Summary of Equations for Non-Generalized 3D Point-footed, Midleg-Mass, Hipped Walker without Yaw

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Variables

x5 --> relative stance roll angle
x6 --> relative stance pitch angle
x7 --> relative swing pitch angle

ϕ --> static stance roll angle
θs --> static stance pitch angle
θns --> static swing pitch angle

■ Constants

w --> hip width
l --> leg length
Mp --> pelvis / hip mass
M --> midleg mass
gamma --> slope angle
Beta --> potential shaping angle
(difference between passive slope angle and actual slope gamma)

Relative Coordinate System (from Derivation) for Passive Walking

▪ Mass/Inertia Matrix

```
q = {{x5[t]}, {x6[t]}, {x7[t]}};
qdot = D[q, t];
```

```
MatrixForm[Mmatrix3x3 =
  {{1/8 (l^2 (6 M + 4 Mp) + 2 (4 M + Mp) w^2 + l^2 ((5 M + 4 Mp) Cos[2 x6[t]] + M (-4 Cos[x7[t]] +
              Cos[2 (x6[t] + x7[t])] - 4 Cos[2 x6[t] + x7[t]]))),
    1/2 l w ((2 M + Mp) Sin[x6[t]] - M Sin[x6[t] + x7[t]]), -1/2 l M w Sin[x6[t] + x7[t]]},
   {1/2 l w ((2 M + Mp) Sin[x6[t]] - M Sin[x6[t] + x7[t]]),
    1/2 l^2 (3 M + 2 Mp - 2 M Cos[x7[t]]), 1/4 l M (l - 2 l Cos[x7[t]])},
   {-1/2 l M w Sin[x6[t] + x7[t]], 1/4 l M (l - 2 l Cos[x7[t]]), l^2 M/4}}]
```

$$\begin{pmatrix} \frac{1}{8}\left(l^2 (6 M + 4 Mp) + 2 (4 M + Mp) w^2 + l^2 ((5 M + 4 Mp) \mathrm{Cos}[2 x6[t]] + M (-4 \mathrm{Cos}[x7[t]] + \mathrm{Cos}[2 (x6[t] \right. \\ \frac{1}{2} l w ((2 M + Mp) \mathrm{Sin}[x6[t]] - M \mathrm{Sin}[x6[t] + x7[t]]) \\ -\frac{1}{2} l M w \mathrm{Sin}[x6[t] + x7[t]] \end{pmatrix}$$

▪ Hipless

```
MatrixForm[Mmatrix3x3 /. w → 0]
```

$$\begin{pmatrix} \frac{1}{8}\left(l^2 (6 M + 4 Mp) + l^2 ((5 M + 4 Mp) \mathrm{Cos}[2 x6[t]] + M (-4 \mathrm{Cos}[x7[t]] + \mathrm{Cos}[2 (x6[t] + x7[t])] - 4 \mathrm{Co} \right. \\ 0 \\ 0 \end{pmatrix}$$

▪ Lagrangian

```
KE = First[First[1 / 2 Transpose[qdot].Mmatrix3x3.qdot]] // FullSimplify
```

$$\frac{1}{16}\left((l^2 (6 M + 4 Mp) + 2 (4 M + Mp) w^2 + l^2 ((5 M + 4 Mp) \mathrm{Cos}[2 x6[t]] + M (-4 \mathrm{Cos}[x7[t]] + \mathrm{Cos}[2 (x6[t] + x7[t])] - 4 \mathrm{Cos}[2 x6[t] + x7[t]]))) x5'[t]^2 + 8 l w\, x5'[t] (((2 M + Mp) \mathrm{Sin}[x6[t]] - M \mathrm{Sin}[x6[t] + x7[t]]) x6'[t] - M \mathrm{Sin}[x6[t] + x7[t]] x7'[t]) + 2 l^2 ((6 M + 4 Mp - 4 M \mathrm{Cos}[x7[t]]) x6'[t]^2 + 2 M (1 - 2 \mathrm{Cos}[x7[t]]) x6'[t] x7'[t] + M x7'[t]^2)\right)$$

```
PE =
  1/2 g (l Cos[x5[t]] ((3 M + 2 Mp) Cos[x6[t]] - M Cos[x6[t] + x7[t]]) - (2 M + Mp) w Sin[x5[t]]);
```

```
Lagrangian = KE - PE
```

$$\begin{aligned}
&-\frac{1}{2} g (l\,\mathrm{Cos}[x5[t]] ((3 M + 2 Mp) \mathrm{Cos}[x6[t]] - M \mathrm{Cos}[x6[t] + x7[t]]) - (2 M + Mp) w \mathrm{Sin}[x5[t]]) + \\
&\frac{1}{16} ((l^2 (6 M + 4 Mp) + 2 (4 M + Mp) w^2 + l^2 ((5 M + 4 Mp) \mathrm{Cos}[2 x6[t]] + \\
&\qquad M (-4 \mathrm{Cos}[x7[t]] + \mathrm{Cos}[2 (x6[t] + x7[t])] - 4 \mathrm{Cos}[2 x6[t] + x7[t]]))) x5'[t]^2 + \\
&\quad 8 l w x5'[t] (((2 M + Mp) \mathrm{Sin}[x6[t]] - M \mathrm{Sin}[x6[t] + x7[t]]) x6'[t] - \\
&\qquad M \mathrm{Sin}[x6[t] + x7[t]] x7'[t]) + \\
&\quad 2 l^2 ((6 M + 4 Mp - 4 M \mathrm{Cos}[x7[t]]) x6'[t]^2 + 2 M (1 - 2 \mathrm{Cos}[x7[t]]) x6'[t] x7'[t] + M x7'[t]^2))
\end{aligned}$$

Hipless

```
Lagrangian /. w → 0 // FullSimplify
```

$$\begin{aligned}
&\frac{1}{16} l (8 g \mathrm{Cos}[x5[t]] (-(3 M + 2 Mp) \mathrm{Cos}[x6[t]] + M \mathrm{Cos}[x6[t] + x7[t]]) + \\
&\quad l (6 M + 4 Mp + (5 M + 4 Mp) \mathrm{Cos}[2 x6[t]] + \\
&\qquad M (-4 \mathrm{Cos}[x7[t]] + \mathrm{Cos}[2 (x6[t] + x7[t])] - 4 \mathrm{Cos}[2 x6[t] + x7[t]])) x5'[t]^2 + \\
&\quad 2 l ((6 M + 4 Mp - 4 M \mathrm{Cos}[x7[t]]) x6'[t]^2 + 2 M (1 - 2 \mathrm{Cos}[x7[t]]) x6'[t] x7'[t] + M x7'[t]^2))
\end{aligned}$$

Equations of Motion

```
eq1 = D[D[Lagrangian, x5'[t]], t] - D[Lagrangian, x5[t]] // FullSimplify
eq2 = D[D[Lagrangian, x6'[t]], t] - D[Lagrangian, x6[t]] // FullSimplify
eq3 = D[D[Lagrangian, x7'[t]], t] - D[Lagrangian, x7[t]] // FullSimplify
```

$$\begin{aligned}
&\frac{1}{16} \\
&(8 g (-(2 M + Mp) w \mathrm{Cos}[x5[t]] + l (-(3 M + 2 Mp) \mathrm{Cos}[x6[t]] + M \mathrm{Cos}[x6[t] + x7[t]]) \mathrm{Sin}[x5[t]]) + \\
&\quad 2 l^2 x5'[t] (-2 (5 M + 4 Mp) \mathrm{Sin}[2 x6[t]] x6'[t] + M (4 \mathrm{Sin}[x7[t]] x7'[t] - 2 \mathrm{Sin}[\\
&\qquad 2 (x6[t] + x7[t])] (x6'[t] + x7'[t]) + 4 \mathrm{Sin}[2 x6[t] + x7[t]] (2 x6'[t] + x7'[t]))) + \\
&\quad 2 (l^2 (6 M + 4 Mp) + 2 (4 M + Mp) w^2 + l^2 ((5 M + 4 Mp) \mathrm{Cos}[2 x6[t]] + \\
&\qquad M (-4 \mathrm{Cos}[x7[t]] + \mathrm{Cos}[2 (x6[t] + x7[t])] - 4 \mathrm{Cos}[2 x6[t] + x7[t]]))) x5''[t] + \\
&\quad 8 l w (((2 M + Mp) \mathrm{Cos}[x6[t]] - M \mathrm{Cos}[x6[t] + x7[t]]) x6'[t]^2 - \\
&\qquad 2 M \mathrm{Cos}[x6[t] + x7[t]] x6'[t] x7'[t] - M \mathrm{Cos}[x6[t] + x7[t]] x7'[t]^2 + \\
&\qquad ((2 M + Mp) \mathrm{Sin}[x6[t]] - M \mathrm{Sin}[x6[t] + x7[t]]) x6''[t] - M \mathrm{Sin}[x6[t] + x7[t]] x7''[t]))
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{8} l (4 g \mathrm{Cos}[x5[t]] (-(3 M + 2 Mp) \mathrm{Sin}[x6[t]] + M \mathrm{Sin}[x6[t] + x7[t]]) + \\
&\quad l ((5 M + 4 Mp) \mathrm{Sin}[2 x6[t]] + M (\mathrm{Sin}[2 (x6[t] + x7[t])] - 4 \mathrm{Sin}[2 x6[t] + x7[t]])) x5'[t]^2 + \\
&\quad 8 l M \mathrm{Sin}[x7[t]] x6'[t] x7'[t] + \\
&\quad 4 (l M \mathrm{Sin}[x7[t]] x7'[t]^2 + w ((2 M + Mp) \mathrm{Sin}[x6[t]] - M \mathrm{Sin}[x6[t] + x7[t]]) x5''[t] + \\
&\qquad l (3 M + 2 Mp - 2 M \mathrm{Cos}[x7[t]]) x6''[t]) + 2 l M (1 - 2 \mathrm{Cos}[x7[t]]) x7''[t])
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{8} l M (4 \mathrm{Sin}[x6[t] + x7[t]] (-g \mathrm{Cos}[x5[t]] + w x5''[t]) + \\
&\quad l ((2 \mathrm{Sin}[x7[t]] - \mathrm{Sin}[2 (x6[t] + x7[t])] + 2 \mathrm{Sin}[2 x6[t] + x7[t]]) x5'[t]^2 + \\
&\qquad 4 \mathrm{Sin}[x7[t]] x6'[t]^2 + (-2 + 4 \mathrm{Cos}[x7[t]]) x6''[t] - 2 x7''[t]))
\end{aligned}$$

■ Collision Guard

```
height = -w Sin[x5[t]] + 2 l Cos[x5[t]] Sin[x6[t] + x7[t]/2] Sin[x7[t]/2] +
  l (-Sin[x6[t]] + Sin[x6[t] + x7[t]]) Tan[gamma]
```

```
-w Sin[x5[t]] + 2 l Cos[x5[t]] Sin[x6[t] + x7[t]/2] Sin[x7[t]/2] +
 l (-Sin[x6[t]] + Sin[x6[t] + x7[t]]) Tan[gamma]
```

```
Avect = {{D[height, x5[t]], D[height, x6[t]], D[height, x7[t]]}};
holonomicTraj = First[First[Avect.qdot]] // FullSimplify
```

```
-(w Cos[x5[t]] + 2 l Sin[x5[t]] Sin[x6[t] + x7[t]/2] Sin[x7[t]/2]) x5′[t] +
 l (2 Cos[x5[t]] Cos[x6[t] + x7[t]/2] Sin[x7[t]/2] +
    (-Cos[x6[t]] + Cos[x6[t] + x7[t]]) Tan[gamma]) x6′[t] +
 l (Cos[x5[t]] Sin[x6[t] + x7[t]] + Cos[x6[t] + x7[t]] Tan[gamma]) x7′[t]
```

The guard is the zero-level set of the height function:

```
height == 0
```

```
-w Sin[x5[t]] + 2 l Cos[x5[t]] Sin[x6[t] + x7[t]/2] Sin[x7[t]/2] +
  l (-Sin[x6[t]] + Sin[x6[t] + x7[t]]) Tan[gamma] == 0
```

and the negative region of the holonomic constraint's trajectory:

```
holonomicTraj < 0
```

```
-(w Cos[x5[t]] + 2 l Sin[x5[t]] Sin[x6[t] + x7[t]/2] Sin[x7[t]/2]) x5′[t] +
  l (2 Cos[x5[t]] Cos[x6[t] + x7[t]/2] Sin[x7[t]/2] +
     (-Cos[x6[t]] + Cos[x6[t] + x7[t]]) Tan[gamma]) x6′[t] +
  l (Cos[x5[t]] Sin[x6[t] + x7[t]] + Cos[x6[t] + x7[t]] Tan[gamma]) x7′[t] < 0
```

■ Impact Equations

```
{x5impact = x5[t], x6impact = x6[t] + x7[t], x7impact = -x7[t]};
```

```
{x5dotimpact =
  -((Cos[2 x6[t]] ((l^2 M (M + 4 Mp) + 4 (M + Mp)^2 w^2) Cos[2 x7[t]] - M (-2 l^2 (M + 2 Mp) + 4
              (M + Mp) w^2 + l^2 M Cos[4 x7[t]])) +
         2 (l^2 M (3 M + 4 Mp) - (2 M^2 + M Mp - 2 Mp^2) w^2 + 2 M^2 (-l^2 + w^2) Cos[2 x7[t]] +
            4 l^2 (M + Mp) Cos[x6[t]]^2 Cos[x7[t]] (-3 M - 4 Mp + 2 M Cos[2 x7[t]]) +
            (-2 (l^2 M (M + 2 Mp) + 2 (M + Mp)^2 w^2) Cos[x7[t]] + l^2 (-2 (M + Mp) (-3 M - 4 Mp +
                  2 M Cos[2 x7[t]]) + M^2 Cos[3 x7[t]])) Sin[2 x6[t]] Sin[x7[t]])) x5′[t] +
```

$$4\,l\,w\left(\left(\left(2\,Mp\,(5\,M+4\,Mp+(6\,M+4\,Mp)\,Cos[x7[t]])\,Sin[x6[t]]\,Sin\left[\frac{x7[t]}{2}\right]^2-(2\,M+Mp)\,Cos[x6[t]]\,(-M+2\,(M+2\,Mp)\,Cos[x7[t]])\,Sin[x7[t]]\right)x6'[t]+M\,(2\,M\,Cos[x6[t]]\,Sin[x7[t]]+Mp\,Sin[x6[t]+x7[t]])\,x7'[t]\right)\right)\Big/$$
$$(Cos[2\,x6[t]]\,((l^2\,(13\,M^2+32\,M\,Mp+16\,Mp^2)+4\,Mp\,(2\,M+Mp)\,w^2)\,Cos[2\,x7[t]]+M\,(-2\,l^2\,M+4\,(M+2\,Mp)\,w^2-l^2\,(5\,M+4\,Mp)\,Cos[4\,x7[t]]))+2\,(l^2\,(3\,M+2\,Mp)\,(3\,M+4\,Mp)+(2\,M^2+7\,M\,Mp+2\,Mp^2)\,w^2+2\,M\,(-l^2\,(3\,M+2\,Mp)+Mp\,w^2)\,Cos[2\,x7[t]]+4\,l^2\,M\,Cos[x6[t]]^2\,(-2\,(M+2\,Mp)\,Cos[x7[t]]+M\,Cos[3\,x7[t]])+(-2\,(l^2\,(M+2\,Mp)\,(5\,M+4\,Mp)+2\,Mp\,(2\,M+Mp)\,w^2)\,Cos[x7[t]]+l^2\,M\,(6\,M+8\,Mp-4\,M\,Cos[2\,x7[t]]+(5\,M+4\,Mp)\,Cos[3\,x7[t]]))\,Sin[2\,x6[t]]\,Sin[x7[t]])),$$

$$x6dotimpact=-(2\,w\,(8\,l^2\,(M+Mp)\,Cos[x6[t]]^2\,(M-Mp\,Cos[2\,x7[t]])\,Sin[x6[t]]+Cos[x7[t]]\,((-2\,l^2\,(2\,M^2+2\,M\,Mp+Mp^2)-2\,M\,(M+2\,Mp)\,w^2+l^2\,M^2\,Cos[2\,x7[t]])\,Sin[x6[t]]+(l^2\,(M^2+4\,M\,Mp+2\,Mp^2)-2\,M\,(M+Mp)\,w^2-4\,l^2\,(M+Mp)^2\,Cos[2\,x7[t]])\,Sin[3\,x6[t]])+Cos[x6[t]]\,(M\,(3\,l^2\,M-2\,(2\,M+Mp)\,w^2)-4\,(M+Mp)\,((l^2\,(M+Mp)+M\,w^2)\,Cos[2\,x6[t]]+2\,l^2\,(M+Mp\,Cos[2\,x6[t]])\,Cos[x7[t]])-l^2\,(-5\,M^2-8\,M\,Mp-4\,Mp^2+8\,(M+Mp)^2\,Cos[2\,x6[t]])\,Cos[2\,x7[t]])\,Sin[x7[t]])$$
$$x5'[t]+l\,((Cos[2\,x6[t]]\,(M\,(l^2\,(5\,M+8\,Mp)+4\,M\,w^2)-(l^2\,(11\,M^2+18\,M\,Mp+8\,Mp^2)+4\,(2\,M^2+4\,M\,Mp+Mp^2)\,w^2)\,Cos[x7[t]]+l^2\,(3\,M\,(3\,M+4\,Mp)\,Cos[2\,x7[t]]-(M+2\,Mp)\,(5\,M+4\,Mp)\,Cos[3\,x7[t]]))+2\,(l^2\,M\,(5\,M+6\,Mp)+M\,(2\,M+Mp)\,w^2-2\,(4\,l^2\,(M+Mp)^2+(2\,M^2+5\,M\,Mp+Mp^2)\,w^2)\,Cos[x7[t]]+2\,l^2\,M\,(M+2\,Mp)\,Cos[2\,x7[t]]+(l^2\,(7\,M^2+14\,M\,Mp+8\,Mp^2)+2\,Mp\,(2\,M+Mp)\,w^2+l^2\,(-3\,M\,(3\,M+4\,Mp)\,Cos[x7[t]]+(M+2\,Mp)\,(5\,M+4\,Mp)\,Cos[2\,x7[t]]))\,Sin[2\,x6[t]]\,Sin[x7[t]]))\,x6'[t]+M\,(l^2\,(6\,M+4\,Mp)+2\,(2\,M+Mp)\,w^2+M\,(l^2+4\,w^2)\,Cos[2\,x6[t]]+l^2\,(-4\,M\,Cos[x7[t]]+(5\,M+4\,Mp)\,Cos[2\,(x6[t]+x7[t])]-4\,M\,Cos[2\,x6[t]+x7[t]]))\,x7'[t]))\,/$$
$$(l\,(Cos[2\,x6[t]]\,((l^2\,(13\,M^2+32\,M\,Mp+16\,Mp^2)+4\,Mp\,(2\,M+Mp)\,w^2)\,Cos[2\,x7[t]]+M\,(-2\,l^2\,M+4\,(M+2\,Mp)\,w^2-l^2\,(5\,M+4\,Mp)\,Cos[4\,x7[t]]))+2\,(l^2\,(3\,M+2\,Mp)\,(3\,M+4\,Mp)+(2\,M^2+7\,M\,Mp+2\,Mp^2)\,w^2+2\,M\,(-l^2\,(3\,M+2\,Mp)+Mp\,w^2)\,Cos[2\,x7[t]]+4\,l^2\,M\,Cos[x6[t]]^2\,(-2\,(M+2\,Mp)\,Cos[x7[t]]+M\,Cos[3\,x7[t]])+(-2\,(l^2\,(M+2\,Mp)\,(5\,M+4\,Mp)+2\,Mp\,(2\,M+Mp)\,w^2)\,Cos[x7[t]]+l^2\,M\,(6\,M+8\,Mp-4\,M\,Cos[2\,x7[t]]+(5\,M+4\,Mp)\,Cos[3\,x7[t]]))\,Sin[2\,x6[t]]\,Sin[x7[t]]))),$$

$$x7dotimpact=\Big(w\,((l^2\,(21\,M^2+24\,M\,Mp+4\,Mp^2)+8\,Mp\,(M+Mp)\,w^2)\,Sin[x6[t]]+$$
$$8\,l^2\,M\,(M+Mp)\,Sin[3\,x6[t]]+5\,l^2\,M^2\,Sin[x6[t]-2\,x7[t]]+4\,l^2\,M\,Mp\,Sin[x6[t]-2\,x7[t]]-24\,l^2\,M^2\,Sin[x6[t]-x7[t]]-36\,l^2\,M\,Mp\,Sin[x6[t]-x7[t]]-16\,l^2\,Mp^2\,Sin[x6[t]-x7[t]]+4\,M^2\,w^2\,Sin[x6[t]-x7[t]]-l^2\,M^2\,Sin[3\,x6[t]-x7[t]]+l^2\,M^2\,Sin[x6[t]+x7[t]]-8\,M^2\,w^2\,Sin[x6[t]+x7[t]]-8\,M\,Mp\,w^2\,Sin[x6[t]+x7[t]]-17\,l^2\,M^2\,Sin[3\,x6[t]+x7[t]]-28\,l^2\,M\,Mp\,Sin[3\,x6[t]+x7[t]]-12\,l^2\,Mp^2\,Sin[3\,x6[t]+x7[t]]-4\,M^2\,w^2\,Sin[3\,x6[t]+x7[t]]-4\,M\,Mp\,w^2\,Sin[3\,x6[t]+x7[t]]-8\,l^2\,M^2\,Sin[x6[t]+2\,x7[t]]-4\,l^2\,M\,Mp\,Sin[x6[t]+2\,x7[t]]+8\,M^2\,w^2\,Sin[x6[t]+2\,x7[t]]-4\,Mp^2\,w^2\,Sin[x6[t]+2\,x7[t]]+7\,l^2\,M^2\,Sin[3\,x6[t]+2\,x7[t]]+8\,l^2\,M\,Mp\,Sin[3\,x6[t]+2\,x7[t]]+8\,M^2\,w^2\,Sin[3\,x6[t]+2\,x7[t]]+12\,M\,Mp\,w^2\,Sin[3\,x6[t]+2\,x7[t]]+4\,Mp^2\,w^2\,Sin[3\,x6[t]+2\,x7[t]]+5\,l^2\,M^2\,Sin[x6[t]+3\,x7[t]]+8\,l^2\,M\,Mp\,Sin[x6[t]+3\,x7[t]]+4\,l^2\,Mp^2\,Sin[x6[t]+3\,x7[t]]+l^2\,(M+2\,Mp)\,(3\,M+2\,Mp)\,Sin[3\,x6[t]+4\,x7[t]])\,x5'[t]+$$

```
l (4 (2 (-(M + Mp) (2 l^2 (M + 2 Mp) - 3 Mp w^2 + (8 l^2 (M + Mp) + 2 (2 M - Mp) w^2) Cos[x7[t]] -
          4 l^2 M Cos[2 x7[t]]) - Cos[2 x6[t]] (-3 l^2 M (M + Mp) + Mp (5 M + 3 Mp) w^2 +
          (l^2 (M + Mp) (3 M + 4 Mp) + 2 (2 M + Mp) (M + 2 Mp) w^2) Cos[x7[t]] + l^2 (M + Mp)
          ((M + 4 Mp) Cos[2 x7[t]] + (5 M + 4 Mp) Cos[3 x7[t]]))) Sin[x7[t]/2]^2 +
      (-(l^2 (M + Mp) (9 M + 4 Mp) + 2 (2 M + Mp) (M + 2 Mp) w^2) Cos[x7[t]] +
        (M + Mp) (l^2 (5 M + 4 Mp) + (2 M + Mp) w^2 + l^2 ((9 M + 4 Mp) Cos[2 x7[t]] -
              (5 M + 4 Mp) Cos[3 x7[t]]))) Sin[2 x6[t]] Sin[x7[t]]) x6′[t] +
    M (2 l^2 (5 M + 2 Mp) + 2 (2 M + Mp) w^2 + M (5 l^2 + 4 w^2) Cos[2 x6[t]] -
      l^2 M Cos[2 x6[t] - x7[t]] - 16 l^2 M Cos[x7[t]] - 8 l^2 Mp Cos[x7[t]] -
      8 M w^2 Cos[x7[t]] + 4 l^2 M Cos[2 x7[t]] + 9 l^2 M Cos[2 (x6[t] + x7[t])] +
      4 l^2 Mp Cos[2 (x6[t] + x7[t])] - 10 l^2 M Cos[2 x6[t] + x7[t]] -
      4 l^2 Mp Cos[2 x6[t] + x7[t]] - 8 M w^2 Cos[2 x6[t] + x7[t]] -
      4 Mp w^2 Cos[2 x6[t] + x7[t]] - l^2 (5 M + 4 Mp) Cos[2 x6[t] + 3 x7[t]]) x7′[t])) /
 (l (Cos[2 x6[t]] ((l^2 (13 M^2 + 32 M Mp + 16 Mp^2) + 4 Mp (2 M + Mp) w^2) Cos[2 x7[t]] +
        M (-2 l^2 M + 4 (M + 2 Mp) w^2 - l^2 (5 M + 4 Mp) Cos[4 x7[t]])) +
     2 (l^2 (3 M + 2 Mp) (3 M + 4 Mp) + (2 M^2 + 7 M Mp + 2 Mp^2) w^2 + 2 M (-l^2 (3 M + 2 Mp) + Mp w^2)
        Cos[2 x7[t]] + 4 l^2 M Cos[x6[t]]^2 (-2 (M + 2 Mp) Cos[x7[t]] + M Cos[3 x7[t]]) +
       (-2 (l^2 (M + 2 Mp) (5 M + 4 Mp) + 2 Mp (2 M + Mp) w^2) Cos[x7[t]] + l^2 M (6 M + 8 Mp -
             4 M Cos[2 x7[t]] + (5 M + 4 Mp) Cos[3 x7[t]])) Sin[2 x6[t]] Sin[x7[t]])))};
```

Converting to Static Coordinate System

Conversion from relative to static coordinates:

```
{x6[t] → -θs[t], x6'[t] → -θs'[t], x7[t] → (-θns[t] + θs[t]),
 x7'[t] → (-θns'[t] + θs'[t]), x5[t] → ϕ[t], x5'[t] → ϕ'[t]}
```

▪ Variables

```
ϕ --> stance roll angle
θns --> swing pitch angle
θs --> stance pitch angle
```

▪ Lagrangian

```
q = {{ϕ[t]}, {θns[t]}, {θs[t]}};
qdot = D[q, t];
```

```
KE = KE /. {x6[t] → -θs[t], x6'[t] → -θs'[t], x7[t] → (-θns[t] + θs[t]),
    x7'[t] → (-θns'[t] + θs'[t]), x5[t] → ϕ[t], x5'[t] → ϕ'[t]} // FullSimplify
```

$$\frac{1}{16}\left(2\,l^2\left(M\,\theta ns'[t]^2 - 4\,M\,Cos[\theta ns[t] - \theta s[t]]\,\theta ns'[t]\,\theta s'[t] + (5\,M + 4\,Mp)\,\theta s'[t]^2\right) + 8\,l\,w\,(-M\,Sin[\theta ns[t]]\,\theta ns'[t] + (2\,M + Mp)\,Sin[\theta s[t]]\,\theta s'[t])\,\phi'[t] + \left(l^2\,(6\,M + 4\,Mp) + 2\,(4\,M + Mp)\,w^2 + l^2\,(M\,Cos[2\,\theta ns[t]] - 8\,M\,Cos[\theta ns[t]]\,Cos[\theta s[t]] + (5\,M + 4\,Mp)\,Cos[2\,\theta s[t]])\right)\phi'[t]^2\right)$$

```
PE = PE /. {x6[t] → -θs[t], x6'[t] → -θs'[t], x7[t] → (-θns[t] + θs[t]),
    x7'[t] → (-θns'[t] + θs'[t]), x5[t] → ϕ[t], x5'[t] → ϕ'[t]} // FullSimplify
```

$$\frac{1}{2}\,g\,(l\,(-M\,Cos[\theta ns[t]] + (3\,M + 2\,Mp)\,Cos[\theta s[t]])\,Cos[\phi[t]] - (2\,M + Mp)\,w\,Sin[\phi[t]])$$

```
Lagrangian = KE - PE /. w → 0
```

$$-\frac{1}{2}\,g\,l\,(-M\,Cos[\theta ns[t]] + (3\,M + 2\,Mp)\,Cos[\theta s[t]])\,Cos[\phi[t]] + \frac{1}{16}\left(2\,l^2\left(M\,\theta ns'[t]^2 - 4\,M\,Cos[\theta ns[t] - \theta s[t]]\,\theta ns'[t]\,\theta s'[t] + (5\,M + 4\,Mp)\,\theta s'[t]^2\right) + \left(l^2\,(6\,M + 4\,Mp) + l^2\,(M\,Cos[2\,\theta ns[t]] - 8\,M\,Cos[\theta ns[t]]\,Cos[\theta s[t]] + (5\,M + 4\,Mp)\,Cos[2\,\theta s[t]])\right)\phi'[t]^2\right)$$

```
Lagrangian /. w → 0
```

$$-\frac{1}{2}\,g\,l\,(-M\,Cos[\theta ns[t]] + (3\,M + 2\,Mp)\,Cos[\theta s[t]])\,Cos[\phi[t]] + \frac{1}{16}\left(2\,l^2\left(M\,\theta ns'[t]^2 - 4\,M\,Cos[\theta ns[t] - \theta s[t]]\,\theta ns'[t]\,\theta s'[t] + (5\,M + 4\,Mp)\,\theta s'[t]^2\right) + \left(l^2\,(6\,M + 4\,Mp) + l^2\,(M\,Cos[2\,\theta ns[t]] - 8\,M\,Cos[\theta ns[t]]\,Cos[\theta s[t]] + (5\,M + 4\,Mp)\,Cos[2\,\theta s[t]])\right)\phi'[t]^2\right)$$

■ Mass/Inertia Matrix

```
MatrixForm[MmatrixStatic = {{1/8 (l^2 (6 M + 4 Mp) + 2 (4 M + Mp) w^2 +
        l^2 (M Cos[2 θns[t]] - 8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])),
     1/2 l w (-M Sin[θns[t]]) , 1/2 l w ((2 M + Mp) Sin[θs[t]]) },
   { 1/2 l w (-M Sin[θns[t]]), l^2 M/4, -1/2 l^2 M Cos[θs[t] - θns[t]]},
   { 1/2 l w ((2 M + Mp) Sin[θs[t]]) ,
    -1/2 l^2 M Cos[θs[t] - θns[t]], l^2 M/4 + l^2 (M + Mp)}}]
```

$$\begin{pmatrix} \frac{1}{8}\,(l^2\,(6\,M + 4\,Mp) + 2\,(4\,M + Mp)\,w^2 + l^2\,(M\,Cos[2\,\theta ns[t]] - 8\,M\,Cos[\theta ns[t]]\,Cos[\theta s[t]] + (5\,M + 4\,Mp) \\ -\frac{1}{2}\,l\,M\,w\,Sin[\theta ns[t]] \\ \frac{1}{2}\,l\,(2\,M + Mp)\,w\,Sin[\theta s[t]] \end{pmatrix}$$

Hipless

```
MatrixForm[MmatrixStatic /. w → 0]
```

$$\begin{pmatrix} \frac{1}{8}\left(l^2 (6 M + 4 Mp) + l^2 (M \text{Cos}[2 \theta ns[t]] - 8 M \text{Cos}[\theta ns[t]] \text{Cos}[\theta s[t]] + (5 M + 4 Mp) \text{Cos}[2 \theta s[t]])\right) \\ 0 \\ 0 \end{pmatrix}$$

```
KE == First[First[1 / 2 Transpose[{{ϕ'[t]}, {θns'[t]}, {θs'[t]}}].
      MmatrixStatic.{{ϕ'[t]}, {θns'[t]}, {θs'[t]}}]] // FullSimplify
```

True

Equations of Motion

```
eq1 = D[D[Lagrangian, ϕ'[t]], t] - D[Lagrangian, ϕ[t]] // FullSimplify
eq2 = D[D[Lagrangian, θns'[t]], t] - D[Lagrangian, θns[t]] // FullSimplify
eq3 = D[D[Lagrangian, θs'[t]], t] - D[Lagrangian, θs[t]] // FullSimplify
```

$$\frac{1}{8} l \Big(4 g (M \text{Cos}[\theta ns[t]] - (3 M + 2 Mp) \text{Cos}[\theta s[t]]) \text{Sin}[\phi[t]] - 4 l \big(M (\text{Cos}[\theta ns[t]] - 2 \text{Cos}[\theta s[t]]) \text{Sin}[\theta ns[t]] \theta ns'[t] + (-2 M \text{Cos}[\theta ns[t]] + (5 M + 4 Mp) \text{Cos}[\theta s[t]]) \text{Sin}[\theta s[t]] \theta s'[t]\big) \phi'[t] + l (6 M + 4 Mp + M \text{Cos}[2 \theta ns[t]] - 8 M \text{Cos}[\theta ns[t]] \text{Cos}[\theta s[t]] + (5 M + 4 Mp) \text{Cos}[2 \theta s[t]]) \phi''[t]\Big)$$

$$\frac{1}{4} l M \Big(2 g \text{Cos}[\phi[t]] \text{Sin}[\theta ns[t]] + l \big(-2 \text{Sin}[\theta ns[t] - \theta s[t]] \theta s'[t]^2 + (\text{Cos}[\theta ns[t]] - 2 \text{Cos}[\theta s[t]]) \text{Sin}[\theta ns[t]] \phi'[t]^2 + \theta ns''[t] - 2 \text{Cos}[\theta ns[t] - \theta s[t]] \theta s''[t]\big)\Big)$$

$$\frac{1}{4} l \Big(-2 g (3 M + 2 Mp) \text{Cos}[\phi[t]] \text{Sin}[\theta s[t]] + l \big(2 M \text{Sin}[\theta ns[t] - \theta s[t]] \theta ns'[t]^2 + (-2 M \text{Cos}[\theta ns[t]] + (5 M + 4 Mp) \text{Cos}[\theta s[t]]) \text{Sin}[\theta s[t]] \phi'[t]^2 - 2 M \text{Cos}[\theta ns[t] - \theta s[t]] \theta ns''[t] + (5 M + 4 Mp) \theta s''[t]\big)\Big)$$

```
odes = Solve[{eq1 == 0, eq2 == 0, eq3 == 0}, {ϕ''[t], θns''[t], θs''[t]}] // FullSimplify
```

```
{{θns''[t] → (4 (5 M + 4 Mp) u1 - 8 M (u1 + u2) Cos[θns[t] - θs[t]] +
      2 g l M Cos[ϕ[t]] (2 (M + Mp) Sin[θns[t]] + (3 M + 2 Mp) Sin[θns[t] - 2 θs[t]]) +
      l^2 M (2 M Sin[2 (θns[t] - θs[t])] θns'[t]^2 - 2 (5 M + 4 Mp) Sin[θns[t] - θs[t]] θs'[t]^2 +
         (-2 (5 M + 4 Mp) Cos[θs[t]]^3 Sin[θns[t]] + 2 Mp Sin[2 θns[t]] +
            Cos[θns[t]] (3 M Sin[θns[t]] + 2 M Sin[θns[t] - 2 θs[t]] + 2
                (5 M + 4 Mp) Cos[θs[t]]^2 Sin[θs[t]])) ϕ'[t]^2)) /
    (l^2 M (-3 M - 4 Mp + 2 M Cos[2 (θns[t] - θs[t])])), θs''[t] →
   (2 (-2 (u1 + u2) + 4 u1 Cos[θns[t] - θs[t]] +
         g l Cos[ϕ[t]] (M Sin[2 θns[t] - θs[t]] - 2 (M + Mp) Sin[θs[t]])) +
      l^2 (2 M Sin[θns[t] - θs[t]] (θns'[t]^2 - 2 Cos[θns[t] - θs[t]] θs'[t]^2) +
         (2 M Cos[θns[t]]^2 Sin[θns[t] - θs[t]] +
            Cos[θs[t]] (-2 M Sin[2 θns[t] - θs[t]] + (3 M + 4 Mp) Sin[θs[t]])) ϕ'[t]^2)) /
    (l^2 (-3 M - 4 Mp + 2 M Cos[2 (θns[t] - θs[t])])), ϕ''[
    t] →
   (4 (2 u3 + g l (-M Cos[θns[t]] + (3 M + 2 Mp) Cos[θs[t]]) Sin[ϕ[t]] +
         l^2 (M (Cos[θns[t]] - 2 Cos[θs[t]]) Sin[θns[t]] θns'[t] +
            (-2 M Cos[θns[t]] + (5 M + 4 Mp) Cos[θs[t]]) Sin[θs[t]] θs'[t]) ϕ'[t])) /
    (l^2 (6 M + 4 Mp + M Cos[2 θns[t]] - 8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]]))}}
```

■ Hipless

```
odes /. w → 0 // FullSimplify
```

```
{{ϕ''[t] → (4 g (-M Cos[θns[t]] + (3 M + 2 Mp) Cos[θs[t]]) Sin[ϕ[t]] +
       4 l (M (Cos[θns[t]] - 2 Cos[θs[t]]) Sin[θns[t]] θns'[t] +
          (-2 M Cos[θns[t]] + (5 M + 4 Mp) Cos[θs[t]]) Sin[θs[t]] θs'[t]) ϕ'[t]) /
    (l (6 M + 4 Mp + M Cos[2 θns[t]] - 8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])),
  θns''[t] → 1 / (l (-3 M - 4 Mp + 2 M Cos[2 (θns[t] - θs[t])]))
    (2 g Cos[ϕ[t]] (2 (M + Mp) Sin[θns[t]] + (3 M + 2 Mp) Sin[θns[t] - 2 θs[t]]) +
     2 l M Sin[2 (θns[t] - θs[t])] θns'[t]^2 - 2 l (5 M + 4 Mp) Sin[θns[t] - θs[t]] θs'[t]^2 +
     l (-2 (5 M + 4 Mp) Cos[θs[t]]^3 Sin[θns[t]] + 2 Mp Sin[2 θns[t]] + Cos[θns[t]]
          (3 M Sin[θns[t]] + 2 M Sin[θns[t] - 2 θs[t]] + 2 (5 M + 4 Mp) Cos[θs[t]]^2 Sin[θs[t]]))
       ϕ'[t]^2), θs''[t] → 1 / (l (-3 M - 4 Mp + 2 M Cos[2 (θns[t] - θs[t])]))
    (2 g Cos[ϕ[t]] (M Sin[2 θns[t] - θs[t]] - 2 (M + Mp) Sin[θs[t]]) + 2 l M Sin[θns[t] - θs[t]]
       (θns'[t]^2 - 2 Cos[θns[t] - θs[t]] θs'[t]^2) + l (2 M Cos[θns[t]]^2 Sin[θns[t] - θs[t]] +
         Cos[θs[t]] (-2 M Sin[2 θns[t] - θs[t]] + (3 M + 4 Mp) Sin[θs[t]])) ϕ'[t]^2)}}
```

■ Collision Guard

```
height = l (-Cos[θns[t]] + Cos[θs[t]]) Cos[ϕ[t]] -
  w Sin[ϕ[t]] + l (-Sin[θns[t]] + Sin[θs[t]]) Tan[gamma]
```

```
l (-Cos[θns[t]] + Cos[θs[t]]) Cos[ϕ[t]] -
 w Sin[ϕ[t]] + l (-Sin[θns[t]] + Sin[θs[t]]) Tan[gamma]
```

```
Avect = {{D[height, ϕ[t]], D[height, θns[t]], D[height, θs[t]]}};
holonomicTraj = First[First[Avect.{{ϕ'[t]}, {θns'[t]}, {θs'[t]}}]] // FullSimplify
```

```
l (Cos[ϕ[t]] Sin[θns[t]] - Cos[θns[t]] Tan[gamma]) θns′[t] +
 l (-Cos[ϕ[t]] Sin[θs[t]] + Cos[θs[t]] Tan[gamma]) θs′[t] -
 (w Cos[ϕ[t]] + l (-Cos[θns[t]] + Cos[θs[t]]) Sin[ϕ[t]]) ϕ′[t]
```

The guard is the zero-level set of the height function:

```
height == 0
```

```
l (-Cos[θns[t]] + Cos[θs[t]]) Cos[ϕ[t]] -
  w Sin[ϕ[t]] + l (-Sin[θns[t]] + Sin[θs[t]]) Tan[gamma] == 0
```

and the negative region of the holonomic constraint's trajectory:

```
holonomicTraj < 0
```

```
l (Cos[ϕ[t]] Sin[θns[t]] - Cos[θns[t]] Tan[gamma]) θns′[t] +
  l (-Cos[ϕ[t]] Sin[θs[t]] + Cos[θs[t]] Tan[gamma]) θs′[t] -
  (w Cos[ϕ[t]] + l (-Cos[θns[t]] + Cos[θs[t]]) Sin[ϕ[t]]) ϕ′[t] < 0
```

■ Hipless

```
height == 0 /. w → 0
holonomicTraj < 0 /. w → 0
```

```
l (-Cos[θns[t]] + Cos[θs[t]]) Cos[ϕ[t]] + l (-Sin[θns[t]] + Sin[θs[t]]) Tan[gamma] == 0
```

```
l (Cos[ϕ[t]] Sin[θns[t]] - Cos[θns[t]] Tan[gamma]) θns′[t] +
  l (-Cos[ϕ[t]] Sin[θs[t]] + Cos[θs[t]] Tan[gamma]) θs′[t] -
  l (-Cos[θns[t]] + Cos[θs[t]]) Sin[ϕ[t]] ϕ′[t] < 0
```

■ Impact Equations

```
{phiimpact = ϕ[t], thetaNSimpact = θs[t], thetaSimpact = θns[t]};
```

```
phidotimpact = x5dotimpact /. {x6[t] → -θs[t], x6'[t] → -θs'[t], x7[t] → (-θns[t] + θs[t]),
    x7'[t] → (-θns'[t] + θs'[t]), x5[t] → ϕ[t], x5'[t] → ϕ'[t]} // FullSimplify
thetaNSdotimpact = -x7dotimpact - x6dotimpact /.
   {x6[t] → -θs[t], x6'[t] → -θs'[t], x7[t] → (-θns[t] + θs[t]),
    x7'[t] → (-θns'[t] + θs'[t]), x5[t] → ϕ[t], x5'[t] → ϕ'[t]} // FullSimplify
thetaSdotimpact = -x6dotimpact /. {x6[t] → -θs[t], x6'[t] → -θs'[t],
    x7[t] → (-θns[t] + θs[t]), x7'[t] → (-θns'[t] + θs'[t]),
    x5[t] → ϕ[t], x5'[t] → ϕ'[t]} // FullSimplify
```

```
(4 l w (M (Mp Sin[θns[t]] + 2 M Cos[θs[t]] Sin[θns[t] - θs[t]]) θns′[t] -
      (M (M + Mp) Sin[2 θns[t] - 3 θs[t]] +
         (M^2 + 4 M Mp + 2 Mp^2) Sin[2 θns[t] - θs[t]] - 2 Mp (M + Mp) Sin[θs[t]]) θs′[t]) +
   (2 l^2 M (3 M + 4 Mp) - 2 (2 M^2 + M Mp - 2 Mp^2) w^2 - 8 l^2 (M + Mp) (3 M + 4 Mp) Cos[θns[t]] Cos[θs[t]] +
     4 M Cos[2 (θns[t] - θs[t])] (M (-l^2 + w^2) + 4 l^2 (M + Mp) Cos[θns[t]] Cos[θs[t]]) -
     M (-2 l^2 (M + 2 Mp) + 4 (M + Mp) w^2 + l^2 M Cos[4 θns[t]]) Cos[2 θs[t]] +
     Cos[2 θns[t]] (2 l^2 M (M + 2 Mp) + 4 (M + Mp)^2 w^2 - l^2 M^2 Cos[4 θs[t]]) -
     l^2 M^2 Sin[4 θns[t]] Sin[2 θs[t]] - l^2 M^2 Sin[2 θns[t]] Sin[4 θs[t]]) ϕ′[t]) /
 (-2 l^2 (3 M + 2 Mp) (3 M + 4 Mp) - 2 (2 M^2 + 7 M Mp + 2 Mp^2) w^2 +
   Cos[2 θns[t]] (-2 l^2 (7 M^2 + 16 M Mp + 8 Mp^2) - 4 Mp (2 M + Mp) w^2 + l^2 M^2 Cos[4 θs[t]]) +
   M (8 l^2 (3 M + 4 Mp) Cos[θns[t]] Cos[θs[t]] -
     4 Cos[2 (θns[t] - θs[t])] (-l^2 (3 M + 2 Mp) + Mp w^2 + 4 l^2 M Cos[θns[t]] Cos[θs[t]]) +
     (2 l^2 M - 4 (M + 2 Mp) w^2 + l^2 (5 M + 4 Mp) Cos[4 θns[t]]) Cos[2 θs[t]] +
     l^2 (5 M + 4 Mp) Sin[4 θns[t]] Sin[2 θs[t]] + l^2 M Sin[2 θns[t]] Sin[4 θs[t]]))

(-l M (-4 l^2 M + l^2 M Cos[θns[t] - 3 θs[t]] +
      2 l^2 Cos[2 θns[t]] (-2 M + (5 M + 4 Mp) Cos[θns[t] - θs[t]]) +
      (l^2 (13 M + 8 Mp) + 4 (4 M + Mp) w^2) Cos[θns[t]] Cos[θs[t]] -
      8 l^2 M Cos[θns[t]]^2 Cos[2 θs[t]] + (l^2 (11 M + 8 Mp) - 4 Mp w^2) Sin[θns[t]] Sin[θs[t]] -
      4 l^2 M Sin[2 θns[t]] Sin[2 θs[t]]) θns′[t] +
   l (-2 l^2 M (3 M + 2 Mp) + 2 (2 M^2 + 7 M Mp + 4 Mp^2) w^2 +
      (l^2 (3 M + 2 Mp) (3 M + 4 Mp) + 4 (M - Mp) (M + Mp) w^2) Cos[2 θs[t]] + 2 l^2 (M + Mp)
       (5 M + 4 Mp) Cos[4 θns[t]] Cos[2 θs[t]] - 8 l^2 M (M + Mp) Cos[3 θns[t]] Cos[3 θs[t]] +
      8 l^2 M Cos[θns[t]] ((M - 2 (M + Mp) Cos[2 θns[t]]) Cos[θs[t]] - (M + Mp) Cos[3 θs[t]]) +
      Cos[2 θns[t]] (-l^2 M (3 M + 2 Mp) + 4 (2 M + Mp) (M + 2 Mp) w^2 +
         2 (M + Mp) (2 (l^2 (6 M + 4 Mp) + (2 M - Mp) w^2) Cos[2 θs[t]] + l^2 M Cos[4 θs[t]])) -
      128 l^2 M (M + Mp) Cos[θns[t]]^2 Cos[θs[t]]^2 Sin[θns[t]] Sin[θs[t]] +
      4 (M + Mp) (l^2 (6 M + 4 Mp) + (2 M - Mp) w^2 + l^2 M Cos[2 θs[t]]) Sin[2 θns[t]] Sin[2 θs[t]] +
      2 l^2 (M + Mp) (5 M + 4 Mp) Sin[4 θns[t]] Sin[2 θs[t]]) θs′[t] +
   w (32 l^2 (M + Mp)^2 Cos[θns[t]]^2 Cos[θs[t]]^2 Sin[θns[t]] +
      (l^2 (3 M^2 + 16 M Mp + 8 Mp^2) + 4 M (4 M + 3 Mp) w^2 + 2 l^2 (M + 2 Mp) (3 M + 2 Mp) Cos[2 θns[t]])
       Cos[θs[t]] Sin[2 θns[t]] - l^2 M^2 Cos[3 θs[t]] Sin[2 θns[t]] +
      (l^2 (17 M^2 + 20 M Mp + 4 Mp^2) + 8 Mp (M + Mp) w^2 + (l^2 M (11 M + 8 Mp) + 4 Mp (3 M + 2 Mp) w^2)
          Cos[2 θns[t]] - l^2 (M + 2 Mp) (3 M + 2 Mp) Cos[4 θns[t]]) Sin[θs[t]] -
      8 l^2 (M + Mp) Cos[θns[t]] (4 M + 3 Mp + (M + Mp) Cos[2 θns[t]]) Sin[2 θs[t]] +
      l^2 M (4 (M + Mp) + M Cos[2 θns[t]]) Sin[3 θs[t]]) ϕ′[t]) /
 (l (2 l^2 (3 M + 2 Mp) (3 M + 4 Mp) + 2 (2 M^2 + 7 M Mp + 2 Mp^2) w^2 +
     2 (l^2 (7 M^2 + 16 M Mp + 8 Mp^2) + 2 Mp (2 M + Mp) w^2) Cos[2 θns[t]] +
     4 l^2 M^2 Cos[θns[t] - 3 θs[t]] -
     M (l^2 M Cos[2 (θns[t] - 2 θs[t])] + l^2 (5 M + 4 Mp) Cos[4 θns[t] - 2 θs[t]] -
        4 (-2 l^2 (M + 2 Mp) Cos[θns[t] - θs[t]] + (-l^2 (3 M + 2 Mp) + Mp w^2)
            Cos[2 (θns[t] - θs[t])] + l^2 M (Cos[3 (θns[t] - θs[t])] + Cos[3 θns[t] - θs[t]]))) +
     2 M (-l^2 M + 2 (M + 2 Mp) w^2) Cos[2 θs[t]] - 4 l^2 M (3 M + 4 Mp) Cos[θns[t] + θs[t]]))
```

$$
\begin{aligned}
&-(-l M (l^2 (6 M+4 Mp)+2 (2 M+Mp) w^2+l^2 (5 M+4 Mp) Cos[2 \theta ns[t]]- \\
&\quad 8 l^2 M Cos[\theta ns[t]] Cos[\theta s[t]]+M (l^2+4 w^2) Cos[2 \theta s[t]]) \theta ns'[t]+ \\
&\quad l (-16 l^2 M (M+2 Mp) Cos[\theta ns[t]]^2 Cos[\theta s[t]]^2+Cos[\theta ns[t]] \\
&\qquad ((l^2 (M+2 Mp) (13 M+8 Mp)+4 (3 M^2+8 M Mp+2 Mp^2) w^2+2 l^2 (M+2 Mp) (5 M+4 Mp) \\
&\qquad\quad Cos[2 \theta ns[t]]) Cos[\theta s[t]]+M (l^2 (M+2 Mp)+4 (M+Mp) w^2) Cos[3 \theta s[t]])+ \\
&\qquad (M+2 Mp) ((l^2 (11 M+8 Mp)+4 M w^2+2 l^2 (5 M+4 Mp) Cos[2 \theta ns[t]]) \\
&\qquad\quad Sin[\theta ns[t]] Sin[\theta s[t]]-4 l^2 M Sin[2 \theta ns[t]] Sin[2 \theta s[t]])+ \\
&\qquad M (l^2 (M+2 Mp)+4 (M+Mp) w^2) Sin[\theta ns[t]] Sin[3 \theta s[t]]) \theta s'[t]+ \\
&\quad w ((l^2 (M+2 Mp) (3 M+2 Mp)+8 M (M+Mp) w^2-2 M (2 l^2 M-2 (2 M+Mp) w^2+l^2 M Cos[2 \theta ns[t]]) \\
&\qquad\quad Cos[2 \theta s[t]]-l^2 M^2 Cos[4 \theta s[t]]) Sin[\theta ns[t]]+ \\
&\qquad 4 l^2 (M+Mp) (2 Cos[\theta s[t]] (Mp+M Cos[2 \theta s[t]]) Sin[2 \theta ns[t]]+ \\
&\qquad\quad (M+Mp) Sin[3 \theta ns[t]]-8 M Cos[\theta ns[t]]^2 Cos[\theta s[t]]^2 Sin[\theta s[t]])+2 M Cos[\theta ns[t]] \\
&\qquad (2 (l^2 M+Mp w^2)+l^2 M (Cos[2 \theta ns[t]]+Cos[2 \theta s[t]])) Sin[2 \theta s[t]]) \phi'[t]) / \\
&(l (-2 l^2 (3 M+2 Mp) (3 M+4 Mp)-2 (2 M^2+7 M Mp+2 Mp^2) w^2+ \\
&\quad Cos[2 \theta ns[t]] (-2 l^2 (7 M^2+16 M Mp+8 Mp^2)-4 Mp (2 M+Mp) w^2+l^2 M^2 Cos[4 \theta s[t]])+ \\
&\quad M (8 l^2 (3 M+4 Mp) Cos[\theta ns[t]] Cos[\theta s[t]]- \\
&\qquad 4 Cos[2 (\theta ns[t]-\theta s[t])] (-l^2 (3 M+2 Mp)+Mp w^2+4 l^2 M Cos[\theta ns[t]] Cos[\theta s[t]])+ \\
&\qquad (2 l^2 M-4 (M+2 Mp) w^2+l^2 (5 M+4 Mp) Cos[4 \theta ns[t]]) Cos[2 \theta s[t]]+ \\
&\qquad l^2 (5 M+4 Mp) Sin[4 \theta ns[t]] Sin[2 \theta s[t]]+l^2 M Sin[2 \theta ns[t]] Sin[4 \theta s[t]])))
\end{aligned}
$$

■ Hipless

```
{phidotimpact, thetaNSdotimpact, thetaSdotimpact} /. w → 0 // FullSimplify
```

$$
\Big\{\frac{(-M Cos[2 \theta ns[t]]+8 (M+Mp) Cos[\theta ns[t]] Cos[\theta s[t]]-M (2+Cos[2 \theta s[t]])) \phi'[t]}{6 M+4 Mp+(5 M+4 Mp) Cos[2 \theta ns[t]]-8 M Cos[\theta ns[t]] Cos[\theta s[t]]+M Cos[2 \theta s[t]]},
$$
$$
\frac{2 M Cos[\theta ns[t]-\theta s[t]] \theta ns'[t]+(M-4 (M+Mp) Cos[2 (\theta ns[t]-\theta s[t])]) \theta s'[t]}{-3 M-4 Mp+2 M Cos[2 (\theta ns[t]-\theta s[t])]},
$$
$$
\frac{M \theta ns'[t]-2 (M+2 Mp) Cos[\theta ns[t]-\theta s[t]] \theta s'[t]}{-3 M-4 Mp+2 M Cos[2 (\theta ns[t]-\theta s[t])]}\Big\}
$$

The last two equations match the impact equations for the 2D-equivalent walker as derived by Bullo/Grizzle's method.

Controlled Hipless Walking on Flat Ground (Static Coordinates)

■ Controlled Lagrangian

```
PE
```

$$\frac{1}{2} g (l (-M Cos[\theta ns[t]]+(3 M+2 Mp) Cos[\theta s[t]]) Cos[\phi[t]]-(2 M+Mp) w Sin[\phi[t]])$$

Two-dimensional potential shaping for flat-ground walking:

```
PE2Dgamma = PE /. {ϕ[t] → 0, θns[t] → θns[t] - Beta, θs[t] -> θs[t] - Beta}
```

$$\frac{1}{2} g l (-M Cos[Beta-\theta ns[t]]+(3 M+2 Mp) Cos[Beta-\theta s[t]])$$

Three-dimensional potential shaping for flat-ground walking and lean compensation:

```
PE3Dmu =
 PE2Dgamma - 1/2 (-c ϕ[t])^2 / (1/8 (l^2 (6 M + 4 Mp) + l^2 (M Cos[2 θns[t]] - 8 M Cos[θns[t]]
        Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]]))) // FullSimplify
```

$$\frac{1}{2} g\, l \left(-M\, \text{Cos}[\text{Beta} - \theta ns[t]] + (3M + 2Mp)\, \text{Cos}[\text{Beta} - \theta s[t]]\right) - \frac{4 c^2 \phi[t]^2}{l^2 \left(6M + 4Mp + M\, \text{Cos}[2\theta ns[t]] - 8M\, \text{Cos}[\theta ns[t]]\, \text{Cos}[\theta s[t]] + (5M + 4Mp)\, \text{Cos}[2\theta s[t]]\right)}$$

```
L3Dmu = KE - PE3Dmu /. w → 0
```

$$-\frac{1}{2} g\, l \left(-M\, \text{Cos}[\text{Beta} - \theta ns[t]] + (3M + 2Mp)\, \text{Cos}[\text{Beta} - \theta s[t]]\right) + \frac{4 c^2 \phi[t]^2}{l^2 \left(6M + 4Mp + M\, \text{Cos}[2\theta ns[t]] - 8M\, \text{Cos}[\theta ns[t]]\, \text{Cos}[\theta s[t]] + (5M + 4Mp)\, \text{Cos}[2\theta s[t]]\right)} + \frac{1}{16}\Big(2 l^2 \left(M\, \theta ns'[t]^2 - 4M\, \text{Cos}[\theta ns[t] - \theta s[t]]\, \theta ns'[t]\, \theta s'[t] + (5M + 4Mp)\, \theta s'[t]^2\right) + \left(l^2 (6M + 4Mp) + l^2 \left(M\, \text{Cos}[2\theta ns[t]] - 8M\, \text{Cos}[\theta ns[t]]\, \text{Cos}[\theta s[t]] + (5M + 4Mp)\, \text{Cos}[2\theta s[t]]\right)\right) \phi'[t]^2\Big)$$

■ Controlled Equations of Motion

```
eq1 = D[D[L3Dmu, ϕ'[t]], t] - D[L3Dmu, ϕ[t]] // FullSimplify
eq2 = D[D[L3Dmu, θns'[t]], t] - D[L3Dmu, θns[t]] // FullSimplify
eq3 = D[D[L3Dmu, θs'[t]], t] - D[L3Dmu, θs[t]] // FullSimplify
```

$$\frac{1}{8}\Bigg(-\frac{64 c^2 \phi[t]}{l^2 \left(6M + 4Mp + M\, \text{Cos}[2\theta ns[t]] - 8M\, \text{Cos}[\theta ns[t]]\, \text{Cos}[\theta s[t]] + (5M + 4Mp)\, \text{Cos}[2\theta s[t]]\right)} + l^2 \Big(-4 \big(M \left(\text{Cos}[\theta ns[t]] - 2\, \text{Cos}[\theta s[t]]\right) \text{Sin}[\theta ns[t]]\, \theta ns'[t] + \left(-2M\, \text{Cos}[\theta ns[t]] + (5M + 4Mp)\, \text{Cos}[\theta s[t]]\right) \text{Sin}[\theta s[t]]\, \theta s'[t]\big)\, \phi'[t] + \left(6M + 4Mp + M\, \text{Cos}[2\theta ns[t]] - 8M\, \text{Cos}[\theta ns[t]]\, \text{Cos}[\theta s[t]] + (5M + 4Mp)\, \text{Cos}[2\theta s[t]]\right) \phi''[t]\Big)\Bigg)$$

$$\frac{1}{4 l^2}\Bigg(M \Bigg(-\frac{64 c^2 \left(\text{Cos}[\theta ns[t]] - 2\, \text{Cos}[\theta s[t]]\right) \text{Sin}[\theta ns[t]]\, \phi[t]^2}{\left(6M + 4Mp + M\, \text{Cos}[2\theta ns[t]] - 8M\, \text{Cos}[\theta ns[t]]\, \text{Cos}[\theta s[t]] + (5M + 4Mp)\, \text{Cos}[2\theta s[t]]\right)^2} + l^3 \Big(-2 g\, \text{Sin}[\text{Beta} - \theta ns[t]] + l \big(-2\, \text{Sin}[\theta ns[t] - \theta s[t]]\, \theta s'[t]^2 + \left(\text{Cos}[\theta ns[t]] - 2\, \text{Cos}[\theta s[t]]\right) \text{Sin}[\theta ns[t]]\, \phi'[t]^2 + \theta ns''[t] - 2\, \text{Cos}[\theta ns[t] - \theta s[t]]\, \theta s''[t]\big)\Big)\Bigg)\Bigg)$$

$$\frac{1}{16}\Bigg(\frac{256 c^2 \left(2M\, \text{Cos}[\theta ns[t]] - (5M + 4Mp)\, \text{Cos}[\theta s[t]]\right) \text{Sin}[\theta s[t]]\, \phi[t]^2}{l^2 \left(6M + 4Mp + M\, \text{Cos}[2\theta ns[t]] - 8M\, \text{Cos}[\theta ns[t]]\, \text{Cos}[\theta s[t]] + (5M + 4Mp)\, \text{Cos}[2\theta s[t]]\right)^2} + 4 l \Big(2 g (3M + 2Mp)\, \text{Sin}[\text{Beta} - \theta s[t]] + l \big(2M\, \text{Sin}[\theta ns[t] - \theta s[t]]\, \theta ns'[t]^2 + \left(-2M\, \text{Cos}[\theta ns[t]] + (5M + 4Mp)\, \text{Cos}[\theta s[t]]\right) \text{Sin}[\theta s[t]]\, \phi'[t]^2 - 2M\, \text{Cos}[\theta ns[t] - \theta s[t]]\, \theta ns''[t] + (5M + 4Mp)\, \theta s''[t]\big)\Big)\Bigg)$$

```
odes = Solve[{eq1 == 0, eq2 == 0, eq3 == 0}, {ϕ''[t], θns''[t], θs''[t]}] // FullSimplify
```

```
{{θns''[t] →
    (-64 c^2 (Cos[θns[t]] ((3 M + 4 Mp) Sin[θns[t]] + 2 M Sin[θns[t] - 2 θs[t]]) - 2 (5 M + 4 Mp)
           Cos[θs[t]]^2 Sin[θns[t] - θs[t]]) ϕ[t]^2 +
        l^3 (6 M + 4 Mp + M Cos[2 θns[t]] - 8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])^2
         (-4 g (M + Mp) Sin[Beta - θns[t]] + 2 g (3 M + 2 Mp) Sin[Beta + θns[t] - 2 θs[t]] +
           l (2 M Sin[2 (θns[t] - θs[t])] θns'[t]^2 - 2 (5 M + 4 Mp) Sin[θns[t] - θs[t]] θs'[t]^2 +
              (-2 (5 M + 4 Mp) Cos[θs[t]]^3 Sin[θns[t]] + 2 Mp Sin[2 θns[t]] +
                 Cos[θns[t]] (3 M Sin[θns[t]] + 2 M Sin[θns[t] - 2 θs[t]] +
                    2 (5 M + 4 Mp) Cos[θs[t]]^2 Sin[θs[t]])) ϕ'[t]^2))) /
     (l^4 (-5 M - 4 Mp + 4 M Cos[θns[t] - θs[t]]^2) (6 M + 4 Mp + M Cos[2 θns[t]] -
           8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])^2),
  θs''[t] → (-32 c^2 (4 M Cos[θns[t]]^2 Sin[θns[t] - θs[t]] - 4 M Cos[θs[t]]
           Sin[2 θns[t] - θs[t]] + (3 M + 4 Mp) Sin[2 θs[t]]) ϕ[t]^2 +
        1/2 l^3 (6 M + 4 Mp + M Cos[2 θns[t]] - 8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])^2
         (8 g (M + Mp) Sin[Beta - θs[t]] - 4 g M Sin[Beta - 2 θns[t] + θs[t]] +
           4 l M Sin[θns[t] - θs[t]] (θns'[t]^2 - 2 Cos[θns[t] - θs[t]] θs'[t]^2) +
           l (M Cos[θs[t]] (Sin[θns[t]] + Sin[3 θns[t]] - 4 Sin[2 θns[t] - θs[t]]) -
              4 M Cos[θns[t]]^3 Sin[θs[t]] + (3 M + 4 Mp) Sin[2 θs[t]]) ϕ'[t]^2)) /
     (l^4 (-5 M - 4 Mp + 4 M Cos[θns[t] - θs[t]]^2) (6 M + 4 Mp + M Cos[2 θns[t]] -
           8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])^2),
  ϕ''[t] → (4 (16 c^2 ϕ[t] + l^4 (6 M + 4 Mp + M Cos[2 θns[t]] - 8 M Cos[θns[t]] Cos[θs[t]] +
             (5 M + 4 Mp) Cos[2 θs[t]]) (M (Cos[θns[t]] - 2 Cos[θs[t]]) Sin[θns[t]] θns'[t] +
             (-2 M Cos[θns[t]] + (5 M + 4 Mp) Cos[θs[t]]) Sin[θs[t]] θs'[t]) ϕ'[t])) /
     (l^4 (6 M + 4 Mp + M Cos[2 θns[t]] - 8 M Cos[θns[t]] Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])^2)}}
```

■ Conserved Quantity Relationship (Note: Initial Condition for ϕdot)

Conserved quantity μ is parameterized in terms of ϕ in order to implement lean compensation:

```
μ = -c ϕ[t]

ϕ'[t] =
  μ / m3D[θ] = -c ϕ[t] / (1/8 (l^2 (6 M + 4 Mp) + l^2 (M Cos[2 θns[t]] - 8 M Cos[θns[t]]
            Cos[θs[t]] + (5 M + 4 Mp) Cos[2 θs[t]])))
```

■ Collision Guard (Invariant of Control)

Assuming $\phi \in (-\pi/2, \pi/2)$, so as to disregard scaling factor Cos[ϕ] that will never bring height to zero, flat ground, and hipless:

```
height == 0 /. {ϕ[t] → 0, w → 0, gamma → 0}
holonomicTraj < 0 /. {ϕ[t] → 0, w → 0, gamma → 0}
```

$l\,(-\mathrm{Cos}[\theta ns[t]] + \mathrm{Cos}[\theta s[t]]) == 0$

$l\,\mathrm{Sin}[\theta ns[t]]\,\theta ns'[t] - l\,\mathrm{Sin}[\theta s[t]]\,\theta s'[t] < 0$

■ Impact Equations (Invariant of Control)

```
{phidotimpact, thetaNSdotimpact, thetaSdotimpact} /. w → 0 // FullSimplify
```

$$\left\{\frac{(-M\,\mathrm{Cos}[2\,\theta ns[t]] + 8\,(M + Mp)\,\mathrm{Cos}[\theta ns[t]]\,\mathrm{Cos}[\theta s[t]] - M\,(2 + \mathrm{Cos}[2\,\theta s[t]]))\,\phi'[t]}{6\,M + 4\,Mp + (5\,M + 4\,Mp)\,\mathrm{Cos}[2\,\theta ns[t]] - 8\,M\,\mathrm{Cos}[\theta ns[t]]\,\mathrm{Cos}[\theta s[t]] + M\,\mathrm{Cos}[2\,\theta s[t]]},\right.$$
$$\frac{2\,M\,\mathrm{Cos}[\theta ns[t] - \theta s[t]]\,\theta ns'[t] + (M - 4\,(M + Mp)\,\mathrm{Cos}[2\,(\theta ns[t] - \theta s[t])])\,\theta s'[t]}{-3\,M - 4\,Mp + 2\,M\,\mathrm{Cos}[2\,(\theta ns[t] - \theta s[t])]},$$
$$\left.\frac{M\,\theta ns'[t] - 2\,(M + 2\,Mp)\,\mathrm{Cos}[\theta ns[t] - \theta s[t]]\,\theta s'[t]}{-3\,M - 4\,Mp + 2\,M\,\mathrm{Cos}[2\,(\theta ns[t] - \theta s[t])]}\right\}$$