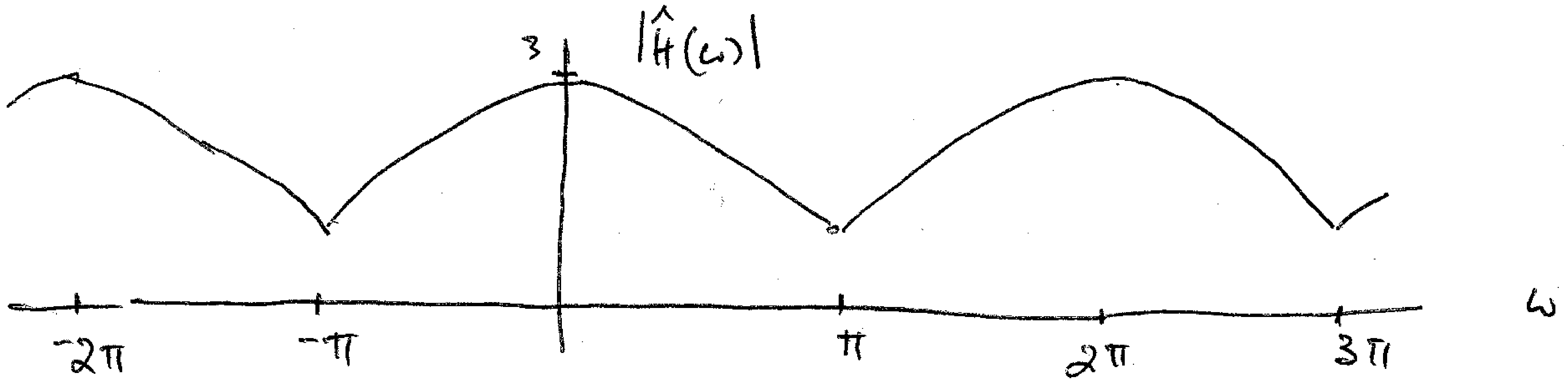
(2) $$\measuredangle \hat{H}(\omega) = -\tan^{-1} \frac{2\sin\omega}{1 + 2\cos\omega}$$

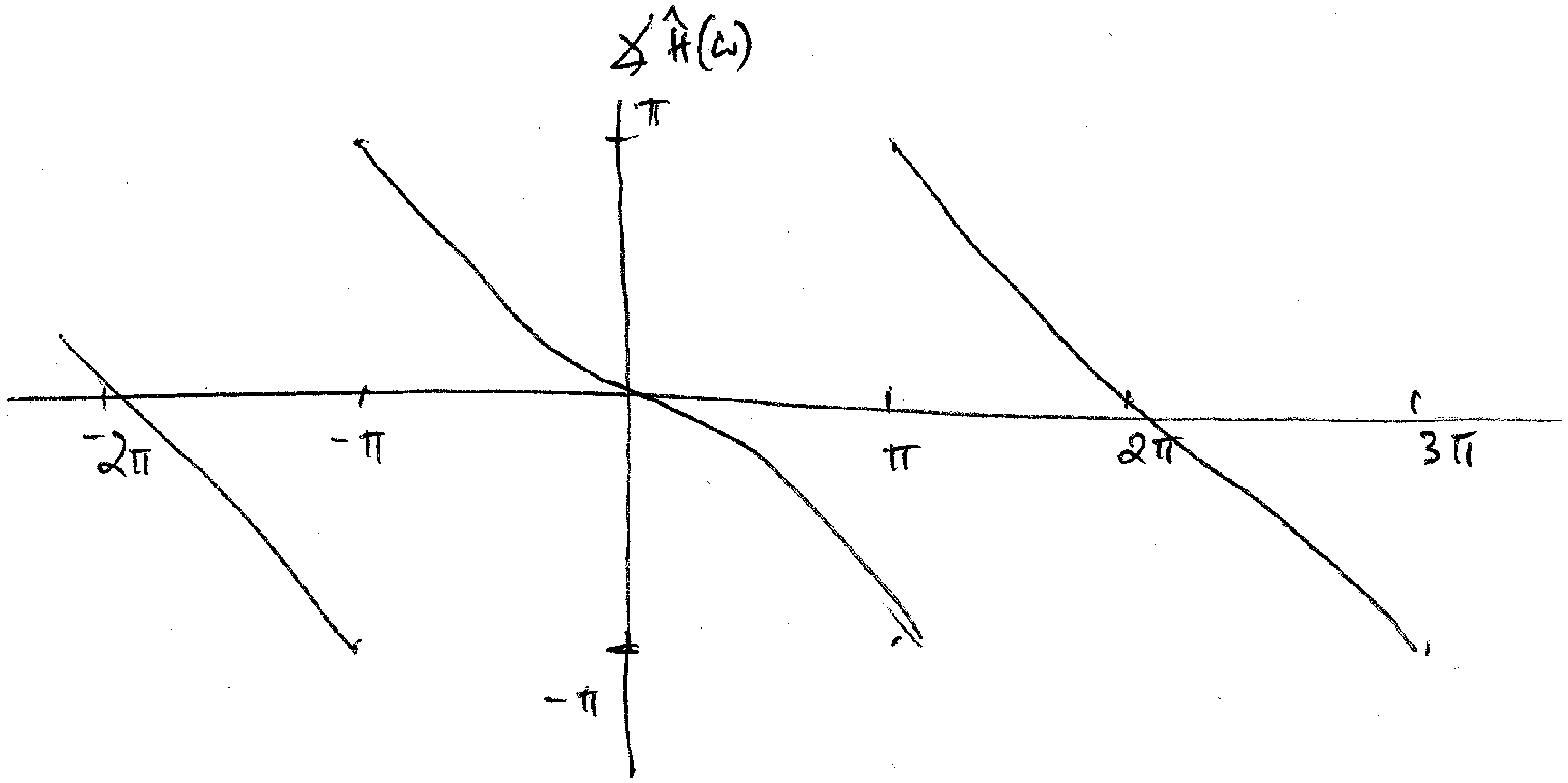
$\sin\omega$ is positive for $0 < \omega < \pi$ and

$(2\cos\omega + 1) > 0$ for $0 < \omega < \frac{2\pi}{3}$

< 0 for $\frac{2\pi}{3} < \omega < \pi$

(j) Use the fact that $|\hat{H}|$ and $\measuredangle \hat{H}$ are periodic with period 2π, $|\hat{H}|$ is even function, $\measuredangle \hat{H}$ is odd to get following plots.

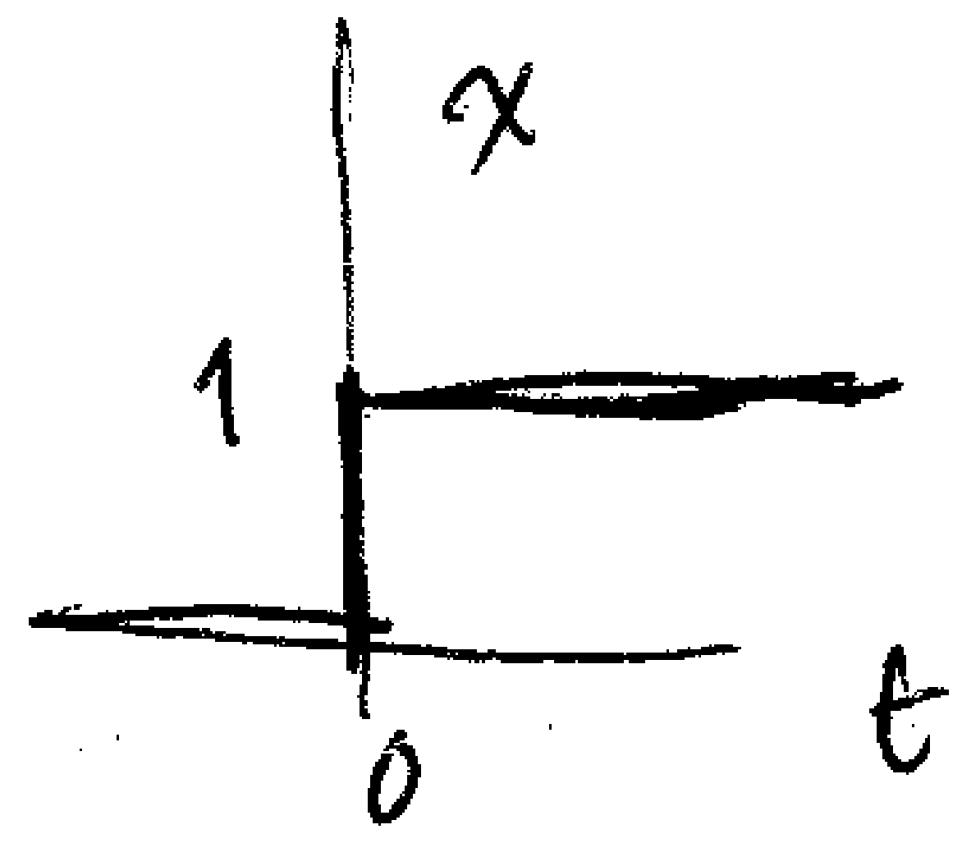


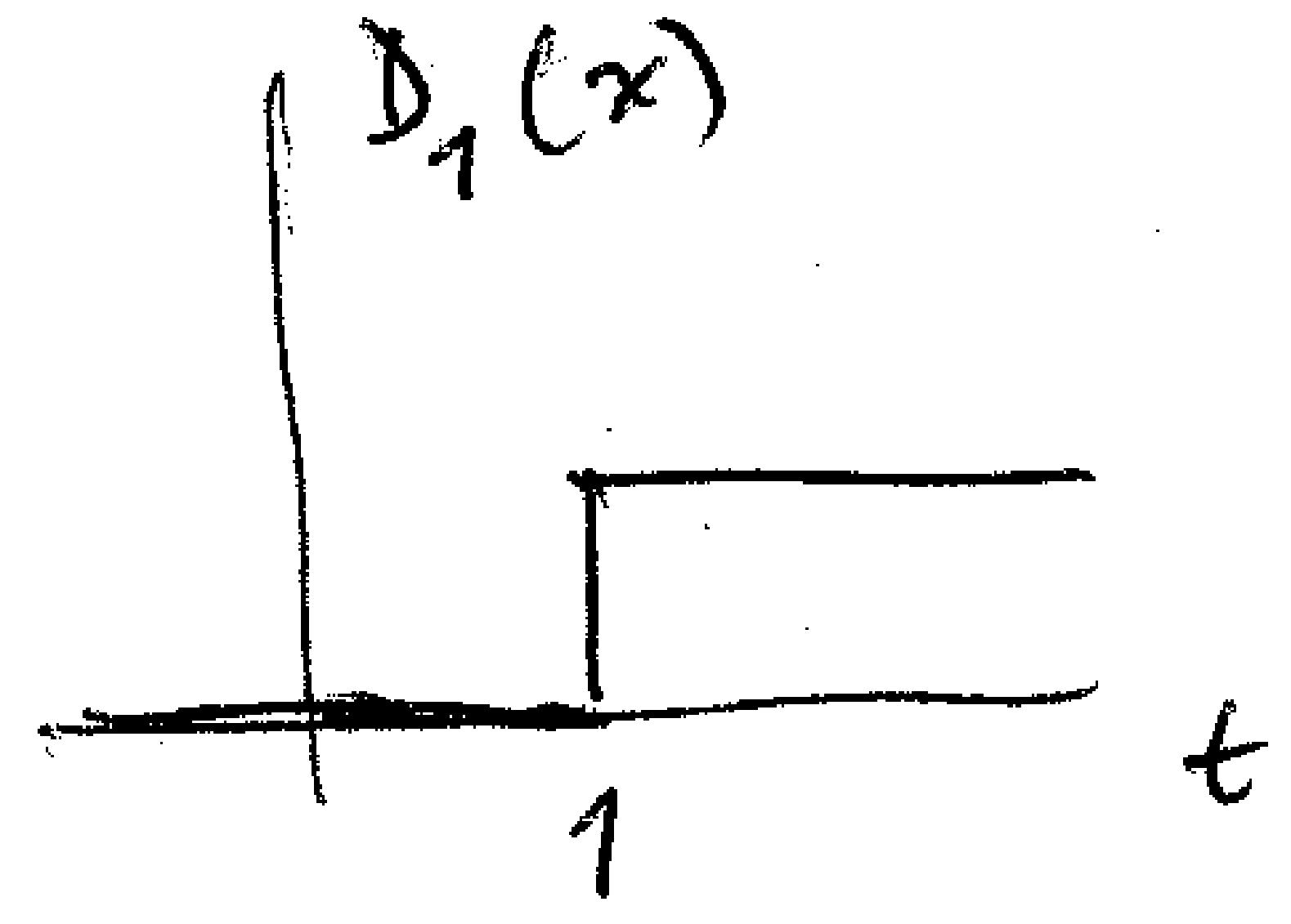


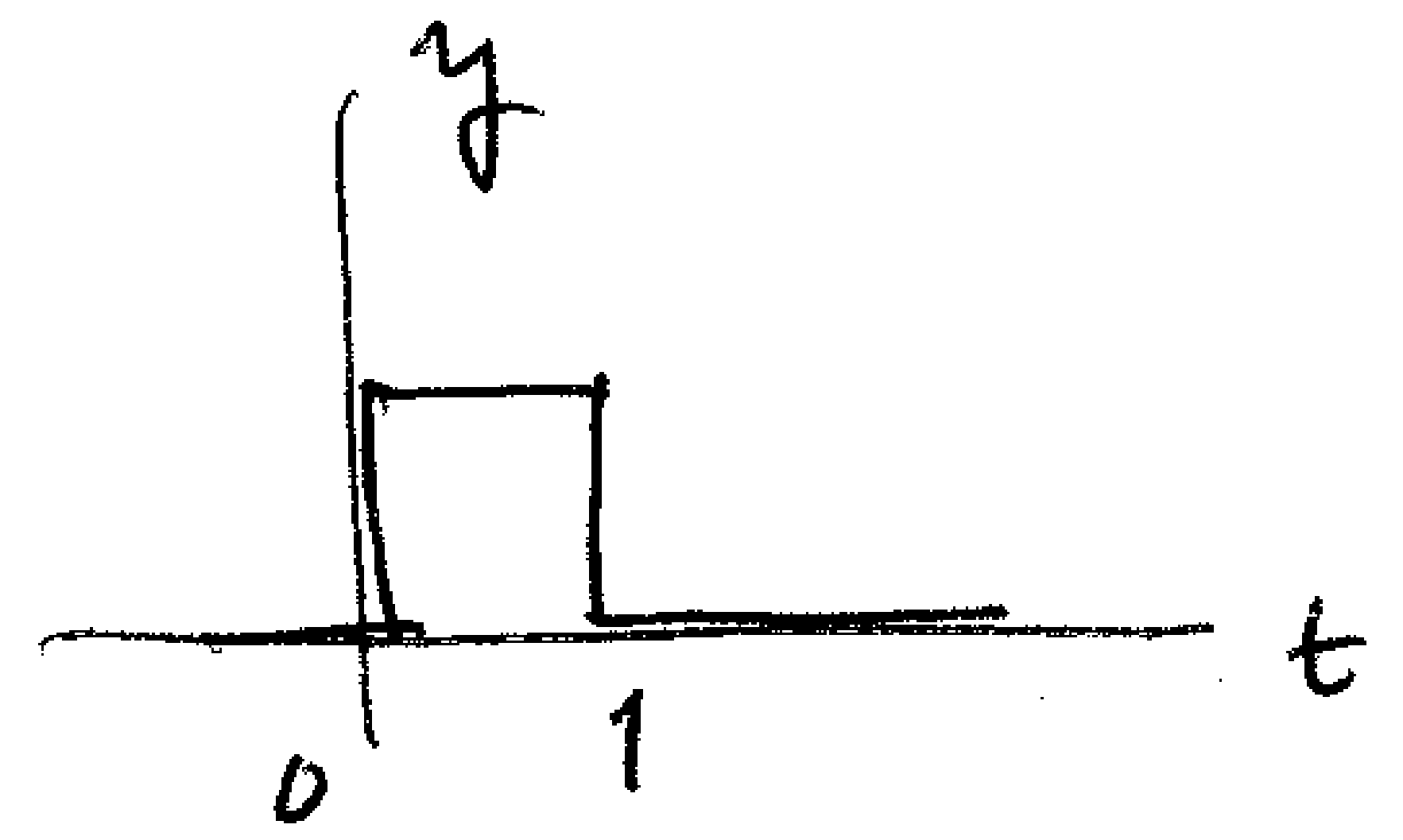
(k) In general, if the input is a sinusoidal signal of frequency ω, the LTI system's response is an output sinusoid of same frequency, with magnitude multiplied by $|\hat{H}(\omega)|$ and phase shifted by $\measuredangle \hat{H}(\omega)$. So in this case,

$$\forall n, \quad y(n) = |\hat{H}(25)| \cos\left(25n + \frac{\pi}{6} + \measuredangle \hat{H}(25)\right) + |\hat{H}(26)| \sin\left(26n + \frac{\pi}{3} + \measuredangle \hat{H}(26)\right).$$

3.

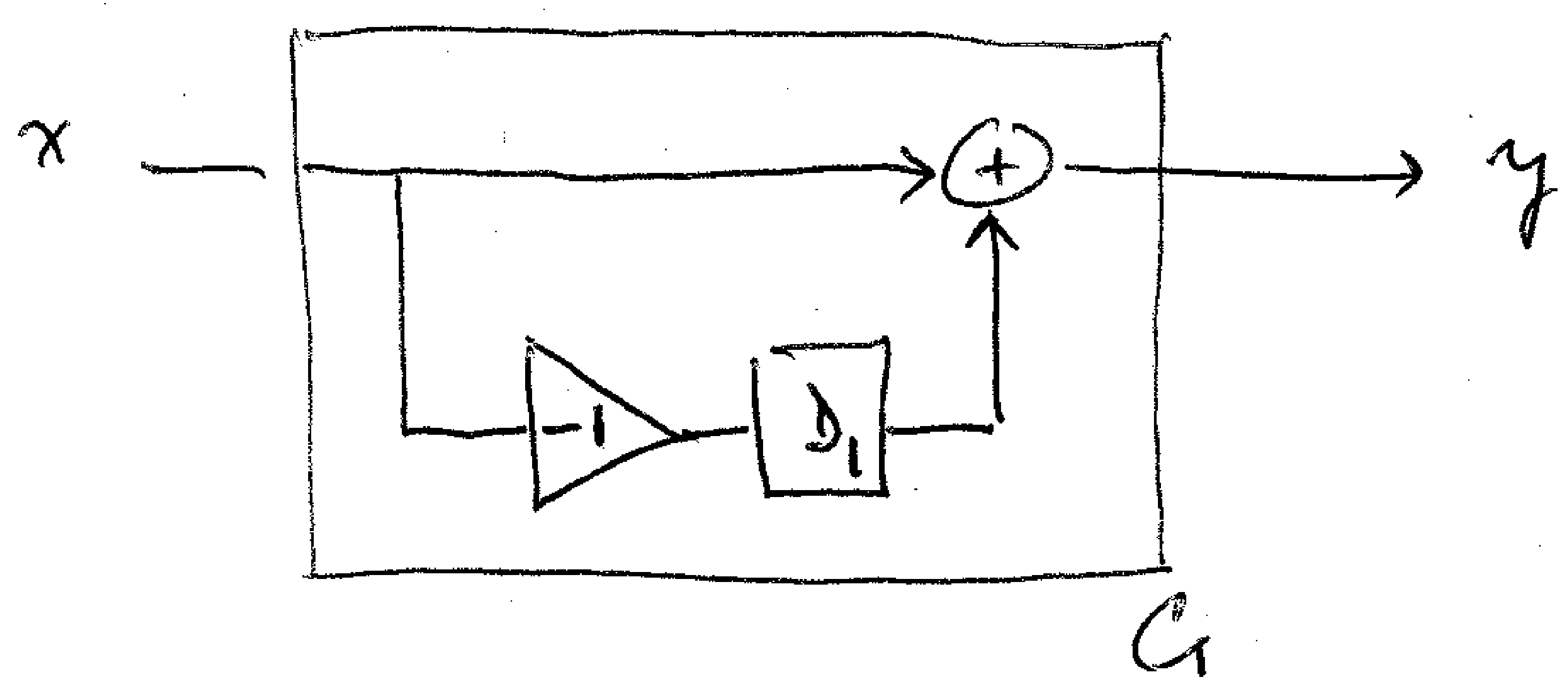






a) $$y = \overset{H(x)=}{\wedge}\, x - D_1(x) \qquad (*)$$

b)

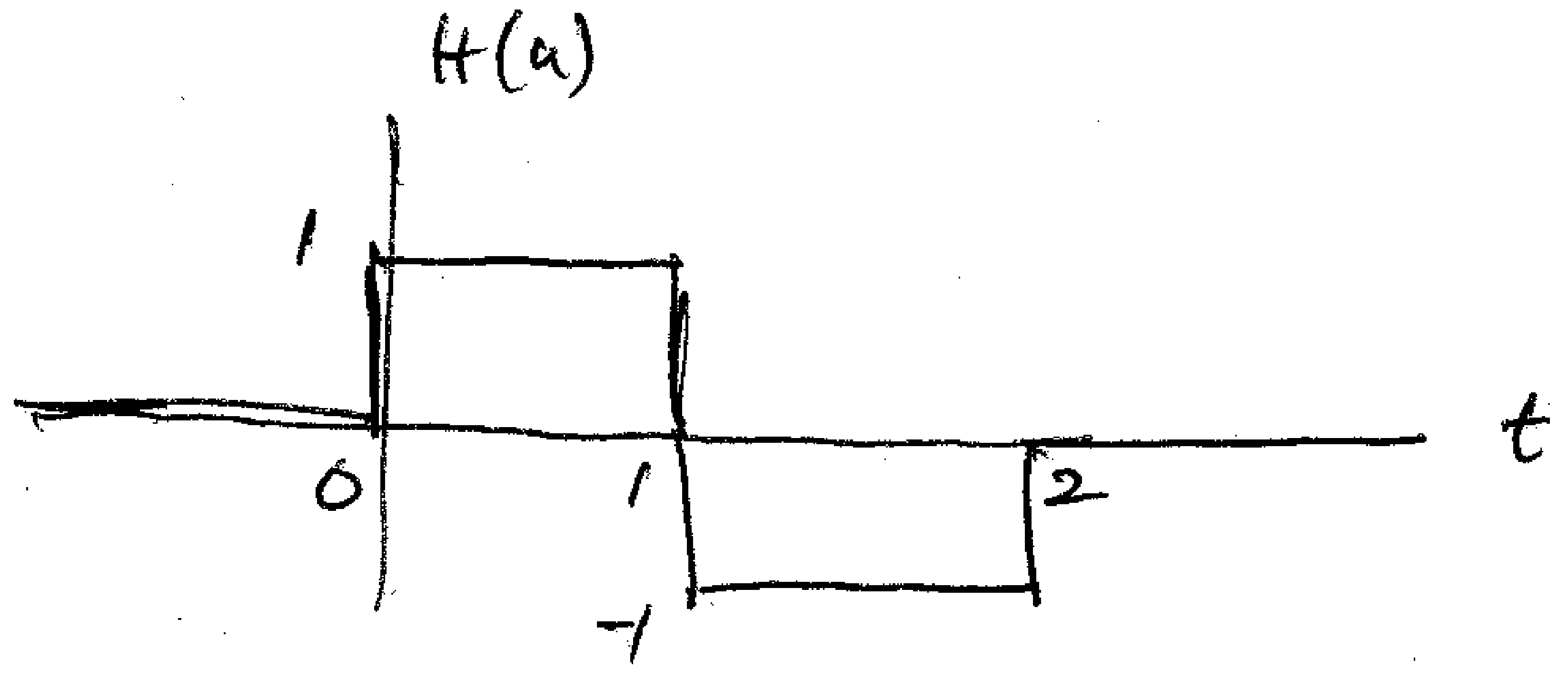


c) Calc the input for this part u,

$$u(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{else.} \end{cases}$$

Then $u = x - D_1(x)$ where x is the unit step. So the response to u is

$$H(u) = H(x - D_1(x))$$
$$= H(x) - H(D_1(x)) \qquad \text{by linearity}$$
$$= H(x) - D_1 H(x) \qquad \text{by time-invariance}$$
$$= y - D_1(y)$$
$$= (x - D_1(x)) - D_1(x - D_1(x))$$
$$= x - D_1(x) - D_1(x) + D_2(x) = x - 2D_1(x) + D_2(x)$$



d) Since for all signals x

$$G(x) = x - D_1(x)$$

we get

$$G(u) = u - D_1(u) = H(u)$$

e) To find $\hat{G}(\omega)$, take input

$\forall n \quad x(n) = e^{i\omega n}$.

The response of G to this input is

$$y(n) = x(n) - D_1(x)(n)$$

$$= e^{i\omega n} - e^{i\omega(n-1)}$$

$$= \left[1 - e^{-i\omega}\right] e^{i\omega n}$$

$$= \hat{G}(\omega) e^{i\omega n}$$

So

$$\hat{G}(\omega) = 1 - e^{-i\omega}$$

4. Call the periodic function Saw, then

$$\text{Saw}(t) = \sum_{-\infty}^{\infty} X_k e^{ik\omega t}$$

where $\omega = \frac{2\pi}{T} = 2\pi$ (T = 1 sec, the period) and

$$X_k = \frac{1}{1} \int_0^1 \text{Saw}(t)\, e^{-ik\omega t}\, dt$$

$$= 2 \int_0^1 t\, e^{-ik2\pi t}\, dt \qquad (*).$$

Now $X_0 = 1$, and for $k \neq 0$

$$\int_0^1 t e^{-ik2\pi t} dt = \int_0 t\, d\left(\frac{e^{-i2\pi kt}}{-i2\pi k}\right)$$

$$= t \cdot \frac{e^{-i2\pi kt}}{-i2\pi kt}\Bigg|_0^1 - \int_0^1 \frac{e^{-i2\pi kt}}{-i2\pi k}\, dt$$

$$= \frac{1}{-i2\pi k} - \frac{1}{(-i2\pi k)^2}\, e^{-i2\pi kt}\Bigg|_0^1$$

$$= \frac{i}{2\pi k} - \frac{1}{(-i2\pi k)^2}(1-1)$$

$$= \frac{i}{2\pi k}$$

Substituting in $(*)$ gives

$$\mathrm{Saw}(t) = 1 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{2i}{2\pi k} e^{ik2\pi t}$$

So the response of H to the signal Saw is

$$\forall t, \quad H(\mathrm{Saw})(t) = \hat{H}(0) \cdot 1 + \sum_{k \neq 0} \frac{2i}{2\pi k} \hat{H}(2\pi k) e^{i2\pi k t}$$

$$= \hat{H}(0) \cdot 1 = \underline{1}$$

since $\hat{H}(2\pi k) = 0$ for $k \neq 0$.

5. a) The response y in general is

$$\forall n \quad y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^k} x(n-k)$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{n} \frac{1}{2^k} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} & , n \geq 0 \\ 0 \quad , n < 0 \end{cases}$$

b) The frequency response is obtained by using the input

$$\forall n \quad x(n) = e^{i\omega n}$$

So

$$\hat{H}(\omega) e^{i\omega n} = \sum_{k=0}^{\infty} \frac{1}{2^k} e^{i\omega(n-k)}$$

$$= \left[\sum_{k=0}^{\infty} \frac{1}{2^k} e^{-i\omega k} \right] e^{i\omega n}$$

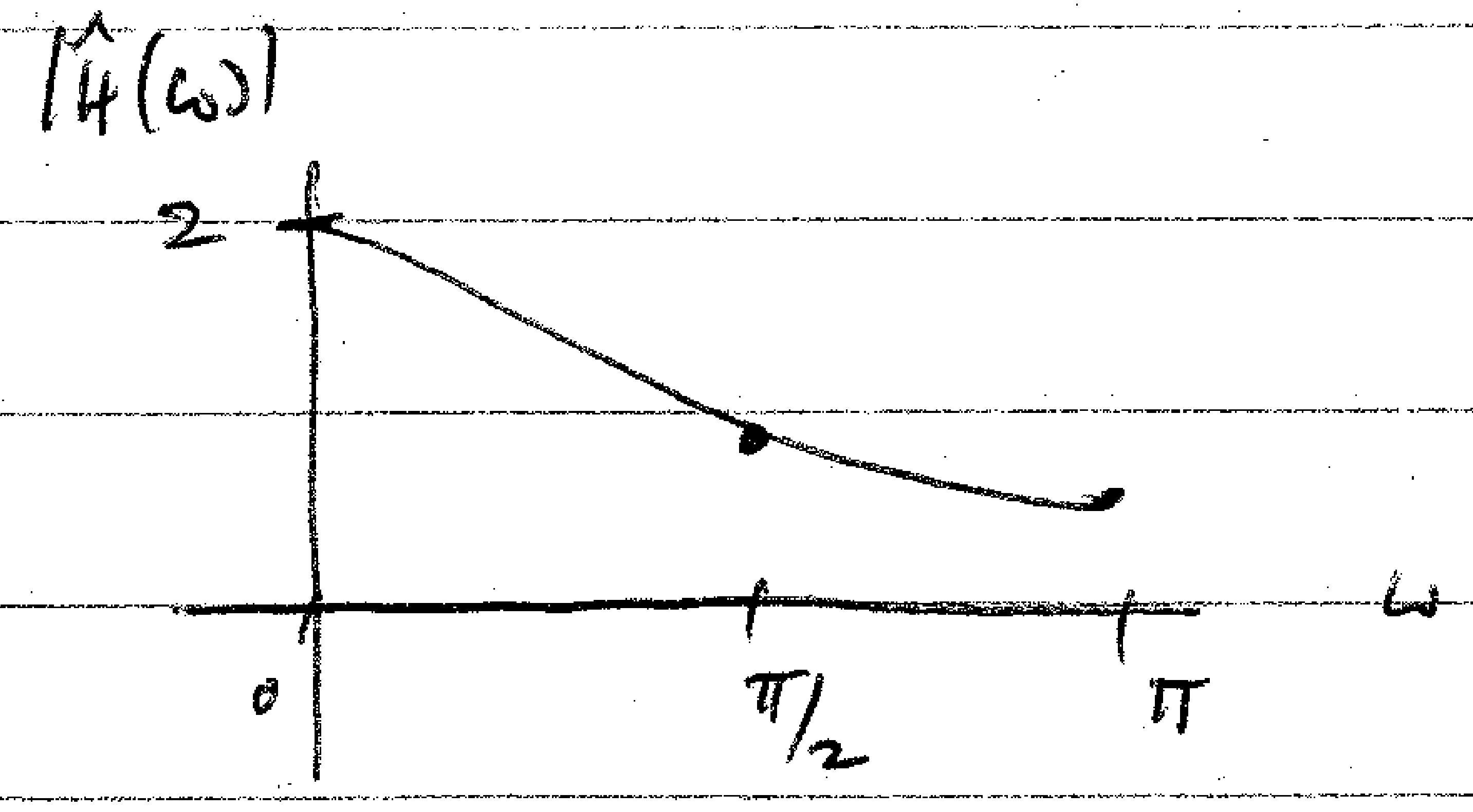
So

$$\hat{H}(\omega) = \sum_{k=0}^{\infty} \left(\frac{e^{-i\omega}}{2} \right)^k$$

$$= \frac{1}{1 - \frac{1}{2} e^{-i\omega}} = \frac{1}{1 - \frac{1}{2}\cos\omega + \frac{i}{2}\sin\omega}$$

$$|\hat{H}(\omega)| = \frac{1}{\left[(1 - \frac{1}{2}\cos\omega)^2 + \frac{1}{4}\sin^2\omega \right]^{1/2}}, \quad \measuredangle \hat{H}(\omega) = -\tan^{-1} \frac{\frac{1}{2}\sin\omega}{1 - \frac{1}{2}\cos\omega}$$

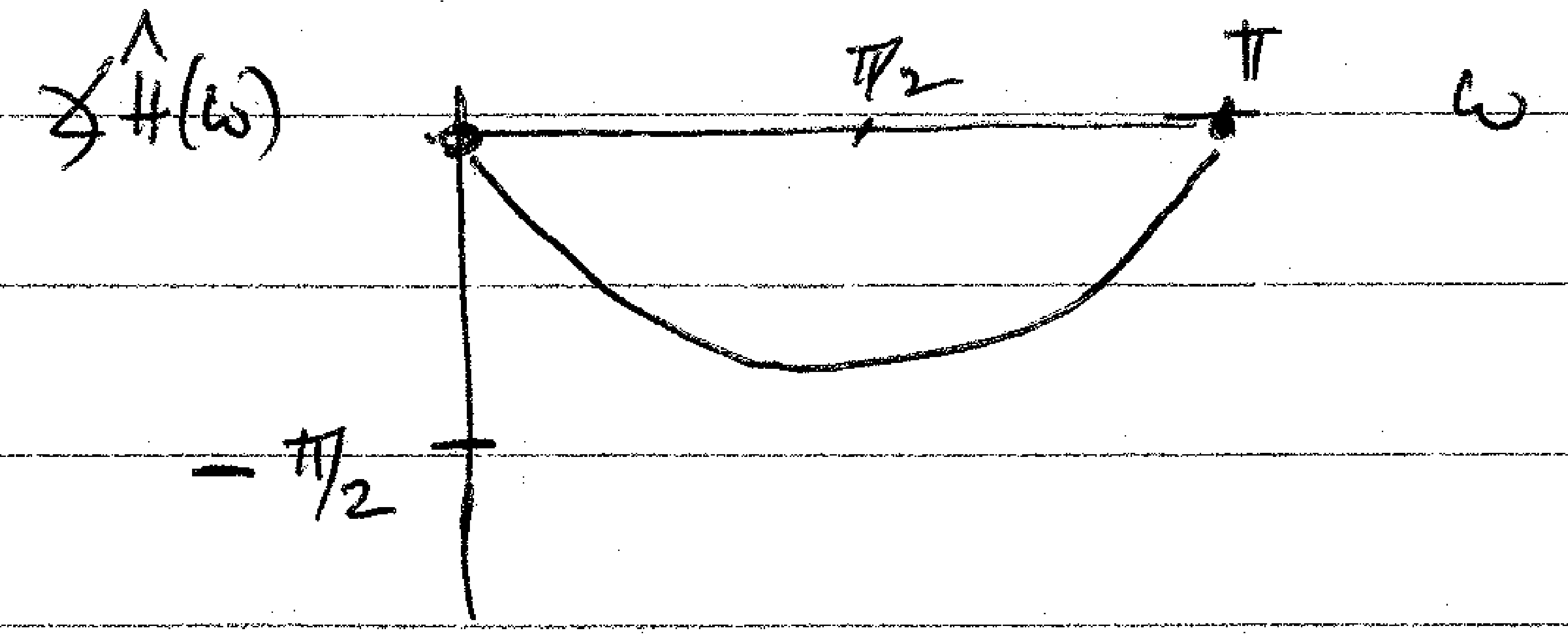
$$= \frac{1}{\left(\frac{5}{4} - \cos\omega \right)^{1/2}}$$



Note $\hat{H}(0) = 2$

$\hat{H}(\pi/2) = 2/\sqrt{5}$

$\hat{H}(\pi) = 2/3$



Note $\measuredangle \hat{H}(0) = 0 = \measuredangle \hat{H}(\pi)$

$\measuredangle \hat{H}(\pi/2) = -\tan^{-1} \frac{1}{2}$

c) $$\hat{G}(\omega) = \left[\hat{H}(\omega)\right]^2 = \left(\frac{1}{1-0.5e^{-i\omega}}\right)^2$$

d) The closed-loop frequency response is

$$\frac{\hat{H}(\omega)}{1+\hat{H}(\omega)} = \frac{1}{2-0.5e^{-i\omega}}$$