EECS 20. Practice Problems No. 2.

You have the following sources of practice problems:

- Quizzes you have taken.
- Problem sets you have turned in.
- The exam archive on the web from prior semesters.
- The practice problems for the first midterm.
- The labs you have done.
- The problem session problems.
- The following problems, mostly drawn from the problem sessions.

Note that when looking at the exam archive, you need to be aware that the order in which material was covered has changed. In particular, prior to Spring of 2000, state machines were done last rather than first. In the problems below, references to figures and sections are references to your reader, except when the figure is included here.

1. **E** Find θ so that

$$Re\{(1+i)e^{i\theta}\} = -1.$$

2. **T** Factor the polynomial $z^5 + 2$ as

$$z^5 + 2 = \prod_{k=1}^{5} (z - \alpha_k),$$

expressing the α_k in polar coordinates.

3. **C** How would you define $\sqrt{1+i}$? More generally, how would you define $\sqrt{z}$ for any complex number z?

4. **T** The logarithm of a complex number z is written $\log z$ or $\log(z)$. It can be defined as an infinite series, or as the inverse of the exponential, i.e. define $\log z = w$, if $e^w = z$. Using the latter definition, find the logarithm of the following complex numbers:

$$1,\ -1,\ i,\ -i,\ 1+i$$

More generally, if $z \neq 0$ is expressed in polar coordinates, what is $\log z$? For which complex numbers z is $\log z$ not defined?

5. (a) Write $\cos^3(\omega t)$ in terms of cosines and sines of multiples of ωt.

   (b) Express in polar form all the **distinct** roots of the equation $z^5 = 2$.

   (c) Find A and θ so that

$$A \sin(\omega t + \theta) = \sin(\omega t) + \cos(\omega t).$$

   (d) Express the fraction $\frac{1+j\omega}{1-j\omega}$ in rectangular and polar forms.

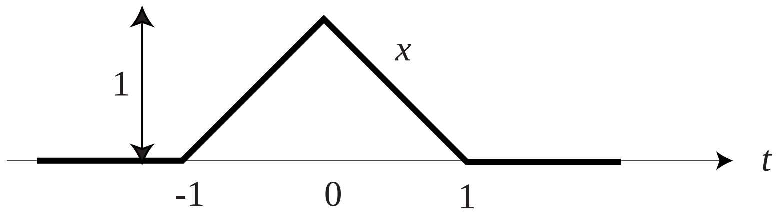


Figure 1: The graph of x for problem 6

6. The function $x : Reals \to Reals$ is described by its graph in Figure 1. Define the periodic signal $y : Reals \to Reals$ by

$$\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} x(t - kp).$$

   (a) What is the period of y?

   (b) Carefully sketch y for $p = 2$ and $p = 4$.

   (c) Since y is periodic, it has an exponential Fourier series expression

$$\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_0 t}. \tag{1}$$

   What is ω_0? What are its units?

   (d) Compute the Fourier series coefficient Y_0 in (1) for the case $p = 2$ and $p = 4$.

7. A single-input, single-output linear difference equation system is expressed in the form

$$\begin{aligned} x(n+1) &= Ax(n) + bu(n) \\ y(n) &= c^T x(n) + du(n) \end{aligned}$$

   where A is a $N \times N$ matrix, b, c are $N \times 1$ column vectors, $d, u(n), y(n)$ are scalars.

   (a) Write down the expression for the zero-state impulse response, i.e. what is the output when the initial state $x(0) = 0$, and $u(n) = \delta(n)$, the Kronecker delta function.

   (b) Suppose

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c^T = [0 \quad 1] \quad d = 1$$

   Find the zero-state impulse response.
   Hint: for all $n \geq 0$,

$$A^n = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix},$$

8. **E** In (6.10) we defined periodic for continuous-time signals.

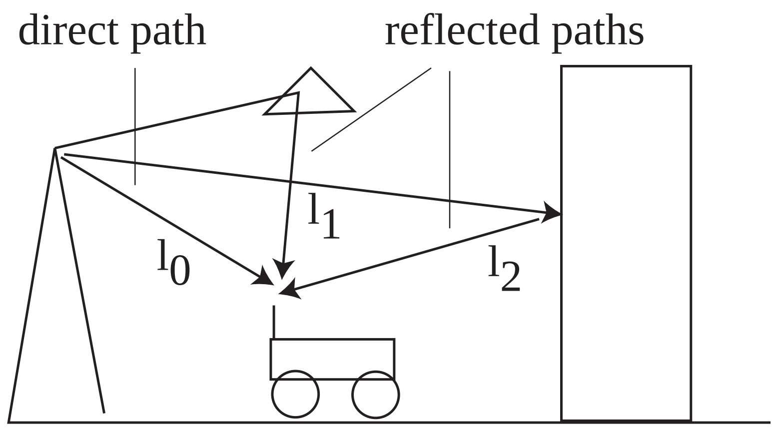


Figure 2: The direct and two reflected paths from transmitter to receiver

(a) Define **finite** and **periodic** for images.

(b) Define **finite** and **periodic** for discrete-time signals, where the domain is *Ints*.

9. **T** In this problem, we examine a practical application of the mathematical result in exercise 6.4. In particular, we consider multipath interference, a common problem with wireless systems where multiple paths from a transmitter to a receiver can result in destructive interference of a signal.

When a transmitter sends a radio signal to a receiver, the received signal consists of the direct path plus several reflected paths. In figure 2, the transmitter is on a tower at the left of the figure, the receiver is on the vehicle, and there are three paths: the direct path is l_0 meters long, the path reflected from a hill (the little triangle in the middle) is l_1 meters long, and the path reflected from the building (the rectangle on the right) is l_2 meters long.

Suppose the transmitted signal is a f Hz sinusoid $x\colon Reals \to Reals$,

$$\forall\, t \in Reals, \quad x(t) = A\cos(2\pi f t)$$

So the received signal is y such that $\forall\, t \in Reals$,

$$y(t) = \alpha_0 A\cos(2\pi f(t - \frac{l_0}{c})) + \alpha_1 A\cos(2\pi f(t - \frac{l_1}{c})) + \alpha_2 A\cos(2\pi f(t - \frac{l_2}{c})). \quad (2)$$

Here, $0 \le \alpha_i \le 1$ are numbers that represent the attenuation (or reduction in signal amplitude) of the signal, and $c = 3 \times 10^8$ m/s is the speed of light.[1] Answer the following questions.

(a) Explain why the description of y given in (2) is a reasonable model of the received signal.

(b) What would be the description if instead of the 3 paths as shown in figure 2, there were 10 paths (one direct and 9 reflected).

[1] In reality, the reflections are more complicated than the model here.

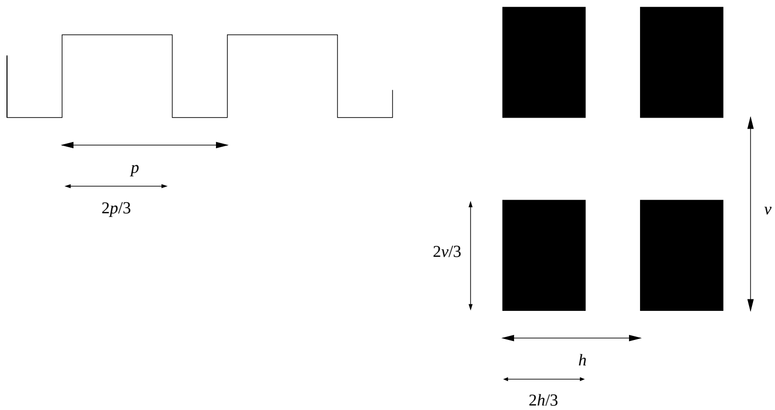


Figure 3: A periodic square wave (left) and a periodic pattern (right)

(c) The signals received over the different paths cause different phase shifts, ϕ_i, so the signal y (with three paths) can also be written as

$$\forall\, t \in Reals, \quad y(t) = \sum_{k=0}^{2} \alpha_k A \cos(2\pi f t - \phi_k)$$

What are the ϕ_k? Give an expression in terms of f, l_k, and c.

(d) Let $\Phi = \max\{\phi_1 - \phi_0, \phi_2 - \phi_0\}$ be the largest difference in the phase of the received signals and let $L = \max\{l_1 - l_0, l_2 - l_0\}$ be the maximum path length difference. What is the relationship between Φ, L, f?

(e) Suppose for simplicity that there is only one reflected path of distance l_1, i.e. take $\alpha_2 = 0$ in the expressions above. Then $\Phi = \phi_1 - \phi_0$. When $\Phi = \pi$, the reflected signal is said to *destroy* the direct signal. Explain why the term "destroy" is appropriate. (This phenomenon is called destructive interference.)

(f) In the context of mobile radio shown in the figure, typically $L \leq 500$m. For what values of f is $\Phi \leq \pi/10$? (Note that if $\Phi \leq \pi/10$ the signals will not interact destructively by much.)

(g) For the two-path case, drive an expression that relates the frequencies f that interfere destructively to the path length difference $L = l_1 - l_0$.

10. **C** Suppose the periodic square wave shown on the left in Figure 3 has the Fourier series representation

$$A_0 + \sum_{k=0}^{\infty} A_k \cos(2\pi k t/p + \phi_k)$$

Use this to obtain a Fourier series representation of the two-dimensional pattern of rectangles on the right. Note that the vertical and horizontal periods l, w are different.

11. **E** Plot the function s: *Reals* $\to$ *Reals* given by

$$\forall\, x \in \textit{Reals}, \quad s(x) = \textit{Im}\{e^{(-x+i2\pi x)}\}.$$

You are free to choose a reasonable interval for your plot, but be sure it includes $x = 0$.

12. **C** Consider a system *Squarer*: $[Reals \to Reals] \to [Reals \to Reals]$, where if $y = Squarer(x)$ then

$$\forall\, t \in Reals, \quad y(t) = x^2(t).$$

   (a) Show that this system is memoryless.

   (b) Show that this system is not linear.

   (c) Show that this system is time invariant.

   (d) Suppose that the input x is given by

   $$\forall\, t \in Reals, \quad x(t) = cos(\omega t),$$

   for some fixed ω. Show that the output y contains a component at frequency 2ω.

13. **E** Consider a continuous-time system *TimeScale*: $[Reals \to Reals] \to [Reals \to Reals]$, where if $y = TimeScale(x)$ then

$$\forall\, t \in Reals, \quad y(t) = x(2t).$$

   (a) Is *TimeScale* linear? Justify your answer.

   (b) Is *TimeScale* time-invariant? Justify your answer.

14. **E** Suppose that the frequency response of a discrete-time LTI system *Filter* is given by

$$H(\omega) = |\sin(\omega)|$$

where ω has units of radians/sample. Suppose the input is the discrete-time signal x given by $\forall\, n \in Ints,\ \ x(n) = 1$. Give a *simple* expression for $y = Filter(x)$.

15. **T** Consider a continuous-time LTI system S. Suppose that when the input is given by

$$x(t) = \begin{cases} \sin(\pi t) & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

then the output $y = S(x)$ is given by

$$y(t) = \begin{cases} \sin(\pi t) & 0 \le t < 1 \\ \sin(\pi(t-1)) & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$$

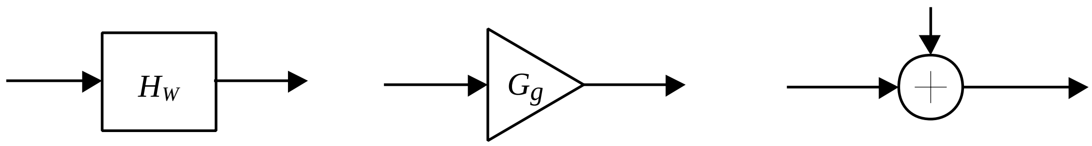
for all $t \in Reals$.

   (a) Carefully sketch these two signals.

(b) Give an expression and a sketch for the output of the same system if the input is

$$x(t) = \begin{cases} \sin(\pi t) & 0 \le t < 1 \\ -\sin(\pi(t-1)) & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}.$$

16. **T** Suppose you are given the building blocks shown below for building block diagrams:



These blocks are defined as follows:

- An LTI system H_W: [*Reals* $\to$ *Reals*] $\to$ [*Reals* $\to$ *Reals*] which has a rectangular frequency response given by

  $$\forall\, \omega \in \textit{Reals}, \quad H(\omega) = \begin{cases} 1 & -W < \omega < W \\ 0 & \text{otherwise} \end{cases}$$

  where W is a parameter you can set.
- A gain block G_g: [*Reals* $\to$ *Reals*] $\to$ [*Reals* $\to$ *Reals*] where if $y = g(x)$, then

  $$\forall\, t \in \textit{Reals}, \quad y(t) = gx(t)$$

  where $g \in$ *Reals* is a parameter you can set.
- An adder, which can add two continuous-time signals. Specifically, *Add*: [*Reals* $\to$ *Reals*] $\times$ [*Reals* $\to$ *Reals*] $\to$ [*Reals* $\to$ *Reals*] such that if $y = \textit{Add}(x_1, x_2)$ then

  $$\forall\, t \in \textit{Reals}, \quad y(t) = x_1(t) + x_2(t).$$

Use these building blocks to construct a system with the frequency response shown below: