EECS 20N: Structure and Interpretation of Signals and Systems QUIZ 2
Department of Electrical Engineering and Computer Sciences 16 March 2006
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name ______________ FIRST Name Solution

Lab Time ______________

- **(5 Points)** Print your name and lab time in legible, block lettering above.
- This quiz should take up to 20 minutes to complete. You will be given at least 20 minutes—up to the end of today's lecture hour—to work on the quiz.
- **This quiz is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the quiz. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct.
- **The quiz printout consists of pages numbered 1 through 4.** When you are prompted by the teaching staff to begin work, verify that your copy of the quiz is free of printing anomalies and contains all of the four numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your quiz. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this quiz.

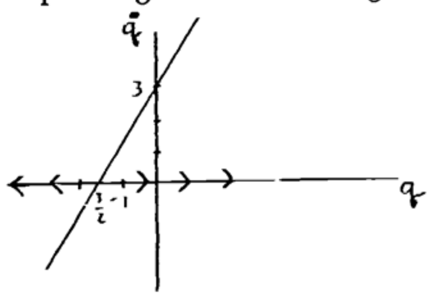
Problem	Points	Your Score
Name	5	5
1	14	14
2	14	14
3	12	12
Total	**45**	45

Q2.1 (14 Points) Consider a causal, autonomous system described in part by the following state-evolution equation:

$$\dot{q} = 2q + 3,$$

where the state $q \in \mathbb{R}$.

(a) On the same graph, sketch (i) $\dot{q}$ as a function of q, and (ii) the corresponding vector field along the q-axis.



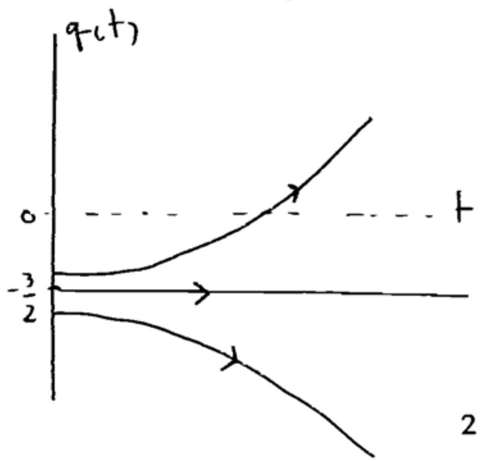
(b) Determine all the fixed points and classify their stability (e.g., stable, half-stable, unstable).

$q^* = -\frac{3}{2}$ is an unstable fixed point.

Slope of $\dot{q}$ vs. q is positive at $-\frac{3}{2}$.

Vector field is going out from $-\frac{3}{2}$.

(c) Provide a well-labeled phase portrait of the system. The phase portrait you provide must include as many qualitatively-distinct trajectories (and their corresponding initial states q_0) as can be discerned from the information given about the system.

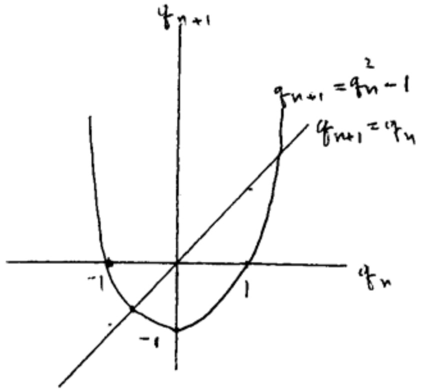


Q2.2 (14 points) Consider a discrete-time, autonomous system described in part by the following state-evolution equation:

$$q_{n+1} = q_n^2 - 1\,,$$

where the state $q \in \mathbb{R}$. It may be useful to know that $\sqrt{5} \approx 2.2$.

(a) Sketch q_{n+1} as a function of q_n, and determine all the fixed points graphically and numerically.



Find fixed point:

$q^* = q^{*2} - 1$

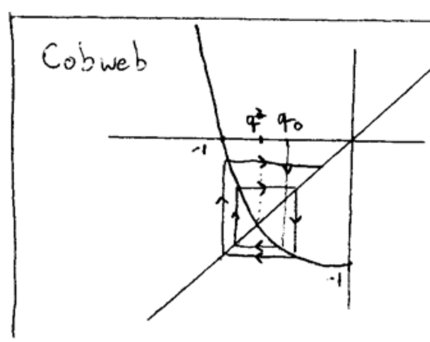
$q^* = \dfrac{1 \pm \sqrt{5}}{2}$

(b) Determine the stability of each fixed point either graphically (using a cobweb diagram) or mathematically. Be sure to explain your reasoning succinctly, but clearly and convincingly.

Mathematically, we can do derivative test for $f(q) = q^2 - 1 \Rightarrow f'(q) = 2q$

$\left| f'\left(\frac{1+\sqrt{5}}{2}\right)\right| = |1+\sqrt{5}| > 1 \Rightarrow$ unstable

$\left| f'\left(\frac{1-\sqrt{5}}{2}\right)\right| = |1-\sqrt{5}| > 1 \Rightarrow$ unstable

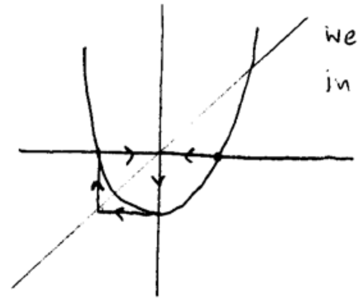


The trajectory goes outward.

(c) Explain the behavior of the system, if the initial state is $q_0 = 1$.

n	0	1	2	3	4
q_n	1	0	-1	0	-1

q_n oscilates between 0 and -1



we can see this in cobweb too.

Q2.3 (12 points) Consider a real, causal, second-order autonomous system described by the state-evolution equation

$$\dot{\boldsymbol{q}} = \boldsymbol{A}\boldsymbol{q}\,,$$

where

$$\boldsymbol{A} = \begin{bmatrix} -1 & \mu \\ +1 & -1 \end{bmatrix}.$$

The tunable parameter $\mu \in \mathbb{R}$. Let $(\lambda_1, \boldsymbol{v}_1)$ and $(\lambda_2, \boldsymbol{v}_2)$ denote the modes of the system. Note that depending on μ, the system may not have a second eigenvector $\boldsymbol{v}_2$, but we will not concern ourselves here with that possibility.

(a) Without any complicated mathematical manipulation, determine a value of μ that produces $\lambda_1 = 0$. What is the corresponding second eigenvalue λ_2?

$|A| = 1-\mu = \lambda_1\lambda_2 = 0 \Rightarrow \quad \mu = 1$

$\text{trace}(A) = -1-1 = -2 = \lambda_1 + \lambda_2 \quad \Rightarrow \lambda_2 = -2$

(b) Is the system Lyapunov stable? Explain your reasoning succinctly, but clearly and convincingly.

(3b)

We give everyone full points for this question since there was a typo in the definition of Lyapunov stability in homework 4.

The correct definition is

A fixed point q^* is Lyapunov stable if

$\forall \epsilon > 0 \ \exists \delta > 0$ such that $\| q(0) - q^* \| < \delta \Rightarrow \forall t \geq 0, \| q(t) - q^* \| < \epsilon$

For this problem, all fixed points are Lyapunov stable.

$$\dot{q} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} q \qquad \lambda_1 = 0 \qquad \lambda_2 = -2$$

$$\underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \underline{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Let $q^* = \begin{bmatrix} q \\ q \end{bmatrix}$ be a fixed point (all fixed points are along direction of $\underline{v_1}$)

and $q(0) = c_1 \underline{v_1} + c_2 \underline{v_2}$

Then $\| q(0) - q^* \| = \sqrt{(c_1 \| \underline{v_1} \| - \| q^* \|)^2 + (c_2 \| \underline{v_2} \|)^2}$

And $\forall t \geq 0$

$$\| q(t) - q^* \| = \sqrt{(c_1 e^{\lambda_1 t} \| v_1 \| - \| q^* \|)^2 + (c_2 e^{\lambda_2 t} \| \underline{v_2} \|)^2}$$

$$= \sqrt{(c_1 \| \underline{v_1} \| - \| q^* \|)^2 + (c_2 e^{-2t} \| v_2 \|)^2}$$

$$\leq \| q(0) - q^* \|$$

Hence, for any fixed point q^* of the system and any $\epsilon > 0$, there exists some $\delta > 0$ (here we can use ϵ itself) such that $\| q(0) - q^* \| < \delta$ implies that $\| q(t) - q^* \| < \epsilon \ \forall t \geq 0$, and we can conclude that the system is Lyapunov stable.