

1 (a)

$$s(n+1) = As(n) + Bx(n) \quad s(0) = \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix}$$

Zero-Input State Response  $\rightarrow x(n) = 0, \forall n$

$$\begin{aligned} s(n+1) &= As(n); \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ s(1) &= As(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix} \\ &= \begin{bmatrix} s_1(0) + s_2(0) \\ s_2(0) \end{bmatrix} \\ s(2) &= As(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1(0) + s_2(0) \\ s_2(0) \end{bmatrix} \\ &= \begin{bmatrix} s_1(0) + 2s_2(0) \\ s_2(0) \end{bmatrix} \\ s(n) &= \begin{bmatrix} s_1(0) + ns_2(0) \\ s_2(0) \end{bmatrix} \end{aligned}$$

(b)

$$h(n) = c^T A^{n-1} b, \quad n \geq 1; \quad h(0) = d = 0$$

$$A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \quad \forall n \geq 0$$

$$h(1) = c^T A^0 b = [1 \ 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1 \ 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0]$$

$$h(2) = c^T A^1 b = [1 \ 0] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1 \ 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1]$$

$$h(n) = c^T A^{n-1} b = [1 \ 0] \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1 \ n-1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [n-1]$$

(c)

$$s(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; x(n) = \delta(n-1) = \begin{cases} 1, & n=1 \\ 0, & \text{otherwise} \end{cases}$$

$$\forall n \geq 0, \quad y(n) = c^T A^n s(0) + \sum_{k=0}^n h(n-k)x(k)$$

$$\begin{aligned} y(0) &= c^T A^0 s(0) + h(0)x(0) = c^T A^0 s(0) \\ &= [1 \ 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [1 \ 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1] \end{aligned}$$

$$\begin{aligned} y(1) &= c^T A^1 s(0) + \sum_{k=0}^1 h(1-k)x(k) = c^T A^1 s(0) + h(0) \\ &= [1 \ 0] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [0] \\ &= [1 \ 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [0] = [2] \end{aligned}$$

$$\begin{aligned} y(n) &= c^T A^n s(0) + \sum_{k=0}^n h(1-k)x(k) = c^T A^n s(0) + h(n-1) \\ &= [1 \ 0] \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h[n-1] \\ &= [1 \ n] \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h[n-1] = [1+n] + [n-2] = [2n-1] \end{aligned}$$

**2**

$$y(n) - y(n-1) = x(n) - 2x(n-1)$$

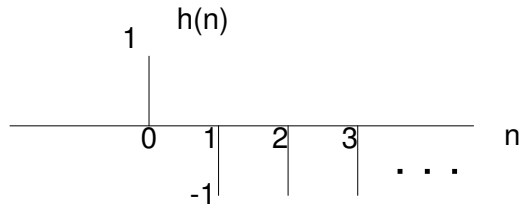
$$y(n) = x(n) - 2x(n-1) + y(n-1)$$

(a)

$$\begin{aligned} s(n) &= \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} \\ s(n+1) &= \begin{bmatrix} y(n) \\ x(n) \end{bmatrix} = A \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + Bx(n) \\ &= \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}}_A \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_B x(n) \\ y(n) &= C \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + dx(n) \\ &= \underbrace{\begin{bmatrix} 1 & -2 \end{bmatrix}}_C \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + \underbrace{1}_d x(n) \end{aligned}$$

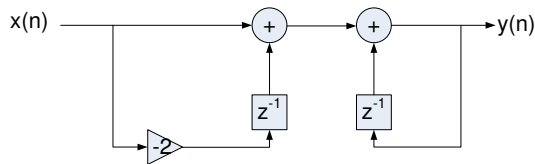
Zero State Impulse Response

$$\begin{aligned} s(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}; x(n) = \delta(n) \\ y(0) = d = 1 & \quad s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \\ y(1) = Cs(1) & \quad s(1) = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \\ y(2) = Cs(2) & \quad s(2) = As(1) \\ y(3) = Cs(3) & \quad s(3) = A^2s(1) \\ & \quad \vdots \\ y(n) = Cs(n) & \quad s(n) = A^{n-1}s(1) \\ A^n &= \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\ A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ h(0) &= 1 \\ h(1) &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \\ h(2) &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1 \\ & \quad \vdots \\ h(n) &= -1 \end{aligned}$$



(b)

$$y(n] = x[n] - 2x[n - 1] + y[n - 1]$$



(c)

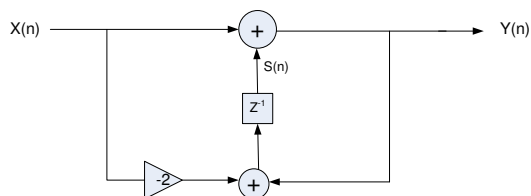
$$s[n] = -2x[n - 1] + y[n - 1]$$

$$s[n + 1] = -2x[n] + y[n] = -x[n] + s[n]$$

$$A = 1, B = -1, C = 1, d = 1$$

Zero-State Impulse Response

$$\begin{aligned} s[0] &= 0 & y[0] &= 1 \\ s[1] &= -1 & y[1] &= -1 \\ s[2] &= -1 & y[2] &= -1 \\ &\vdots & &\vdots \end{aligned}$$



(d) Same as (a)

(e) Frequency response

$$\begin{aligned}
 x(n) &= e^{j\omega n}, \quad y(n) = H(\omega)e^{j\omega n} \\
 H(\omega)e^{j\omega n} &= e^{j\omega n} - 2e^{j\omega(n-1)} + H(\omega)e^{j\omega(n-1)} \\
 H(\omega)(1 - e^{-j\omega}) &= 1 - 2e^{-j\omega} \\
 H(\omega) &= \frac{1 - 2e^{-j\omega}}{1 - e^{-j\omega}}
 \end{aligned}$$

Using impulse response:

$$\begin{aligned}
 h(n) &= \delta(n) - u(n-1) \\
 \delta(n) &\xrightarrow{DTFT} 1 \\
 u(n) &\xrightarrow{DTFT} \frac{1}{1 - e^{-j\omega}} \\
 u(n-1) &\xrightarrow{DTFT} \frac{e^{-j\omega}}{1 - e^{-j\omega}} \\
 H(\omega) &= 1 - \frac{e^{-j\omega}}{1 - e^{-j\omega}} = \frac{1 - 2e^{-j\omega}}{1 - e^{-j\omega}}
 \end{aligned}$$

3 (a)

$$s(n+1) = As(n) + Bx(n) \quad s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$s(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = As(1) + Bx(1)$$

$$s(1) = As(0) + Bx(0) = A \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) + Bx(0) = Bx(0)$$

$$s(2) = A(Bx(0)) + Bx(1) \longrightarrow \text{plug in } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(0) \right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(1)$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times (0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times (1)$$

$$x(0) = [1]$$

$$x(1) = [1]$$

(b)

$$s(0) = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$s(1) = As(0) + Bx(0) = A \left( \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \right) + Bx(0) = Bx(0)$$

$$\begin{aligned} s(2) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = As(1) + Bx(1) \\ &= A \left( A \left( \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \right) + Bx(0) \right) + Bx(1) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(0) \right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(1) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} s_1 + s_2 \\ s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x(0) \end{bmatrix} \right) + \begin{bmatrix} 0 \\ x(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} s_1 + s_2 \\ s_2 + x(0) \end{bmatrix} \right) + \begin{bmatrix} 0 \\ x(1) \end{bmatrix} \\ &= \begin{bmatrix} 2s_1 + s_2 + x(0) \\ s_1 + s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x(1) \end{bmatrix} \\ &= \begin{bmatrix} 2s_1 + s_2 + x(0) \\ s_1 + s_2 + x(1) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

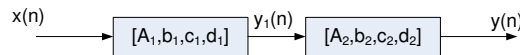
$$2s_1 + s_2 + x(0) = 0$$

$$s_1 + s_2 + x(1) = 0$$

$$x(0) = -2s_1 - s_2$$

$$x(1) = -s_1 - s_2$$

4



$$\begin{aligned}
y_1(n) &= c_1x(n) + d_1s_1(n) \\
y(n) &= c_2y_1(n) + d_2s_2(n) \\
&= c_2(c_1x(n) + d_1s_1(n)) + d_2s_2(n) \\
&= c_1c_2x(n) + c_2d_1s_1(n) + d_2s_2(n) \\
s &= \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix}, C = c_1c_2, D = \begin{bmatrix} c_2d_1 & d_2 \end{bmatrix} \\
s_1(n+1) &= A_1s_1(n) + b_1x_2(n) \\
s_2(n+1) &= A_2s_2(n) + b_2y_1(n) \\
&= A_2s_2(n) + b_2[c_1x(n) + d_1s_1(n)] \\
&= c_1b_2x(n) + b_2d_1s_1(n) + A_2s_2(n) \\
\begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ (b_2d_1) & A_2 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \begin{bmatrix} b_1 \\ c_1b_2 \end{bmatrix} x(n) \\
A &= \begin{bmatrix} A_1 & 0 \\ b_2d_1 & A_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ c_1b_2 \end{bmatrix}
\end{aligned}$$

5

$$\begin{aligned}
x(n) = \delta(n) + \delta(n-2) \longrightarrow y(n) &= \delta(n) + \delta(n-1) + \delta(n-2) \\
&= x(n) + x(n-1) - x(n-3) + x(n-5) - x(n-7) \dots \\
h(n) &= \delta(n) + \delta(n-1) - \delta(n-3) + \delta(n-5) - \dots
\end{aligned}$$

6

$$x(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

(a)

$$\begin{aligned}
h_1(t) &= \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases} \\
y_1(t) &= \int_0^{\infty} e^{-(t-\tau)}d\tau \quad t - \tau \geq 0 \\
&= e^{-t} \int_0^t e^{\tau}d\tau \\
&= e^{-t} e^{\tau} \Big|_0^t \\
&= e^{-t}(e^t - 1) \\
&= (1 - e^{-t})
\end{aligned}$$

7

(b)

$$\begin{aligned}h_2(t) &= \begin{cases} e^t & t < 0 \\ 0 & t \geq 0 \end{cases} \\y_2(t) &= \int_0^\infty h(t-\tau) d\tau \quad t-\tau < 0 \\ \text{if } t < 0, \quad y_2(t) &= \int_0^\infty e^{t-\tau} d\tau \\ &= -e^t e^{-\tau} \Big|_0^\infty \\ &= -e^t(0-1) \\ &= e^t \\ \text{if } t > 0, \quad y_2(t) &= \int_t^\infty e^{t-\tau} d\tau \\ &= -e^t e^{-\tau} \Big|_t^\infty \\ &= -e^t(0-e^{-t}) \\ &= 1 \\ y_2(t) &= \begin{cases} e^t & t \leq 0 \\ 1 & t > 0 \end{cases}\end{aligned}$$

(c)

$$\begin{aligned}h_3(t) &= \begin{cases} e^t & t < 0 \\ e^{-t} & t \geq 0 \end{cases} \\y_3(t) &= \int_0^\infty h(t-\tau) d\tau \\ \text{if } t < 0, \quad &\int_{-\infty}^t e^t dt = e^t \Big|_{-\infty}^t = e^t - 0 = e^t \\ \text{if } t \geq 0 \quad &\int_{-\infty}^0 e^t dt + \int_0^t e^{-t} dt = e^t \Big|_{-\infty}^0 - e^{-t} \Big|_0^t \\ &= 1 - (e^{-t} - 1) \\ &= 2 - e^{-t} \\ y_3(t) &= \begin{cases} 2 - e^{-t} & t \geq 0 \\ e^t & t < 0 \end{cases}\end{aligned}$$

- 7 (a)  $x(t) = \cos(2\pi t) + \sin(3\pi t)$  Both components of  $x(t)$  are periodic.  $\therefore$  Period of  $x$  is the smallest  $p$  such that  $2\pi p, 3\pi p$  are integer multiples of  $2\pi \Rightarrow p = 2$   
 $\therefore x$  periodic with period 2.



$\Rightarrow$  Angular freq  $\omega_0 = 2\pi/p = \pi$  rad/s

Using

$$\begin{aligned}\cos(2\pi t) &= \frac{e^{i2\pi t} + e^{-i2\pi t}}{2} \\ \sin(3\pi t) &= \frac{e^{i3\pi t} - e^{-i3\pi t}}{2i}\end{aligned}$$

we get the exponential Fourier Series representation of  $x$  as:

$$x(t) = \frac{i}{2}e^{-i3\pi t} + \frac{1}{2}e^{-i2\pi t} + \frac{1}{2}e^{i2\pi t} - \frac{i}{2}e^{i3\pi t}$$

Noting that  $3\pi t = 3\omega_0 t$ ,  $2\pi t = 2\omega_0 t$ , we get

$$x_m = \begin{cases} 1/2 & \text{when } m = \pm 3 \\ i/2 & \text{when } m = -3 \\ -i/2 & \text{when } m = 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned}z(t) &= \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases} \\ \text{CTFT } z(\omega) &= \int_{-\infty}^{\infty} z(t)e^{-i\omega t} dt \\ &= \int_0^{\infty} e^{-t}e^{-i\omega t} dt \\ &= \int_0^{\infty} e^{-(1+i\omega)t} dt \\ &= \frac{e^{-(1+i\omega)t}}{-(1+i\omega)} \Big|_0^{\infty} \\ &= \frac{1}{1+i\omega} \\ \therefore z(\omega) &= \frac{1}{1+i\omega}\end{aligned}$$

Using the frequency shifting property of CTFT, we get

$$y(\omega) = z(\omega - \omega_0) = \frac{1}{1+i(\omega - \omega_0)}$$

(c) (i) DTFT symmetric about  $\omega = 0$

$$\Rightarrow x(\omega) = x(-\omega)$$

DTFT real  $\Rightarrow x(\omega) = x^k(\omega)$  combining, we get  $x(\omega) = x^k(\omega)$ . Taking inverse DTFT, we get

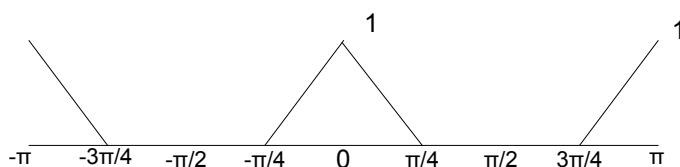
$$x(n) = x^*(n) \Rightarrow x(n) \text{ real } \forall n$$

$\Rightarrow x$  is realvalued.

(ii)  $y(k) = x(k/2)$  if  $k$  even; =0 otherwise.

$$\begin{aligned} y(\omega) &= \sum_{n=-\infty}^{\infty} y(n)e^{-i\omega n} \\ &= \sum_{n \text{ even}} y(n)e^{-i\omega n} \\ &= \sum x(n/2)e^{-i\omega n} \\ &= \sum_{l=-\infty}^{\infty} x(l)e^{i2\omega l} \\ &= \sum_{l=-\infty}^{\infty} x(l)e^{i(2\omega)l} \\ &= x(2\omega) \end{aligned}$$

$$\therefore y(\omega) = x(2\omega)$$



Sketch of  $y$ : use the fact that  $x(\omega)$  periodic with period  $2\pi$ .

8

$$\begin{aligned}
 \forall n \in \mathbb{N}, x(n) &= (0.5)^{|n|} \\
 x(\omega) &= \sum_{m=-\infty}^{\infty} x(m)e^{-i\omega m} \\
 &= \sum_{m=-\infty}^{\infty} (0.5)^{|m|}e^{-i\omega m} \\
 &= \sum_{m=1}^{\infty} (0.5)^m(e^{-i\omega m} + e^{i\omega m}) + (0.5)^0 \\
 &= \sum_{m=0}^{\infty} (0.5)^m((e^{-i\omega})^m + (e^{i\omega})^m) - (0.5)^0 \\
 &= \frac{1}{1 - (0.5)(e^{-i\omega})} + \frac{1}{1 - (0.5)(e^{i\omega})} - 1 \\
 &\quad \text{[Using geometric series sum]} \\
 &= \frac{e^{i\omega}}{e^{i\omega} - 0.5} + \frac{1}{1 - 0.5e^{i\omega}} - 1
 \end{aligned}$$

9

$$\forall n \quad x(n) = \frac{1}{2\pi} \int_0^{2\pi} x(\omega)e^{i\omega n} d\omega$$

(a)

$$x(n) = e^{i\omega_0 n} = \frac{1}{2\pi} \int_0^{2\pi} x(\omega)e^{i\omega n} d\omega$$

$x(\omega)$  needs to pick out  $e^{i\omega n}$  at  $\omega = \omega_0$  with proper scaling.

$$x(\omega) = 2\pi\delta(\omega - \omega_0) \quad \forall \omega \in [0, 2\pi]$$

Note that DTFT  $X(\omega)$  is always periodic with period  $2\pi$ . Here we find expression for  $x(\omega)$  only in  $[0, 2\pi]$

(b)

$$y(n) = \cos(\omega_0, n) = \frac{1}{2}e^{i\omega_0 n} + \frac{1}{2}e^{i(-\omega_0)n}$$

$\omega_0 \in [0, 2\pi]$  but  $-\omega_0 \notin [0, 2\pi]$ , so use the fact that  $e^{i(-\omega_0)n} = e^{i(-\omega_0 n + 2\pi n)}$ .

( $e^{i\omega n}$  is periodic in  $\omega$  with period  $2\pi$ )

$$\begin{aligned}\therefore y(n) &= \frac{1}{2}e^{i\omega n} + \frac{1}{2}e^{i(2\pi-\omega_0)n} \\ &= \frac{1}{2\pi} \int_0^{2\pi} Y(\omega)e^{i\omega n} d\omega \\ Y(\omega) &= \pi\delta(\omega - (2\pi - \omega_0)) \quad \forall \omega \in [0, 2\pi), \text{ periodic period } 2\pi\end{aligned}$$

(c)

$$\begin{aligned}Z(n) &= e^{i\omega n}, \quad \omega_0 = 2\pi + \frac{\pi}{4} \\ Z(n) &= e^{i2\pi n} e^{i\frac{\pi}{4}n} \\ &= e^{i\frac{\pi}{4}n} \\ \therefore Z(\omega) &= 2\pi\delta(\omega - \frac{\pi}{4}) \quad \forall \omega \in [0, 2\pi), \text{ periodic period } 2\pi\end{aligned}$$

10 (a)

$$x(t) = \cos(20\pi t) = \frac{1}{2}e^{i20\pi t} + \frac{1}{2}e^{-i20\pi t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega)e^{i\omega t} d\omega$$

$x(\omega)$  needs to pick out  $e^{i\omega t}$  at  $\omega = 20\pi$  and  $-20\pi$ , with some scaling  
 $x(\omega) = \pi\delta(\omega - 20\pi) + \pi\delta(\omega + 20\pi)$

$$\begin{aligned}p(t) &= \begin{cases} 1 & -10 < t < 10 \\ 0 & \text{otherwise} \end{cases} \\ p(\omega) &= \int_{-\infty}^{\infty} p(t)e^{-i\omega t} dt \\ &= \int_{-10}^{10} e^{-i\omega t} dt \\ &= \frac{e^{-i\omega t}}{-i\omega} \Big|_{-10}^{10} \\ &= \frac{1}{i\omega} (e^{i10\omega} - e^{-i10\omega}) \\ p(\omega) &= \frac{2\sin(10\omega)}{\omega}\end{aligned}$$

$$y(t) = x(t)p(t) \Rightarrow Y(\omega) = \frac{1}{2\pi}(x * p)(\omega)$$

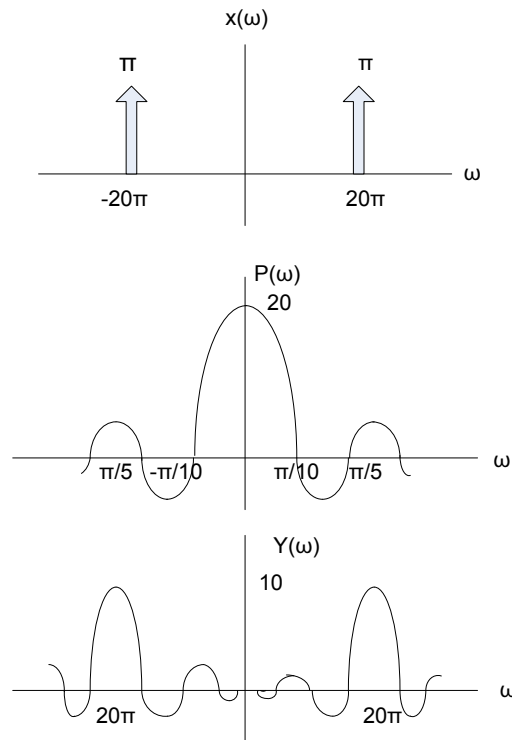
Note that  $x(\omega)$  is sum of scaled, shifted impulses, so  $(x * p)(\omega)$  would be

just  $p(\omega)$  shifted and scaled.

$$\begin{aligned}
 Y(\omega) &= \frac{1}{2\pi}(\pi p(\omega - 20\pi) + \pi p(\omega + 20\pi)) \\
 &= \frac{\sin(10(\omega - 20\pi))}{\omega - 20\pi} + \frac{\sin(10(\omega + 20\pi))}{\omega + 20\pi}
 \end{aligned}$$

Unit of  $\omega$  is rad/s

(b) Note that  $x(\omega), Y(\omega)$  are all real, so we can simply plot them (as opposed to plotting magnitude and phase in case of complex number)

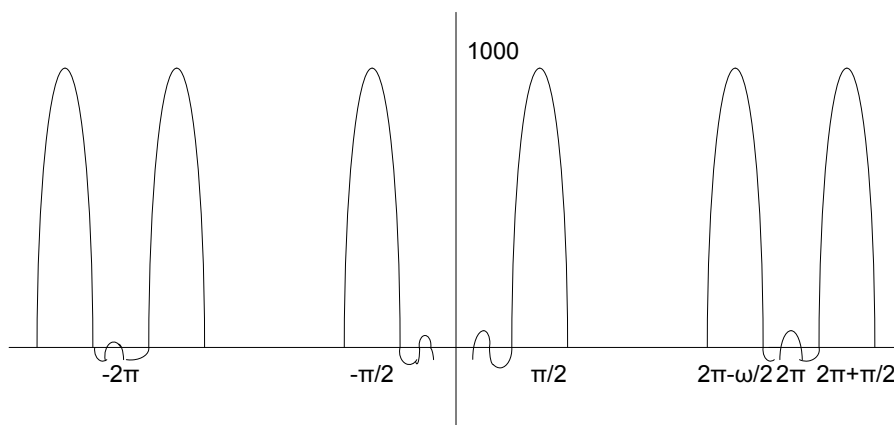


( $\pi$  is the strength(area under) of the impulse)

$p(0)$  is actually undefined (divided by zero), but the limit is  $\lim_{\omega \rightarrow 0} \frac{2\sin(10\omega)}{\omega}$   
 (by L'Hopital rule)  $= \frac{2\cos(10\omega)10}{1} \Big|_{\omega=0} = 20$

$Y(0) = 0$  because  $p(\omega) = 0$  when  $\omega = \frac{\pi}{10}k, k \in \mathbb{Z}$  and  $20\pi$  is divisible by  $\frac{\pi}{10}$

- (c) Unit of  $\omega$  in discrete-time is rad/sample . Sampling has an effect of replicating  $Y(\omega)$  every  $2\pi f_s = 200\pi$  rad/s and scale the frequency by  $f_s$ , so frequency  $2\pi$  rad/sample in discrete time corresponds to frequency  $2\pi f_s$  in continuous time.  $z(\omega) = 100 \sum_{k=-\infty}^{\infty} Y(100(\omega - 2\pi k))$
- (d) Note that there will be some tails from sines at other location although the effects are not significant. periodic with period  $2\pi$



11 (a)

$$t_1 = \cos(10\pi t) \xrightarrow{F.T} x_1(\omega) = \pi\delta(\omega \pm 10\pi) \quad \omega_{B_1} = 10\pi$$

(b)

$$t_2 = \cos(20\pi t) \xrightarrow{F.T} x_2(\omega) = \pi\delta(\omega \pm 20\pi) \quad \omega_{B_2} = 20\pi$$

$$t_3 = \cos(30\pi t) \xrightarrow{F.T} x_3(\omega) = \pi\delta(\omega \pm 30\pi) \quad \omega_{B_3} = 30\pi$$

$$t_4 = \cos(10\pi t) + \cos(20\pi t) + \cos(30\pi t) \xrightarrow{F.T}$$

$$x_4(\omega) = \pi(\delta(\omega \pm 10\pi) + \delta(\omega \pm 20\pi) + \delta(\omega \pm 30\pi)) \quad \omega_{B_4} = 30\pi$$

(c)

$$\begin{aligned}\omega_s &= 30\pi \text{rad/s} \leftrightarrow f_s = 15\text{Hz} \\ y_1(n) &= \cos(10\pi \frac{n}{15}) = \cos(\frac{2}{3}\pi n) \\ y_2(n) &= \cos(\frac{4}{3}\pi n) = \cos((-2\pi + \frac{4}{3}\pi)) \\ &= \cos(\frac{-2}{3}\pi n) = \cos(\frac{2}{3}\pi n) \\ y_3(n) &= \cos(2\pi n) = 1 \\ y_4(n) &= 2\cos(\frac{2}{3}\pi n) + 1\end{aligned}$$

(d)

$$\begin{aligned}z_k(t)? \quad z_k(t)|_{t=\frac{n}{15}} &= y_k(n) \\ z_1(t) &= \cos(\frac{2}{3}\pi \cdot 15t) = \cos(10\pi t) \quad \omega_B = 10\pi \\ z_2(t) &= \cos(10\pi t) \quad \omega_B = 10\pi \\ z_3(t) &= 1 \quad \omega_B = 0 \\ z_4(t) &= 2\cos(10\pi t) + 1 \quad \omega_B = 10\pi\end{aligned}$$

Note:  $z(t) = \cos((\omega + 2\pi k)f_s t)$  for any  $k \in \mathbb{Z}$  will have the same  $z(n) = \cos(\omega n)$  after sampling at  $f_s$ . By making sure  $|\omega + 2\pi k| < \pi$ , we always have  $z(t)$  with  $\omega_B < \frac{f_s}{2}$ .

- 12 (a) Note: If  $x(n)$  is  $z(t)$  sampled at sampling period  $T$ ,  $y(n)$  is just  $z(t)$  sampled at sampling period  $< T$ .

Give the explanation on pg437 and final equation on pg440 on textbook,

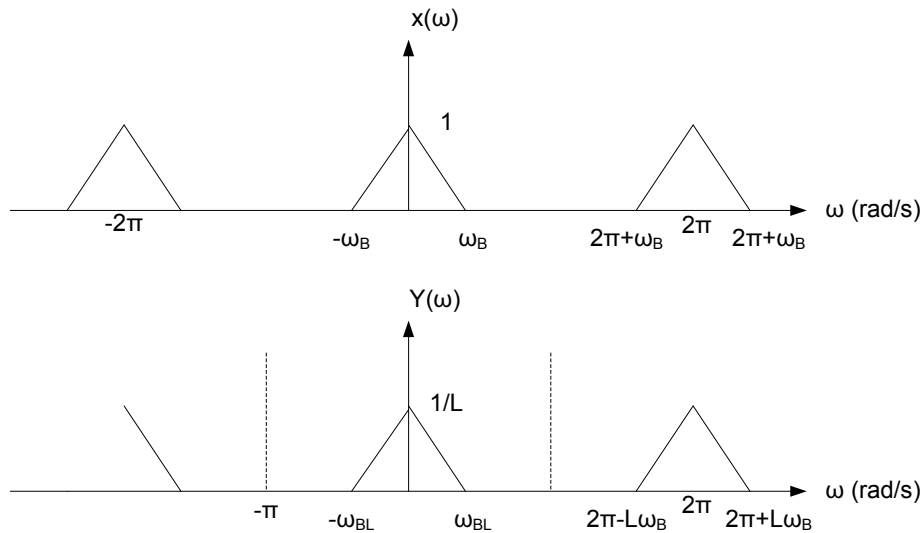
$$\begin{aligned}X(\omega) &= \frac{1}{T} \sum_{t=-\infty}^{\infty} Z(\frac{\omega}{T} - \frac{2\pi k}{T}) \\ Y(\omega) &= \frac{1}{LT} \sum_{t=-\infty}^{\infty} Z(\frac{\omega}{LT} - \frac{2\pi k}{LT})\end{aligned} \quad (1)$$

where  $X(\omega), Y(\omega)$  are DTFT of  $X(n)$  and  $Y(n)$ , and  $Z(\omega)$  is CTFT of

$Z(t)$ , Now let  $k = i + rL$  where  $-\infty < r < \infty$  and  $0 \leq i \leq L - 1$

$$\begin{aligned}
 (1) \implies Y(\omega) &= \frac{1}{LT} \sum_{i=0}^{L-1} \sum_{r=-\infty}^{\infty} Z\left(\frac{\omega}{LT} - \frac{2\pi(i+rL)}{LT}\right) \\
 &= \frac{1}{L} \sum_{i=0}^{L-1} \frac{1}{T} \sum_{r=-\infty}^{\infty} Z\left(\frac{\omega - 2\pi i}{LT} - \frac{2\pi r}{T}\right) \\
 &= \frac{1}{L} \sum_{i=0}^{L-1} X\left(\frac{\omega - 2\pi i}{L}\right)
 \end{aligned}$$

(b) We can reconstruct  $x$  only when there is no overlapping among  $X\left(\frac{\omega - 2\pi i}{L}\right)$



Therefore the condition is:  $\omega_B L < \pi \Leftrightarrow \omega_B < \frac{\pi}{L}$  bandwidth of  $x(n)$  is smaller than  $\frac{\pi}{L}$

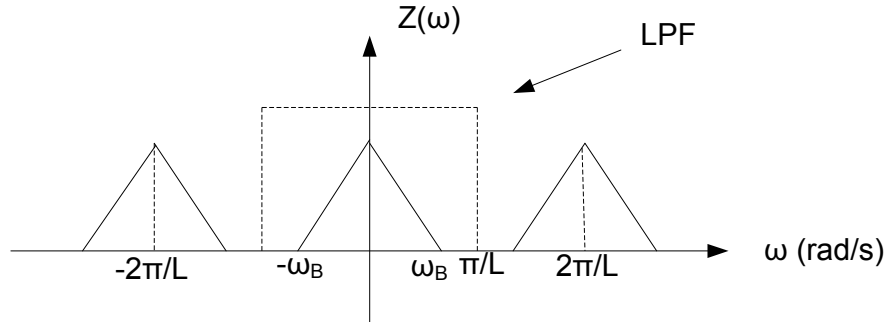


(c) Intuitively, we need to amplify  $y(n)$  by  $L$  and increase the sampling rate.

$$\text{Let } Z(n) = \begin{cases} Ly(n/L) & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$Z(n) = L \sum_{k=-\infty}^{\infty} y(k)\delta(n - kL)$$

$$\begin{aligned} \text{Then } Z(\omega) &= \sum_{n=-\infty}^{\infty} Z(n)e^{-i\omega n} \\ &= L \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} y(k)\delta(n - kL)e^{-i\omega n} \\ &= L \sum_{k=-\infty}^{\infty} y(k)e^{-i\omega kL} \\ &= LY(\omega L) \end{aligned}$$



clearly, after a low-pass filter with cut-off frequency at  $\frac{\pi}{L}$ ,  $\hat{Z}(\omega) = X(\omega)$ , and  $\hat{Z}(n) = X(n)$ .

13 (a)

$$W(n) = Y(nT)$$

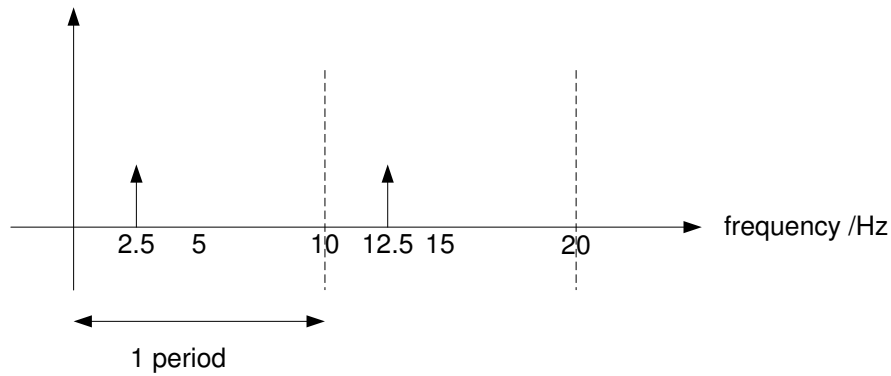
$$U(t) = \sum_n W(n)\delta(t - nT) = \sum_n Y(nT)\delta(t - nT)$$

$$W(\omega) = \frac{1}{T} \sum_k Y\left(\frac{\omega - 2\pi k}{T}\right)$$

$$U(\omega) = W(\omega T) = \frac{1}{T} \sum_k Y\left(\omega - \frac{2\pi k}{T}\right)$$

Note the convention that 'n' is for discrete time and 't' is for continuous time.

- (b) We have  $z(t)$  as the perfect reconstruction of  $Y(t)$ .  $Z(\omega) = Y(\omega) = X(\omega)H(\omega)$
- (c)  $T = 0.1s \Rightarrow$  sampling frequency =  $10Hz$ .  $X(t) = \sin(25\pi t) + \sin(5\pi t)$ .  $\therefore Y(t) = Z(t) = \sin(5\pi t)$
- (d)  $H(\omega) = 1 \forall \omega \Rightarrow Y(\omega)$  is not band-limited as before. Do we have aliasing? However, at a sampling frequency of  $10Hz$ ,  $12.5Hz$  is indistinguishable from  $2.5Hz$ .



$\therefore Z(t) = \sin(5\pi t)$  as before.

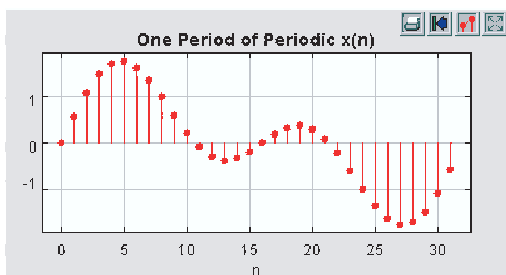
15 (a)

$$\begin{aligned} \text{show } \cos(u+v) &= \cos u \cos v - \sin u \sin v \\ \cos u \cos v &= \frac{1}{4}(e^{ju} + e^{-ju})(e^{jv} + e^{-jv}) \\ &= \frac{1}{4}(e^{j(u+v)} + e^{-j(u+v)} + e^{-j(u-v)} + e^{j(u-v)}) \end{aligned} \quad (1)$$

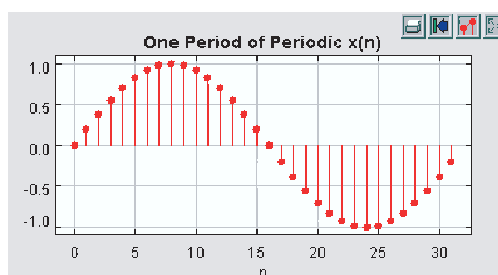
$$\begin{aligned} \sin u \sin v &= -\frac{1}{4}(e^{ju} - e^{-ju})(e^{jv} - e^{-jv}) \\ &= -\frac{1}{4}(e^{j(u+v)} + e^{-j(u+v)} - e^{-j(u-v)} - e^{j(u-v)}) \end{aligned} \quad (2)$$

$$\begin{aligned} (1) - (2) &= \frac{1}{2}(e^{j(u+v)} + e^{-j(u+v)}) \\ &= \cos(u+v) \end{aligned}$$

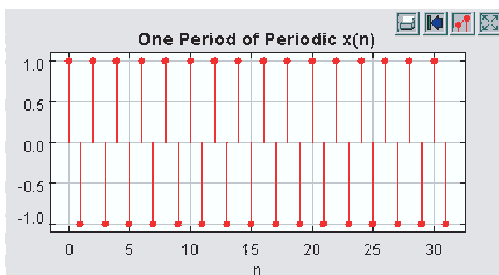
16 Suppose that  $x$  is a discrete-time signal with period  $p = 32$ . Below are plotted six possible such signals  $x$ . For each of these, match one of the six plots on the next page, or indicate that none matches.



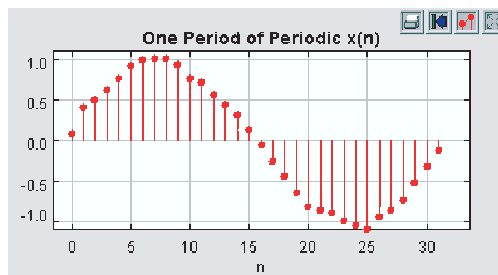
matching FS:     d    



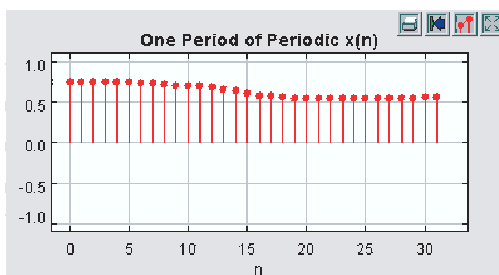
matching FS:     a    



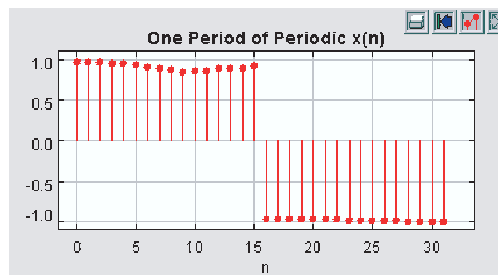
matching FS:     n    



matching FS:     c    



matching FS:     e    



matching FS:     b    

17 (a)

$$x(n) = 1 + \cos\left(\frac{\pi}{4}(n - p_1)\right) + \sin\left(\frac{\pi}{2}(n - p_2)\right)$$

$$\left. \begin{array}{l} p_1 = 8 \\ p_2 = 4 \end{array} \right\} \Rightarrow p = 8 \text{ sec}$$

$$\omega_0 = \frac{2\pi}{p} = \frac{\pi}{4} \text{rads/sec}$$

(b)

$$\begin{aligned} x(n) &= 1 + \cos\left(\frac{n\pi}{4}\right) + \cos\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) \\ &= A_0 + \sum_{k=1}^2 A_k \cos(k\omega_0 n + \phi_k) \\ k &= p/2 = 4; \\ A_0 &= 1, A_1 = 1, A_2 = 1, A_3 = A_4 = 0; \\ \phi_1 &= 0, \phi_2 = -\pi/2, \phi_3 = \phi_4 = 0 \end{aligned}$$

(c)

$$\begin{aligned} x(n) &= \sum_{k=0}^{p-1} x_k e^{ik\omega_0 n} & k &= 4 \\ x_k &= \begin{cases} A_0 & \text{if } k = 0 \\ A_k e^{i\phi_k/2} & \text{if } k \in \{1, \dots, 3\} \\ A_k e^{i\phi_k/2} + A_k e^{-i\phi_k/2} = A_k \cos(\phi) & \text{if } k = 4 \\ A_{8-k} e^{-i\phi_{8-k}/2} & \text{if } k \in \{5, \dots, 7\} \end{cases} \\ x_0 &= 1 \\ x_1 &= A_1 e^{i\phi_1/2} = 1/2 \\ x_2 &= \frac{A_2 e^{i\phi_2}}{2} = \frac{e^{-\frac{\pi}{2}i}}{2} = \frac{\cos(\pi/2) - i \sin(\pi/2)}{2} = \frac{1}{2i} \\ x_3 &= 0 \\ x_4 &= 0 \\ x_5 &= 0 \\ x_6 &= -\frac{1}{2i} \\ x_7 &= \frac{1}{2} \end{aligned}$$

(d)

$$\begin{aligned} y(n) &= 1 + \sin\left(\frac{\pi n}{4}\right) - \cos\left(\frac{\pi n}{2}\right) \\ H'(\omega) &= 1 + iH(\omega), \quad H'(\omega) = \begin{cases} 2 & \text{if } 0 < \omega < \pi \\ 1 & \text{if } \omega = 0 \text{ or } \omega = \pi \\ 1 & \text{if } -\pi < \omega < 0 \end{cases} \end{aligned}$$

(f)

$$y(n) = 1 + e^{i\pi n/4} - ie^{i\pi n/2}$$

or

$$y(n) = 1 + e^{i\pi n/4} + ie^{i(\pi n/2 - \pi/2)}$$

18

$$\forall t \in, \quad c(t) = 2 \cos(\omega_c t)$$

$$\forall t \in, \quad x(t) = 2 \cos(\omega_x t)$$

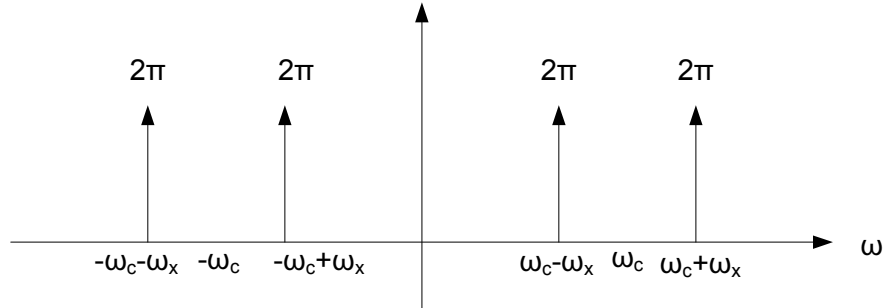
(a)

$$y(t) = c(t)x(t)$$

$$Y(\omega) = \frac{X(\omega - \omega_c) + X(\omega + \omega_c)}{2} \times 2$$

$$X(\omega) = \pi[\delta(\omega - \omega_x) + \delta(\omega + \omega_x)] \times 2$$

$$Y(\omega) = 2\pi[\delta(\omega - (\omega_c + \omega_x)) + \delta(\omega - (\omega_c - \omega_x)) + \delta(\omega + (\omega_c - \omega_x)) + \delta(\omega + (\omega_c + \omega_x))]$$



(b)

$$U(\omega) = Y(\omega)H(\omega)$$

$$= 2\pi[\delta(\omega - (\omega_c - \omega_x)) + \delta(\omega - (\omega_c + \omega_x))]$$

$$u(t) = e^{j(\omega_c - \omega_x)t} + e^{j(\omega_c + \omega_x)t}, \quad \omega_c = 2\pi 8000, \omega_x = 2\pi 400$$

(c)

$$u'(n) = e^{j\omega(2\pi 8000 - 2\pi 400) \cdot (\frac{n}{8000})} + e^{j\omega(2\pi 8000 + 2\pi 400) \cdot (\frac{n}{8000})}$$

$$= e^{j\omega(2\pi n - 2\pi \frac{n}{20})} + e^{j\omega(2\pi n + 2\pi \frac{n}{20})}$$

$$= e^{j\omega(-\frac{\pi n}{10})} + e^{j\omega(\frac{\pi n}{10})}; \quad e^{j\omega(2\pi n)} = 1$$

$$= 2 \cos(\frac{n\pi}{10})$$

(d)

$$z = \text{Ideal Interpolator}_T(u') \quad T = \frac{1}{8000} \text{ s}$$

$$Z(\omega) = U(\omega)S(\omega)$$

$$S(\omega) = \begin{cases} T & \text{if } |\omega| \leq \frac{\pi}{T} \\ 0 & \text{if } |\omega| > \frac{\pi}{T} \end{cases}$$

$$W(t) = \sum_{n=-\infty}^{\infty} u'(n)\delta(t - nT)$$

$$W(\omega) = u'(\omega T) \quad Z(\omega) = W(\omega)S(\omega) = u'(\omega T)S(\omega)$$

$$Z(\omega) = \begin{cases} Tu'(\omega T) & \text{if } \frac{-\pi}{T} < \omega \leq \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

