EECS 20. Solutions to Practice Problems No. 3.

8.4 (a) No, since the response to an impulse includes non-zero samples earlier than time zero.

(b) The frequency response is the DTFT of the impulse response,

$$\begin{aligned} H(\omega) &= \sum_{m=-\infty}^{\infty} h(m)e^{-i\omega m} \\ &= \sum_{m=-\infty}^{\infty} (\delta(m-1)/2 + \delta(m+1)/2)e^{-i\omega m} \\ &= (e^{-i\omega} + e^{i\omega})/2 \\ &= \cos(\omega). \end{aligned}$$

This is periodic with period 2π because

$$\forall\, \omega \in \textit{Reals}, \quad \cos(\omega + 2\pi) = \cos(\omega).$$

(c) The fundamental frequency $\omega_0 = \pi/2$, in units of radians per sample. To get the Fourier series coefficients, just write the signal as a sum of complex exponentials,

$$x(n) = (1/2)e^{-i\pi n} + (i/2)e^{-i\pi n/2} + 2 - (i/2)e^{i\pi n/2} + (1/2)e^{-i\pi n},$$

from which we can read off the coefficients,

$$\begin{aligned} X_{-2} &= 1/2 \\ X_{-1} &= i/2 \\ X_0 &= 2 \\ X_1 &= -i/2 \\ X_2 &= 1/2. \end{aligned}$$

The rest of the coefficients are zero.

(d) The Fourier series coefficients of the output will be the above Fourier series coefficients multiplied by $H(\omega)$ for the corresponding value of ω. This yields

$$\begin{aligned} y(n) &= -(1/2)e^{-i\pi n} + 2 - (1/2)e^{i\pi n} \\ &= 2 - \cos(\pi n). \end{aligned}$$

8.5 We can calculate the CTFT of the impulse response,

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \\ &= \int_{0}^{3} (1/3)e^{-i\omega t}dt \\ &= (1 - e^{-i3\omega})/(3i\omega). \end{aligned}$$

The following Matlab code plots the magnitude response:

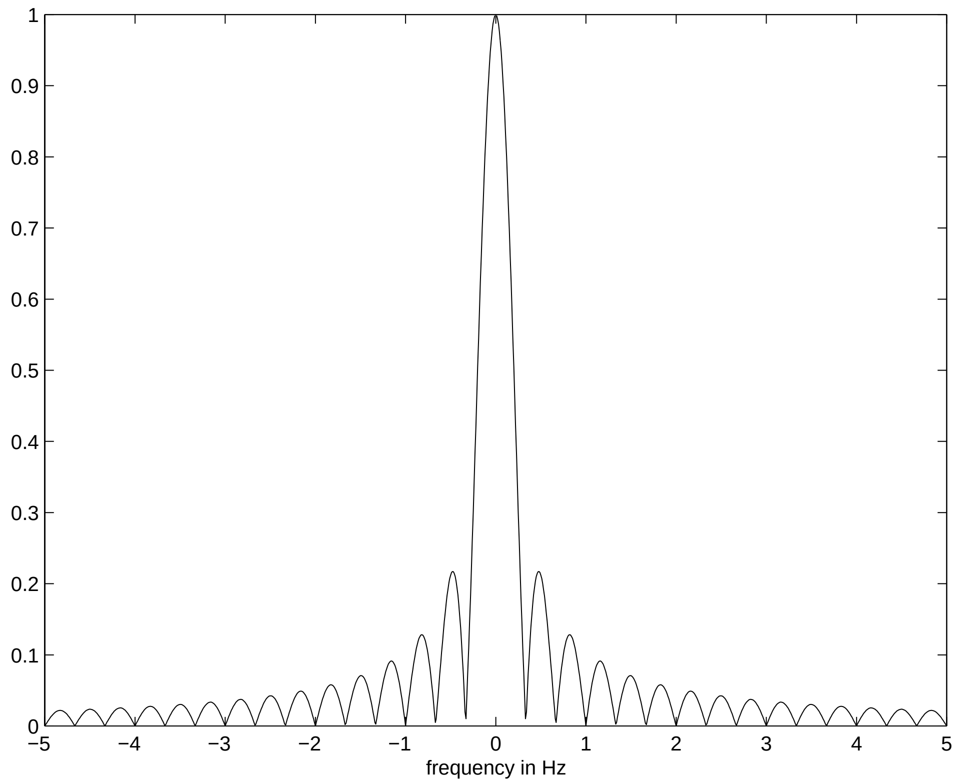


Figure 1: Magnitude response of a 3-second continuous-time moving average.

```
f = [-5:1/100:5];
H = (1-exp(-i*3*2*pi*f))./(3*i*2*pi*f);
plot(f,abs(H));
```

Note that this gives a "Warning: Divide by zero" at frequency 0, but generates a correct plot anyway. You can use L'Hopital's rule to find that the value at frequency zero is 1. The plot is shown in figure 1.

8.6 (a) Using convolution,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} (\delta(\tau-1)+\delta(\tau-2))x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \delta(\tau-1)x(t-\tau)d\tau + \int_{-\infty}^{\infty} \delta(\tau-2)x(t-\tau)d\tau \\ &= = x(t-1)+x(t-2), \end{aligned}$$

using the sifting rule.

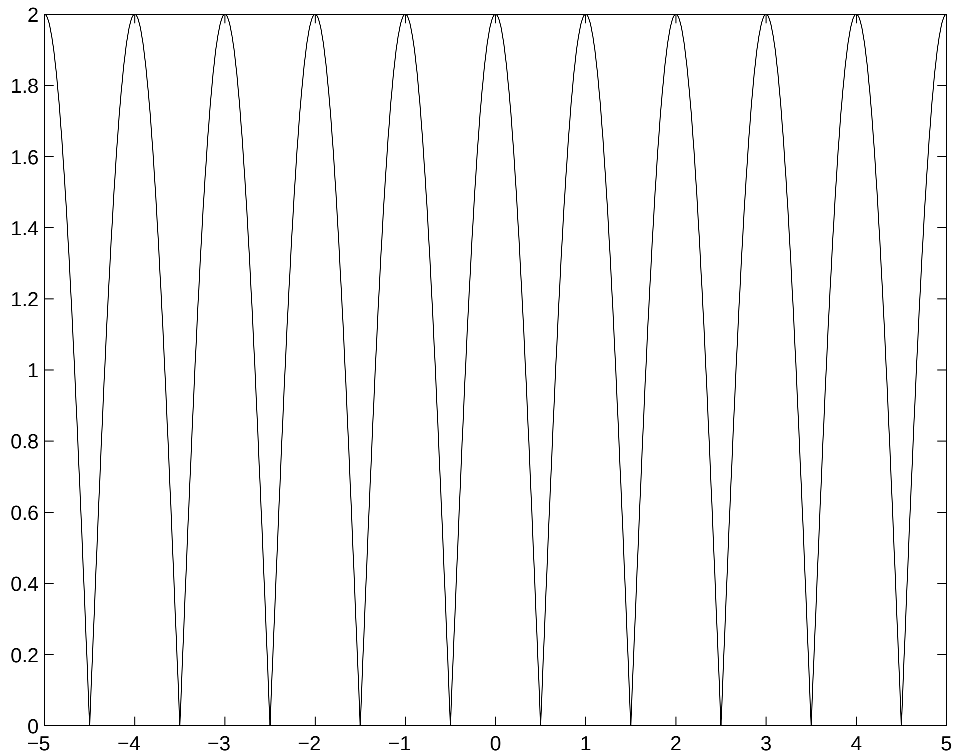


Figure 2: The magnitude frequency response of an LTI system with impulse response $h(t) = \delta(t - 1) + \delta(t - 2)$.

(b) The frequency response is the CTFT of the impulse response,

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} (\delta(t-1) + \delta(t-2)) e^{-i\omega t} dt \\ &= e^{-i\omega} + e^{-i2\omega}, \end{aligned}$$

using the sifting rule.

(c) The following Matlab code creates the plot:

```
f = [-5:1/100:5];
H = (exp(-i*2*pi*f)+exp(-i*2*2*pi*f));
plot(f,abs(H));
```

which yields the plot shown in figure 2.

9.8 (a) Note that

$$X(-\omega) = i \sin(-K\omega) = -i \sin(K\omega) = X^*(-\omega),$$

using the fact that $\sin(\theta) = -\sin(-\theta)$. Thus, X is conjugate symmetric, which implies that x is real.

(b) Using Euler's relation,

$$X(\omega) = (e^{iK\omega} - e^{-iK\omega})/2.$$

We can recognize the inverse DTFT of each of these terms to get

$$x(n) = (\delta(n+K) - \delta(n-K))/2$$

where δ is the Kronecker delta function.

9.9 First, note that y is periodic with period p, just as x is. Its Fourier series coefficients are given by the formula

$$\begin{aligned} Y_m &= \frac{1}{p}\int_0^p y(t)e^{-im\omega_0 t}dt \\ &= \frac{1}{p}\int_0^p x(t-\tau)e^{-im\omega_0 t}dt \\ &= \frac{1}{p}\int_{-\tau}^{p-\tau} x(t)e^{-im\omega_0 (t+\tau)}dt \\ &= e^{-im\omega_0\tau}\frac{1}{p}\int_{-\tau}^{p-\tau} x(t)e^{-im\omega_0 t}dt \\ &= e^{-im\omega_0\tau}\frac{1}{p}\int_0^p x(t)e^{-im\omega_0 t}dt \\ &= e^{-im\omega_0\tau}X_m, \end{aligned}$$

where we have changed variables in the integral (replacing t with $t-\tau$), and then changed the limits from $-\tau$ to $p-\tau$ to 0 to p. The change of limits is valid because we are integrating over one cycle of a periodic function, so it does not matter where the integral begins. The end result is

$$Y_m = e^{-im\omega_0\tau}X_m,$$

so just as with a CTFT, a time delay affects Fourier series coefficients by multiplying them by a complex exponential.

9.10 Use the inverse CTFT,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega_0 t}d\omega$$

$$
\begin{aligned}
&= \frac{T}{2\pi} \int\limits_{-\pi/T}^{\pi/T} e^{i\omega_0 t} d\omega \\
&= \frac{T}{2\pi i t}[e^{it\pi/T} - e^{-it\pi/T}] \\
&= \frac{\sin(t\pi/T)}{t\pi/T}.
\end{aligned}
$$

9.11 Use the CTFT,

$$
\begin{aligned}
Y(\omega) &= \int\limits_{-\infty}^{\infty} y(t) e^{-i\omega t} dt \\
&= \int\limits_{-\infty}^{\infty} X(t) e^{-i\omega t} dt
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{1}{2\pi} Y(-\omega) &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(t) e^{i\omega t} dt \\
&= x(\omega),
\end{aligned}
$$

recognizing this as an inverse CTFT with symbols ω and t swapped. Thus,

$$\frac{1}{2\pi} Y(-\omega) = x(\omega)$$

which implies that

$$Y(\omega) = 2\pi x(-\omega).$$

9.12 Define

$$y(t) = X(t) = 2\pi \frac{\sin(at)}{at}.$$

From exercise , with π/T replaced by a,

$$Y(\omega) = \begin{cases} (2\pi)\pi/a, & \text{if } |\omega| \leq a \\ 0, & \text{if } |\omega| > a \end{cases}$$

From exercise ,

$$Y(\omega) = 2\pi x(-\omega)$$

so

$$x(t) = \frac{1}{2\pi} Y(-t).$$

Hence,

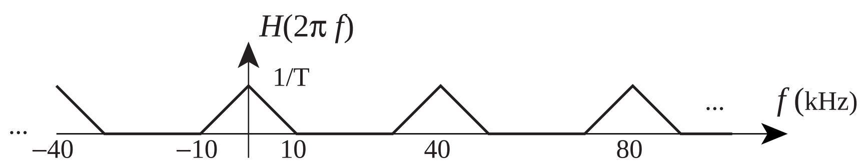
$$x(t) = \begin{cases} \pi/a, & \text{if } |t| \leq a \\ 0, & \text{if } |t| > a \end{cases}$$

10.2 Note that $\cos(\theta) = \cos(-\theta)$. Therefore,

$$\cos(-2\pi 440 nT + \phi) = \cos(2\pi 440 nT - \phi).$$

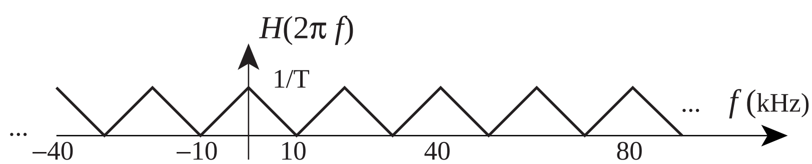
Thus, $f = 440$ and $\theta = -\phi$.

10.6 (a) The sketch is shown below:



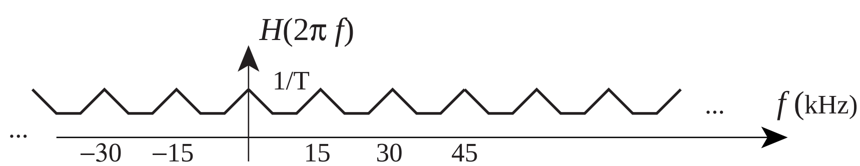
The height of each of the peaks is $1/T$, which in this case is 40,000.

(b) The sketch is shown below:



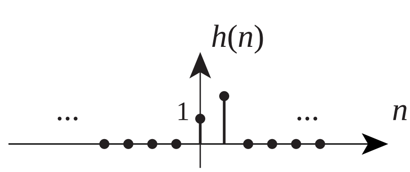
The height of each of the peaks is $1/T$, which in this case is 20,000.

(c) The sketch is shown below:



The height of each of the peaks is $1/T$, which in this case is 15,000. Notice that the overlapping CTFTs caused aliasing distortion.

1. (a) The impulse response is shown below:



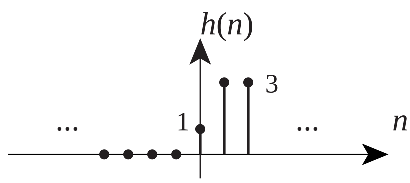
(b) Use convolution to relate the input and output

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= x(n) + 2x(n-1), \end{aligned}$$

using the sifting rule. When the input is the unit step, this becomes

$$y(n) = u(n) + 2u(n-1) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ 3 & \text{if } n \geq 1 \end{cases}$$

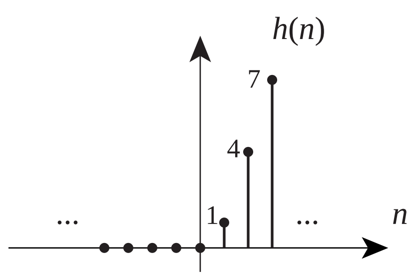
Here is a plot:



(c) If the input is r, then the output is

$$y(n) = r(n) + 2r(n-1) = \begin{cases} 0 & \text{if } n \leq 0 \\ 3n-2 & \text{if } n \geq 1 \end{cases}$$

Here is a plot:



(d) The frequency response is the DTFT of the impulse response,

$$\begin{aligned} H(\omega) &= \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k} \\ &= 1 + 2e^{-i\omega}. \end{aligned}$$

(e) For all $\omega \in$ *Reals*,

$$\begin{aligned} H(\omega + 2\pi) &= 1 + 2e^{-i(\omega+2\pi)} \\ &= 1 + 2e^{-i\omega}e^{-i2\pi} \\ &= 1 + 2e^{-i\omega}, \text{ since } e^{-i2\pi} = 1 \\ &= H(\omega). \end{aligned}$$

(f)

$$\begin{aligned} H(-\omega) &= 1 + 2e^{i\omega} \\ &= (1 + 2e^{-i\omega})^* \\ &= H^*(\omega). \end{aligned}$$

(g) The magnitude response is

$$\begin{aligned} |H(\omega)| &= |1 + 2e^{-i\omega}| \\ &= |1 + 2\cos(\omega) - 2i\sin(\omega)| \\ &= \sqrt{(1 + 2\cos(\omega))^2 + (2\sin(\omega))^2} \\ &= \sqrt{1 + 4\cos(\omega) + 4\cos^2(\omega) + 4\sin^2(\omega)} \\ &= \sqrt{5 + 4\cos(\omega)}. \end{aligned}$$

We have used the facts that for real numbers a and b,

$$|a + ib| = \sqrt{a^2 + b^2}$$

and for any $\omega \in$ *Reals*,

$$\cos^2(\omega) + \sin^2(\omega) = 1.$$

(h) The phase response is

$$\begin{aligned} \angle H(\omega) &= \angle(1 + 2e^{-i\omega}) \\ &= \angle(1 + 2\cos(\omega) - 2i\sin(\omega)) \\ &= \tan^{-1}(-2\sin(\omega)/(1 + 2\cos(\omega))) \\ &= -\tan^{-1}(2\sin(\omega)/(1 + 2\cos(\omega))). \end{aligned}$$

We have used the fact that for real numbers a and b,

$$\angle(a + ib) = \tan^{-1}(b/a).$$

(i) The output will be

$$y(n) = |H(\pi/2)|\cos(\pi n/2 + \pi/6 + \angle H(\pi/2)) + |H(\pi)|\sin(\pi n + \pi/3 + \angle H(\pi)).$$

In this case,

$$H(\pi/2) = 1 - 2i$$

and

$$H(\pi) = -1.$$

So

$$|H(\pi/2)| = \sqrt{5}, \quad \angle H(\pi/2) = -\tan^{-1}(2) \approx 1.107$$

and

$$|H(\pi)| = 1, \quad \angle H(\pi) = \pi.$$

Hence,

$$y(n) = \sqrt{5}\cos(\pi n/2 + \pi/6 + 1.107) + \sin(\pi n + \pi/3 + \pi).$$