EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 14 February 2006
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Nonlinear FIRST Name Mister

Lab Time 365/24/7

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This examshould take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided $8.5'' \times 11''$ sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (25 Points) Every inhabitant of the planet Zirth is called a Zirthling. Data collected by our ZirthRover has revealed the existence of the following non-empty categories (subsets) of living Zirthlings:

- F: A Zirthling belonging to this set is called a Fubricon.
- G: A Zirthling belonging to this set is called a Gubricon.
- H: A Zirthling belonging to this set is called a Hubricon.

We know the following about the Zirthlings:

(i) Some Fubricons are Gubricons (i.e., $\exists\, z \in \mathsf{F}$ such that $z \in \mathsf{G}$).

(ii) A Zirthling can*not* be a Fubricon unless it is also a Hubricon.

Consider the assertion

$$\{z \in \mathsf{H} | z \in \mathsf{G}\} = \phi\,,$$

where ϕ denotes the empty set.

From the choices below, select the strongest correct statement about the assertion above. Explain your reasoning succinctly, but clearly and convincingly.

(I) The assertion must be true.

(II) The truth or falsehood of the assertion cannot be determined based on the information given.

(III) $\boxed{\text{The assertion must be false.}}$

(i) If some Fubricons are Gubricons, it means that some Gubricons are Fubricons.

(ii) Let $\mathcal{H}$ denote being a Hubricon and $\mathcal{F}$ denote being a Fubricon. The statement that "a Zirthling can*not* be a Fubricon unless it is also a Hubricon" means that a Zirthling who is not a Hubricon cannot be a Fubricon. Symbolically, this translates to $\neg\mathcal{H} \Rightarrow \neg\mathcal{F}$; the contrapositive of this statement is $\mathcal{F} \Rightarrow \mathcal{H}$, i.e., $\mathsf{F} \subset \mathsf{H}$.

Putting (i) and (ii) together, we note that some Gubricons are Fubricons and every Fubricon must be a Hubricon; therefore, some Gubricons are Hubricons, which means that $\mathsf{G} \cap \mathsf{H} \neq \phi$. Therefore, the assertion $\{z \in \mathsf{H} | z \in \mathsf{G}\} = \phi$ must be false.

MT1.2 (25 Points) Consider the three sets A, B, and C depicted below:

$$\mathsf{A} = \{a\}, \quad \mathsf{B} = \{a, b\}, \quad \mathsf{C} = \{a, b, c\}\,.$$

Determine the truth or falsehood of each of the assertions below. If you find an assertion to be false, provide a counterexample by finding an element in the purported subset which is not a member of the purported superset. If you find an assertion to be true, prove that every element in the purported subset is a member of the purported superset.

(i) $[\mathsf{A} \to \mathsf{C}] \subset [\mathsf{B} \to \mathsf{C}]$.

Circle one: TRUE [✓ FALSE] INDETERMINATE

Explanation:

The set $[\mathsf{B} \to \mathsf{C}]$ consists of $|\mathsf{C}|^{|\mathsf{B}|} = 3^2 = 9$ functions, the graph of each of which is of the form: $\{(a, *), (b, \cdot)\}$, where $* \in \mathsf{C}$ and $\cdot \in \mathsf{C}$. No such function exists in $[\mathsf{A} \to \mathsf{C}]$, because $b \notin \mathsf{A}$; each function in $[\mathsf{A} \to \mathsf{C}]$ has a graph of the form $\{(a, *)$, where $* \in \mathsf{C}$. In other words, functions in $\mathsf{B} \to \mathsf{C}]$ map the element b to something, whereas functions in $[\mathsf{A} \to \mathsf{C}]$ won't know what to do with b.

(ii) $[\mathsf{A} \to \mathsf{B}] \subset [\mathsf{A} \to \mathsf{C}]$.

Circle one: [✓ TRUE] FALSE INDETERMINATE

Explanation:

There are only $|\mathsf{B}|^{|\mathsf{A}|} = 2^1 = 2$ functions in $[\mathsf{A} \to \mathsf{B}]$; their graphs are $\{(a, a)\}$ and $\{(a, b)\}$. The set $[\mathsf{A} \to \mathsf{C}]$ consists of $|\mathsf{C}|^{|\mathsf{A}|} = 3^1 = 3$ functions; their graphs are $\{(a, a)\}$, $\{(a, b)\}$, and $\{(a, c)\}$. Hence, the assertion must be true.

MT1.3 (25 points) Consider a continuous-time (CT) system

$$F : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \{-1, 0, +1\}]\,,$$

which acts as a crude quantizer, described below. The output signal y, produced by the system F in response to an appropriately-defined (but otherwise arbitrary) input signal x, is characterized as follows:

$$\forall\, t \in \mathbb{R}, \quad y(t) \stackrel{\triangle}{=} \begin{cases} -1 & \text{if } x(t) < 0 \\ 0 & \text{if } x(t) = 0 \\ +1 & \text{if } x(t) > 0\,. \end{cases}$$

For each part (a)-(d), select the strongest true assertion from the list. Provide a succinct, but clear and convincing, explanation for each of your selections. If your selection in any part is "(iii)", i.e., that the system can*not* have the particular property in question, you must provide a counterexample.

(a) MEMORYLESSNESS

(i) $\boxed{\text{The system must be memoryless}}$, because there exists a function f such that $y(t) = f(x(t))\,, \forall t \in \mathbb{R},$ and $\forall\, x \in [\mathbb{R} \to \mathbb{R}]$. If this is your selection, specify the function f.

(ii) The system could be memoryless, but does not have to be.

(iii) The system cannot be memoryless.

The function $f : \mathbb{R} \to \{-1, 0, +1\}$ is given precisely by the quantizer, i.e.,

$$\forall\, t \in \mathbb{R}, \quad (F(x))(t) = \begin{cases} -1 & \text{if } x(t) < 0 \\ 0 & \text{if } x(t) = 0 \\ +1 & \text{if } x(t) > 0\,. \end{cases}$$

(b) CAUSALITY

(i) $\boxed{\text{The system must be causal.}}$
(ii) The system could be causal, but does not have to be.
(iii) The system cannot be causal.

Every memoryless system is causal.

(c) TIME INVARIANCE

(i) $\boxed{\text{The system must be time-invariant.}}$
(ii) The system could be time-invariant, but does not have to be.
(iii) The system cannot be time-invariant.

We have shown that $y(t) = f(x(t)), \forall t \in \mathbb{R}$; therefore, $y(t-\tau) = f(x(t-\tau)), \forall \tau \in \mathbb{R}$.

(d) LINEARITY

(i) The system must be linear.
(ii) The system could be linear, but does not have to be.
(iii) $\boxed{\text{The system cannot be linear.}}$

Noting that the system produces a zero output signal in response to a zero input signal, it might be tempting to conclude that it is linear. However, this reasoning is flawed, for there exist nonlinear systems which exhibit the same property, e.g., $y(x) = f(x) = x^2$ (where $y(t) = x^2(t)$). To prove that F is not linear, we simply consider two input signals x_1 and x_2, where $x_1(t) = 1, \forall t$ and $x_2(t) = 2, \forall t$. Each of these two signals yields the same output signal $y(t) = +1$ even though $x_2 = 2\,x_1$.

MT1.4 (30 points) Consider a discrete-time (DT) system

$$G : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]\,.$$

The DT unit impulse signal (i.e., the Kronecker delta function)

$$\begin{aligned} &\delta : \mathbb{Z} \to \mathbb{R} \\ &\forall\, n \in \mathbb{Z}, \quad \delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} \end{aligned}$$

is applied as input to the system, in response to which the system produces the output signal

$$\begin{aligned} &y : \mathbb{Z} \to \mathbb{R} \\ &\forall\, n \in \mathbb{Z}, \quad y(n) = \begin{cases} \alpha & \text{if } n < 0 \\ 2 & \text{if } n = 0 \\ \beta & \text{if } n > 0\,, \end{cases} \end{aligned}$$

where α and β are real constants.

(a) For this part only, assume that the system G is memoryless and $\alpha = \pi$. Then there must exist a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$y(n) = f(x(n))\,,$$

for every $n \in \mathbb{Z}$ and for every input signal $x \in [\mathbb{Z} \to \mathbb{R}]$.

(i) Determine f to the extent possible (you will not be able to specify f completely). Note that as part of determining f, you must also determine β.

Since the system is memoryless, it must be that $f(0) = \alpha = \beta = \pi$ and $f(1) = 2$. This is because $y(n) = f(\delta(n)) = \alpha$ for $n < 0$, where $\delta(n) = 0$. Furthermore, $y(n) = f(\delta(n)) = \beta$ for $n > 0$, where $\delta(n) = 0$. Hence, $\alpha = \beta = \pi$. We also know that $y(0) = f(\delta(0)) = f(1) = 2$. Therefore, the most we can say about the function f is that $f(0) = \pi$ and $f(1) = 2$. For other values, $f(\cdot)$ is not determinable based on what we know.

(ii) Can the system G (of this part (a)) be linear? Explain your reasoning succinctly, but clearly and convincingly.

No! The system G cannot be linear. The memoryless system G produces $y(n) = f(0) = \alpha = \beta = \pi \neq 0, \forall\, n$, in response to the input signal $x : \mathbb{Z} \to \mathbb{R}$, where $x(n) = 0, \forall\, n$.

(b) For this part only, assume that the system G is linear and memoryless. Determine α and β.

For the system to be linear, it must produce a zero output signal in response to a zero input signal. For the system to be linear *and* memoryless, the response signal $y(n)$ must be zero for every sample n at which $x(n) = 0$. Therefore, G is linear *and* memoryless only if $\alpha = \beta = 0$.

(c) For this part only, assume that the system G is linear and causal (but not necessarily memoryless). What constraint(s), if any, must α and β satisfy if the system G is to be both linear and causal?

The system is linear, so a zero input signal must produce a zero output signal.

The system is causal, so if two signals x_1 and x_2 are equal up to, and including, an arbitrary sample n_0, then their corresponding responses y_1 and y_2 must be equal up to, and including, the sample n_0.

The zero signal equals the unit impulse function δ up to, and including, $n = -1$. Hence, the corresponding responses must be equal up to, and including, $n = -1$. This requires that α be zero. The linearity and causality of the system impose no constraint on β. This problem shows that if a causal system is also linear, then the response of the system cannot "precede" the input signal, i.e., the response can not turn nonzero before the input turns nonzero.

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Problem	Points	Your Score
Name	10	10
1	25	25
2	25	25
3	25	25
4	30	30
Total	**115**	115

Do not write above this line. You may use the blank space below for scratch work. Nothing written on this page will be considered in evaluating your work.
