TOWARDS THE GEOMETRIC REDUCTION OF CONTROLLED THREE-DIMENSIONAL BIPEDAL ROBOTIC WALKERS [1]

Aaron D. Ames, [2] Robert D. Gregg, Eric D.B. Wendel and Shankar Sastry

Department of Electrical Engineering and Computer Sciences
University of California, Berkeley
Berkeley, CA 94720

Abstract: The purpose of this paper is to apply methods from geometric mechanics to the analysis and control of bipedal robotic walkers. We begin by introducing a generalization of Routhian reduction, *functional* Routhian Reduction, which allows for the conserved quantities to be functions of the cyclic variables rather than constants. Since bipedal robotic walkers are naturally modeled as hybrid systems, which are inherently nonsmooth, in order to apply this framework to these systems it is necessary to first extend functional Routhian reduction to a hybrid setting. We apply this extension, along with potential shaping and controlled symmetries, to derive a feedback control law that provably results in walking gaits on flat ground for a three-dimensional bipedal walker given walking gaits in two-dimensions.

1. INTRODUCTION

Geometric reduction plays an essential role in understanding physical systems modeled by Lagrangians or Hamiltonians; the simplest being Routhian reduction first discovered in the 1890's (cf. Marsden and Ratiu (1999)). In the case of Routhian reduction, symmetries in the system are characterized by *cyclic* variables, which are coordinates of the configuration space that do not appear in the Lagrangian. Using these symmetries, one can reduce the dimensionality of the phase space (by "dividing" out by the symmetries) and define a corresponding Lagrangian on this reduced phase space. The main result of geometric reduction is that we can understand the behavior of the full-order system in terms of the behavior of the reduced system and vice versa.

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[2] Point of contact, email: `adames@eecs.berkeley.edu`

In classical geometric reduction the conserved quantities used to reduce and reconstruct systems are constants; this indicates that the "cyclic" variables eliminated when passing to the reduced phase space are typically uncontrolled. Yet it is often the case that these variables are the ones of interest—it may be desirable to *control* the cyclic variables while not affecting the reduced order system. This motivates an extension of Routhian reduction to the case when the conserved quantities are functions of the cyclic variables instead of constants.

These concepts motivate our main goal:

> **Goal.** *Develop a feedback control law that results in walking gaits on flat ground for a three-dimensional bipedal robotic walker given walking gaits for a two-dimensional bipedal robotic walker.*

In order to achieve this goal, we begin by considering Lagrangians that are cyclic except for an additional non-cyclic term in the potential energy, i.e., *almost-cyclic* Lagrangians. When Routhian reduction is performed with a function (of a cyclic variable) the result is a Lagrangian on the reduced phase-space: the *functional Routhian*. We are able to show that the dynamics of an almost-cyclic Lagrangian satisfying certain initial conditions project to dynamics of the corresponding functional Routhian, and dynamics of the functional Routhian can be used to reconstruct dynamics of the full-order system. In order to use this result to develop control strategies for bipedal walkers, it first must be generalized to a hybrid setting. That is, after discussing how to explicitly obtain a hybrid system model of a bipedal walker (Section 2), we generalize functional Routhian reduction to a hybrid setting (Section 3), demonstrating that hybrid flows of the reduced and full order system are related in a way analogous to the continuous result.

We then proceed to consider two-dimensional (2D) bipedal walkers. It is well-known that 2D bipedal walkers can walk down shallow slopes without actuation (cf. McGeer (1990), Goswami et al. (1996)). Spong and Bullo (2005) used this observation to develop a positional feedback control strategy that allows for walking on flat ground. In Section 4, we use these results to obtain a hybrid system, $\mathscr{H}^{\mathrm{s}}_{\mathrm{2D}}$, modeling a 2D bipedal robot that walks on flat ground.

In Section 5 we consider three-dimensional (3D) bipedal walkers. Our main result is a positional feedback control law that produces walking gaits in three-dimensions. To obtain this controller we shape the potential energy of the Lagrangian describing the dynamics of the 3D bipedal walker so that it becomes an almost-cyclic Lagrangian, where the cyclic variable is the roll (the unstable component) of the walker. We are able to control the roll through our choice of a non-cyclic term in the potential energy. Since the *functional Routhian* hybrid system obtained by reducing this system is $\mathscr{H}^{\mathrm{s}}_{\mathrm{2D}}$, by picking the "correct" function of the roll, we can force the roll to go to zero for certain initial conditions. That is, we obtain a non-trivial set of initial conditions that provably result in three-dimensional walking. These conclusions are supported by simulations, the code for which can be found at Ames et al. (2006); proofs of the theorems stated in this paper can be found at this location.

2. LAGRANGIAN HYBRID SYSTEMS

We begin this section by defining (simple) hybrid systems and hybrid flows (as introduced in Ames and Sastry (2006)); for more on general hybrid systems, see Lygeros et al. (2003) and the references therein. We then turn our attention to introducing a special class of hybrid systems that will be important when discussing bipedal robots: *unilaterally constrained Lagrangian hybrid systems*. It will be seen that bipedal robotic walkers are naturally modeled by systems of this form.

Definition 1. A *simple hybrid system*[3] is a tuple:

$$\mathscr{H} = (D, G, R, f),$$

where

- D is a smooth manifold, called the *domain*,
- G is an embedded submanifold of D called the *guard*,
- $R : G \to D$ is a smooth map called the *reset map* (or *impact equations*),
- f is a vector field or control system (in which case we call $\mathscr{H}$ a controlled hybrid system) on D, i.e., $\dot{x} = f(x)$ or $\dot{x} = f(x, u)$, respectively.

Hybrid flows. A *hybrid flow* (or *execution*) is a tuple

$$\chi^{\mathscr{H}} = (\Lambda, \mathcal{I}, \mathcal{C}),$$

where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is a finite or infinite indexing set.
- $\mathcal{I} = \{I_i\}_{i\in\Lambda}$ is a *hybrid interval* where $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, N finite. Here, $\tau_i, \tau_{i+1}, \tau_N \in \mathbb{R}$ and $\tau_i \leq \tau_{i+1}$.
- $\mathcal{C} = \{c_i\}_{i\in\Lambda}$ is a collection of integral curves of f, i.e., $\dot{c}_i(t) = f(c_i(t))$ for all $i \in \Lambda$.

We require that the following conditions hold for every $i, i+1 \in \Lambda$,

(i) $$c_i(\tau_{i+1}) \in G,$$
(ii) $$R(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1}).$$

The initial condition for the hybrid flow is $c_0(\tau_0)$.

Lagrangians. Let Q be a *configuration space*, assumed to be a smooth manifold, and TQ the tangent bundle of Q. In this paper, we will consider Lagrangians $L : TQ \to \mathbb{R}$ describing mechanical, or robotic, systems; that is, Lagrangians given in coordinates by:

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q), \qquad (1)$$

where $M(q)$ is the inertial matrix, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy and $V(q)$ is the potential energy. In this case, the Euler-Lagrange equations yield the equations of motion for the system:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = 0,$$

where $C(q, \dot{q})$ is the *Coriolis matrix* (cf. Murray et al. (1993)) and $N(q) = \frac{\partial V}{\partial q}(q)$. The Lagrangian

[3] So named because of their simple discrete structure, i.e., a simple hybrid system has a single domain, guard and reset map.

vector field, f_L, associated to L takes the familiar form:

$$\begin{aligned}(\dot{q}, \ddot{q}) &= f_L(q, \dot{q}) \\ &= \left(\dot{q}, M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q))\right)\end{aligned}$$

Controlled Lagrangians. We will also be interested in controlled Lagrangians. In this case, the equations of motion for the system have the form:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q) = Bu,$$

where we assume that B is an invertible matrix. The result is a control system of the form:

$$\begin{aligned}(\dot{q}, \ddot{q}) &= f_L(q, \dot{q}, u) \\ &= \left(\dot{q}, M(q)^{-1}(-C(q, \dot{q})\dot{q} - N(q) + Bu)\right).\end{aligned}$$

In the future, it will be clear from context whether for a Lagrangian L we are dealing with a corresponding vector field $(\dot{q}, \ddot{q}) = f_L(q, \dot{q})$ or a control system $(\dot{q}, \ddot{q}) = f_L(q, \dot{q}, u)$.

Unilateral constraints. It is often the case that the set of admissible constraints for a mechanical system is determined by a *unilateral constraint function*, which is a smooth function $h : Q \to \mathbb{R}$ such that $h^{-1}(0)$ is a manifold, i.e., 0 is a regular value of h. For bipedal walkers this function is the height of the non-stance (or swing) foot above the ground. In this case, we can explicitly construct the domain and the guard of a hybrid system:

$$\begin{aligned}D_h &= \{(q, \dot{q}) \in TQ : h(q) \geq 0\}, \\ G_h &= \{(q, \dot{q}) \in TQ : h(q) = 0 \text{ and } dh_q\dot{q} < 0\},\end{aligned}$$

where in coordinates:

$$dh_q = \left(\frac{\partial h}{\partial q_1}(q) \ \cdots \ \frac{\partial h}{\partial q_n}(q) \right).$$

Definition 2. We say that $\mathscr{H} = (D, G, R, f)$ is a *unilaterally constrained Lagrangian hybrid system* w.r.t. a Lagrangian $L : TQ \to \mathbb{R}$ and a unilateral constraint function $h : Q \to \mathbb{R}$ if $D = D_h$, $G = G_h$ and $f = f_L$.

Impact Equations. In order to determine the impact equations (or reset map) for the hybrid system $\mathscr{H}$, we typically will utilize an additional constraint function. A *kinematic* constraint function is a smooth function $\Upsilon : Q \to \mathbb{R}^{\upsilon}$ ($\upsilon \geq 1$); this function usually describes the position of the end-effector of a kinematic chain, e.g., in the case of bipedal robots, this is the position of the swing foot. Using this kinematic constraint function one obtains a reset map $R(q, \dot{q}) = (q, P_q(\dot{q}))$, where $P_q : T_qQ \to T_qQ$, with

$$\begin{aligned}&P_q(\dot{q}) = \qquad (2)\\ &\dot{q} - M(q)^{-1}d\Upsilon_q^T(d\Upsilon_q M(q)^{-1}d\Upsilon_q^T)^{-1}d\Upsilon_q\dot{q}.\end{aligned}$$

This reset map models a perfectly plastic impact without slipping and was derived using the set-up in Grizzle et al. (2001) together with block-diagonal matrix inversion.

3. FUNCTIONAL ROUTHIAN REDUCTION

In this section, we introduce a variation of classical Routhian reduction termed *functional Routhian reduction.* Although the authors are unaware of similar procedures in the literature, these ideas certainly are related to the methods introduced in Bloch et al. (2001).

Shape space. We begin by considering an abelian Lie group, $\mathbb{G}$, given by:

$$\mathbb{G} = \underbrace{(\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1)}_{m-\text{times}} \times \mathbb{R}^p,$$

with $k = m + p = \dim(\mathbb{G})$; here $\mathbb{S}^1$ is the circle. The starting point for classical Routhian reduction is a configuration space of the form $Q = S \times \mathbb{G}$, where S is called the *shape space*; we denote an element $q \in Q$ by $q = (\theta, \varphi)$ where $\theta \in S$ and $\varphi \in \mathbb{G}$. Note that we have a projection map $\pi : TS \times T\mathbb{G} \to TS$ where $(\theta, \dot{\theta}, \varphi, \dot{\varphi}) \mapsto (\theta, \dot{\theta})$.

Almost-Cyclic Lagrangians. We will be interested (in the context of bipedal walking) in Lagrangians of a very special form. We say that a Lagrangian $L_\lambda : TS \times T\mathbb{G} \to \mathbb{R}$ is *almost-cyclic* if, in coordinates, it has the form:

$$\begin{aligned}&L_\lambda(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = \qquad (3)\\ &\frac{1}{2}\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix}^T \begin{pmatrix} M_\theta(\theta) & 0 \\ 0 & M_\varphi(\theta) \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} - V_\lambda(\theta, \varphi),\end{aligned}$$

where

$$V_\lambda(\theta, \varphi) = \widetilde{V}(\theta) - \frac{1}{2}\lambda(\varphi)^T M_\varphi^{-1}(\theta)\lambda(\varphi)$$

for some function $\lambda : \mathbb{G} \to \mathbb{R}^k$. Here $M_\theta(\theta) \in \mathbb{R}^{n \times n}$ and $M_\varphi(\theta) \in \mathbb{R}^{k \times k}$ are both symmetric positive definite matrices; here $n = \dim(S)$.

Momentum maps. Fundamental to reduction is the notion of a momentum map $J : TQ \to \mathbb{R}^k$, which makes explicit the conserved quantities in the system. In the framework we are considering here,

$$J(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = \frac{\partial L_\lambda}{\partial \dot{\varphi}}(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = M_\varphi(\theta)\dot{\varphi}.$$

Typically, one sets the momentum map equal to a constant $\mu \in \mathbb{R}^k$; this defines the conserved quantities of the system. In our framework, we will breach this convention and set J equal to a function: this motivates the name *functional Routhian reduction.*

Functional Routhians. For an almost-cyclic Lagrangian L_λ as given in (3), define the corresponding *functional Routhian* $\widetilde{L} : TS \to \mathbb{R}$ by:

$$\widetilde{L}(\theta, \dot{\theta}) = \left[L_\lambda(\theta, \dot{\theta}, \varphi, \dot{\varphi}) - \lambda(\varphi)^T \dot{\varphi} \right] \Big|_{J(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = \lambda(\varphi)}$$

Because $J(\theta, \dot{\theta}, \varphi, \dot{\varphi}) = \lambda(\varphi)$ implies that $\dot{\varphi} = M_{\varphi}^{-1}(\theta)\lambda(\varphi)$, by direct calculation, the functional Routhian is given by:

$$\widetilde{L}(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M_{\theta}(\theta)\dot{\theta} - \widetilde{V}(\theta).$$

That is, any Lagrangian of the form given in (1) is the functional Routhian of an almost-cyclic Lagrangian.

We can relate solutions of the Lagrangian vector field $f_{\widetilde{L}}$ to solutions of the Lagrangian vector field f_{L_λ} and vice versa (in a way analogous to the classical Routhian reduction result, see Marsden and Ratiu (1999)).

Theorem 1. *Let L_λ be an almost-cyclic Lagrangian, and $\widetilde{L}$ the corresponding functional Routhian. Then $(\theta(t), \dot{\theta}(t), \varphi(t), \dot{\varphi}(t))$ is a solution to the vector field f_{L_λ} on $[t_0, t_F]$ with*

$$\dot{\varphi}(t_0) = M_{\varphi}^{-1}(\theta(t_0))\lambda(\varphi(t_0)),$$

if and only if $(\theta(t), \dot{\theta}(t))$ is a solution to the vector field $f_{\widetilde{L}}$ and $(\varphi(t), \dot{\varphi}(t))$ satisfies:

$$\dot{\varphi}(t) = M_{\varphi}^{-1}(\theta(t))\lambda(\varphi(t)).$$

We now have the necessary material needed to introduce our framework for hybrid functional Routhian reduction.

Definition 3. If $\mathscr{H}_\lambda = (D_h, G_h, R, f_{L_\lambda})$ is a unilaterally constrained Lagrangian hybrid system, $\mathscr{H}_\lambda$ is *almost-cyclic* if the following conditions hold:

- $Q = S \times \mathbb{G}$
- $h : Q = S \times \mathbb{G} \to \mathbb{R}$ is *cyclic*,
$$\frac{\partial h}{\partial \varphi} = 0,$$
and so can be viewed as a function $\tilde{h} : S \to \mathbb{R}$.
- $L_\lambda : TS \times T\mathbb{G} \to \mathbb{R}$ is almost-cyclic,
- $\pi_\varphi(R(\theta, \dot{\theta}, \varphi, \dot{\varphi})) = \varphi$, where π_φ is the projection onto the φ-component.
- The following diagram commutes:

$$\begin{array}{ccccc} & & \mathbb{R}^k & & \\ & \overset{J|_{G_h}}{\nearrow} & & \overset{J|_{D_h}}{\nwarrow} & \\ G_h & & \xrightarrow{\quad R \quad} & & D_h \\ \downarrow \pi & & & & \downarrow \pi \\ G_{\tilde{h}} & & \xrightarrow{\quad \widetilde{R} \quad} & & D_{\tilde{h}} \end{array}$$

for some map $\widetilde{R} : G_{\tilde{h}} \to D_{\tilde{h}}$.

Hybrid functional Routhian. If $\mathscr{H}_\lambda = (D_h, G_h, R, f_{L_\lambda})$ is an almost-cyclic unilaterally constrained Lagrangian hybrid system, we can associate to this hybrid system a reduced hybrid system, termed a *functional Routhian* hybrid system, denoted by $\widetilde{\mathscr{H}}$ and defined by:

$$\widetilde{\mathscr{H}} := (D_{\tilde{h}}, G_{\tilde{h}}, \widetilde{R}, f_{\widetilde{L}}).$$

The following theorem quantifies the relationship between $\mathscr{H}_\lambda$ and $\widetilde{\mathscr{H}}$.

Theorem 2. *Let $\mathscr{H}_\lambda$ be a cyclic unilaterally constrained Lagrangian hybrid system, and $\widetilde{\mathscr{H}}$ the associated functional Routhian hybrid system. Then $\chi^{\mathscr{H}_\lambda} = (\Lambda, \mathcal{I}, \{(\theta_i, \dot{\theta}_i, \varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda})$ is a hybrid flow of $\mathscr{H}_\lambda$ with*

$$\dot{\varphi}_0(\tau_0) = M_{\varphi}^{-1}(\theta_0(\tau_0))\lambda(\varphi_0(\tau_0)),$$

if and only if $\chi^{\widetilde{\mathscr{H}}} = (\Lambda, \mathcal{I}, \{\theta_i, \dot{\theta}_i\}_{i\in\Lambda})$ is a hybrid flow of $\widetilde{\mathscr{H}}$ and $\{(\varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda}$ satisfies:

$$\dot{\varphi}_i(t) = M_{\varphi}^{-1}(\theta_i(t))\lambda(\varphi_i(t)),$$
$$\varphi_{i+1}(\tau_{i+1}) = \varphi_i(\tau_{i+1}).$$

4. CONTROLLED SYMMETRIES APPLIED TO 2D BIPEDAL WALKERS

In this section, we begin by studying the standard model of a two-dimensional bipedal robotic walker walking down a slope (walkers of this form have been well-studied by McGeer (1990) and Goswami et al. (1996), to name a few). We then use controlled symmetries to shape the potential energy of the Lagrangian describing this model so that it can walk stably on flat ground.

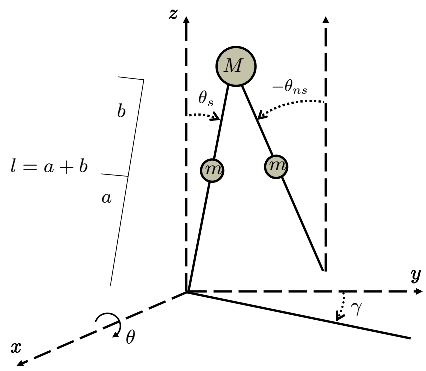

Fig. 1. Two-dimensional bipedal robot.

2D biped model. We begin by introducing a model describing a controlled bipedal robot walking in two-dimensions down a slope of γ degrees. That is, we explicitly construct the controlled hybrid system

$$\mathscr{H}_{2\mathrm{D}}^{\gamma} = (D_{2\mathrm{D}}^{\gamma}, G_{2\mathrm{D}}^{\gamma}, R_{2\mathrm{D}}, f_{2\mathrm{D}}).$$

describing this system.

The configuration space for the 2D biped is $Q_{2\mathrm{D}} = \mathbb{S}^2$ and the Lagrangian describing this system is:

$$L_{2\mathrm{D}}(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M_{2\mathrm{D}}(\theta)\dot{\theta} - V_{2\mathrm{D}}(\theta),$$

where $\theta = (\theta_{\mathrm{ns}}, \theta_{\mathrm{s}})^T$. Table 1 gives $M_{2\mathrm{D}}$ and $V_{2\mathrm{D}}$.

Using the controlled Euler-Lagrange equations, the dynamics for the walker are given by:

$$M_{2D}(\theta)\ddot{\theta} + C_{2D}(\theta,\dot{\theta})\dot{\theta} + N_{2D}(\theta) = B_{2D}u.$$

These equations yield the control system: $(\dot{\theta},\ddot{\theta}) = f_{2D}(\theta,\dot{\theta},u) := f_{L_{2D}}(\theta,\dot{\theta},u)$.

We construct D^{γ}_{2D} and G^{γ}_{2D} by applying the methods outlined in Section 2 to the unilateral constraint function: $h^{\gamma}_{2D}(\theta) = \cos(\theta_s) - \cos(\theta_{ns}) + (\sin(\theta_s) - \sin(\theta_{ns}))\tan(\gamma)$, which gives the height of the foot of the walker above the slope with normalized unit leg length.

Finally, the reset map R_{2D} is given by:

$$R_{2D}(\theta,\dot{\theta}) = \left(S_{2D}\theta, P_{2D}(\theta)\dot{\theta}\right),$$

where S_{2D} and P_{2D} are given in Table 1. Note that this reset map was computed using the methods outlined in Section 2 coupled with the condition that the stance foot is fixed (see Grizzle et al. (2001) for more details).

Setting the control $u = 0$ yields the standard model of a 2D passive bipedal robot walking down a slope. For such a model, it has been well-established (for example, in Goswami et al. (1996)) that for certain γ, $\mathscr{H}^{\gamma}_{2D}$ has a walking gait. For the rest of the paper we pick, once and for all, such a γ.

Controlled Symmetries. Controlled symmetries were introduced in Spong and Bullo (2002) and later in Spong and Bullo (2005) in order to shape the potential of bipedal robotic walkers to allow for stable walking on flat ground based on stable walking down a slope. We will briefly apply the results of this work to derive a feedback control law that yields a hybrid system, $\mathscr{H}^{s}_{2D}$, with stable walking gaits on flat ground.

The main idea of Spong and Bullo (2005) is that inherent symmetries in $\mathscr{H}^{\gamma}_{2D}$ can be used to "rotate the world" (via a group action) to allow for walking on flat ground. Specifically, we have a group action $\Phi : \mathbb{S}^1 \times Q_{2D} \to Q_{2D}$ denoted by:

$$\Phi_{\gamma}(\theta) := (\theta_{ns} - \gamma, \theta_s - \gamma)^T,$$

for $\gamma \in \mathbb{S}^1$. Using this, define the following feedback control law:

$$u = K^{\gamma}_{2D}(\theta) = B^{-1}_{2D}\frac{\partial}{\partial\theta}\left(V_{2D}(\theta) - V_{2D}(\Phi_{\gamma}(\theta))\right).$$

Applying this control law to the control system $(\dot{q},\ddot{q}) = f_{2D}(\theta,\dot{\theta},u)$ yields the dynamical system:

$$(\dot{\theta},\ddot{\theta}) = f^{\gamma}_{2D}(\theta,\dot{\theta}) := f_{2D}(\theta,\dot{\theta},K^{\gamma}_{2D}(\theta))$$

which is just the vector field associated to the Lagrangian

$$L^{\gamma}_{2D}(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T M_{2D}(\theta)\dot{\theta} - V^{\gamma}_{2D}(\theta),$$

where $V^{\gamma}_{2D}(\theta) := V_{2D}(\Phi_{\gamma}(\theta))$. That is, $f^{\gamma}_{2D} = f_{L^{\gamma}_{2D}}$.

Now define, for some γ that results in stable passive walking for $\mathscr{H}^{\gamma}_{2D}$,

$$\mathscr{H}^{s}_{2D} := (D^{0}_{2D}, G^{0}_{2D}, R_{2D}, f^{\gamma}_{2D}),$$

which is a unilaterally constrained Lagrangian hybrid system. In particular, it is related to $\mathscr{H}^{\gamma}_{2D}$ as follows:

Theorem 3. (Spong and Bullo (2005)). *$\chi^{\mathscr{H}^{s}_{2D}} = (\Lambda, \mathcal{I}, \{(\Phi_{\gamma}(\theta_i), \dot{\theta}_i)\}_{i\in\Lambda})$ is a hybrid flow of $\mathscr{H}^{s}_{2D}$ if $\chi^{\mathscr{H}^{\gamma}_{2D}} = (\Lambda, \mathcal{I}, \{(\theta_i, \dot{\theta}_i)\}_{i\in\Lambda})$ is a hybrid flow of $\mathscr{H}^{\gamma}_{2D}$.*

Theorem 3 implies that if $(\theta_0(\tau_0), \dot{\theta}_0(\tau_0))$ is the initial condition of $\mathscr{H}^{\gamma}_{2D}$, then $(\Phi_{\gamma}(\theta_0(\tau_0)), \dot{\theta}_0(\tau_0))$ is the initial condition of $\mathscr{H}^{s}_{2D}$. That is, if $\mathscr{H}^{\gamma}_{2D}$ walks (stably) on a slope, then $\mathscr{H}^{s}_{2D}$ walks (stably) on flat ground.

5. FUNCTIONAL ROUTHIAN REDUCTION APPLIED TO 3D BIPEDAL WALKERS

In this section we construct a control law that results in stable walking for a simple model of a 3D bipedal robotic walker. In order to achieve this goal, we shape the potential energy of this model via feedback control so that when hybrid functional Routhian reduction is carried out, the result is the 2D walker $\mathscr{H}^{s}_{2D}$ introduced in the previous section. We utilize Theorem 2 to demonstrate that this implies that the 3D walker has a walking gait on flat ground (in three dimensions). This is the main contribution of this work.

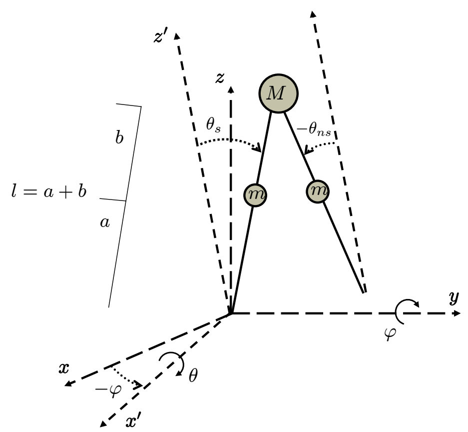

Fig. 2. Three-dimensional bipedal robot.

3D biped model. We now introduce the model describing a controlled bipedal robot walking in three-dimensions on flat ground, i.e., we will explicitly construct the controlled hybrid system describing this system:

$$\mathscr{H}_{3D} = (D_{3D}, G_{3D}, R_{3D}, f_{3D}).$$

The configuration space for the 3D biped is $Q_{3D} = \mathbb{S}^2 \times \mathbb{S}$ and the Lagrangian describing this system is given by:

Additional equations for $\mathscr{H}_{2D}$:

$$M_{2D}(\theta) = \begin{pmatrix} \frac{l^2 m}{4} & -\frac{l^2 m \cos(\theta_s - \theta_{ns})}{2} \\ -\frac{l^2 m \cos(\theta_s - \theta_{ns})}{2} & \frac{l^2 m}{4} + l^2(m+M) \end{pmatrix} \qquad S_{2D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$V_{2D}(\theta) = \frac{1}{2} gl((3m+2M)\cos(\theta_s) - m\cos(\theta_{ns})) \qquad B_{2D} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$P_{2D}(\theta) = \frac{1}{-3m - 4M + 2m\cos(2(\theta_s - \theta_{ns}))} \begin{pmatrix} 2m\cos(\theta_{ns} - \theta_s) & m - 4(m+M)\cos(2(\theta_{ns} - \theta_s)) \\ m & -2(m+2M)\cos(\theta_{ns} - \theta_s) \end{pmatrix}$$

Additional equations for $\mathscr{H}_{3D}$:

$$m_{3D}(\theta) = \frac{1}{8}(l^2(6m+4M) + l^2(m\cos(2\theta_{ns}) - 8m\cos(\theta_{ns})\cos(\theta_s) + (5m+4M)\cos(2\theta_s))$$
$$V_{3D}(\theta, \varphi) = V_{2D}(\theta)\cos(\varphi)$$
$$p_{3D}(\theta) = \frac{-m\cos(2\theta_{ns}) + 8(m+M)\cos(\theta_{ns})\cos(\theta_s) - m(2 + \cos(2\theta_s))}{6m + 4M + (5m+4M)\cos(2\theta_{ns}) - 8m\cos(\theta_{ns})\cos(\theta_s) + m\cos(2\theta_s)}$$

Table 1. Additional equations for $\mathscr{H}_{2D}$ and $\mathscr{H}_{3D}$

$$L_{3D}(\theta, \dot\theta, \varphi, \dot\varphi) = \frac{1}{2}\begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix}^T \begin{pmatrix} M_{2D}(\theta) & 0 \\ 0 & m_{3D}(\theta) \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix} - V_{3D}(\theta, \varphi), \tag{4}$$

where $m_{3D}(\theta)$ is given in the Table 1. Note that, referring to the notation introduced in Section 3, $M_\theta(\theta) = M_{2D}(\theta)$ and $M_\varphi(\theta) = m_{3D}(\theta)$. Also note that L_{3D} is nearly cyclic; it is only the potential energy that prevents its cyclicity. This will motivate the use of a control law that shapes this potential energy.

Using the controlled Euler-Lagrange equations, the dynamics for the walker are given by:

$$M_{3D}(q)\ddot q + C_{3D}(q, \dot q)\dot q + N_{3D}(q) = B_{3D} u,$$

with $q = (\theta, \varphi)$ and

$$B_{3D} = \begin{pmatrix} B_{2D} & 0 \\ 0 & 1 \end{pmatrix}.$$

These equations yield the control system: $(\dot q, \ddot q) = f_{3D}(q, \dot q, u) := f_{L_{3D}}(q, \dot q, u)$.

We construct D_{3D} and G_{3D} by applying the methods outlined in Section 2 to the unilateral constraint function

$$h_{3D}(\theta, \varphi) = h^0_{2D}(\theta) = \cos(\theta_s) - \cos(\theta_{ns}).$$

This function gives the normalized height of the foot of the walker above flat ground with the implicit assumption that $\varphi \in (-\pi/2, \pi/2)$ (which allows us to disregard the scaling factor $\cos(\varphi)$ that would have been present). The result is that h_{3D} is cyclic.

Finally, the reset map R_{3D} is given by:

$$R_{3D}(\theta, \dot\theta, \varphi, \dot\varphi) = \left(S_{2D}\theta, P_{2D}(\theta)\dot\theta, \varphi, p_{3D}(\theta)\dot\varphi\right)$$

where $p_{3D}(\theta)$ is given in Table 1. Note that this map was again computed using the methods outlined in Section 2 coupled with the condition that the stance foot is fixed.

Control law construction. We now proceed to construct a feedback control law for $\mathscr{H}_{3D}$ that makes this hybrid system an almost-cyclic unilaterally constrained Lagrangian hybrid system, $\mathscr{H}^\alpha_{3D}$. We will then demonstrate, using Theorem 2, that $\mathscr{H}^\alpha_{3D}$ has a walking gait by relating it to $\mathscr{H}^s_{2D}$.

Define the feedback control law parameterized by $\alpha \in \mathbb{R}$:

$$\begin{aligned} u &= K^\alpha_{3D}(q) \\ &= B^{-1}_{3D}\frac{\partial}{\partial q}\left(V_{3D}(q) - V^\gamma_{2D}(\theta) + \frac{1}{2}\frac{\alpha^2\varphi^2}{m_{3D}(\theta)}\right) \end{aligned}$$

Applying this control law to the control system $(\dot q, \ddot q) = f_{3D}(q, \dot q, u)$ yields the dynamical system:

$$(\dot q, \ddot q) = f^\alpha_{3D}(q, \dot q) := f_{3D}(q, \dot q, K^\alpha_{3D}(q)),$$

which is just the vector field associated to the almost-cyclic Lagrangian

$$L^\alpha_{3D}(\theta, \dot\theta, \varphi, \dot\varphi) = \frac{1}{2}\begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix}^T \begin{pmatrix} M_{2D}(\theta) & 0 \\ 0 & m_{3D}(\theta) \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix} - V^\alpha_{3D}(\theta), \tag{5}$$

where

$$V^\alpha_{3D}(\theta) = V^\gamma_{2D}(\theta) - \frac{1}{2}\frac{\alpha^2\varphi^2}{m_{3D}(\theta)}.$$

That is, $f^\alpha_{3D} = f_{L^\alpha_{3D}}$.

Let $\mathscr{H}^\alpha_{3D} := (D_{3D}, G_{3D}, R_{3D}, f^\alpha_{3D})$, which is a unilaterally constrained Lagrangian hybrid system.

Applying hybrid functional Routhian reduction. Using the methods outlined in Section 3, there is a momentum map $J_{3D} : TQ_{3D} \to \mathbb{R}$ given by:

$$J_{3D}(\theta, \dot\theta, \varphi, \dot\varphi) = m_{3D}(\theta)\dot\varphi.$$

Setting $J_{3D}(\theta, \dot\theta, \varphi, \dot\varphi) = \lambda(\varphi) = -\alpha\varphi$ implies that

$$\dot\varphi = -\frac{\alpha\varphi}{m_{3D}(\theta)}.$$

The importance of $\mathscr{H}^\alpha_{3D}$ is illustrated by:

Theorem 4. *$\mathscr{H}^{\alpha}_{3D}$ is an almost-cyclic unilaterally constrained Lagrangian hybrid system. Moreover, the following diagram commutes:*

$$
\begin{array}{ccccc}
 & & \mathbb{R}^k & & \\
 & \nearrow^{J_{3D}|_{G_{3D}}} & & \nwarrow^{J_{3D}|_{D_{3D}}} & \\
G_{3D} & & \xrightarrow{R_{3D}} & & D_{3D} \\
\downarrow \pi & & & & \downarrow \pi \\
G_{2D} & & \xrightarrow{R_{2D}} & & D_{2D}
\end{array}
$$

Therefore, $\mathscr{H}^{s}_{2D}$ is the functional Routhian hybrid system associated with $\mathscr{H}^{\alpha}_{3D}$.

This result allows us to prove—using Theorem 2—that the control law used to construct $\mathscr{H}^{\alpha}_{3D}$ in fact results in walking in three-dimensions.

Theorem 5. *$\chi^{\mathscr{H}^{\alpha}_{3D}} = (\Lambda, \mathcal{I}, \{(\theta_i, \dot{\theta}_i, \varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda})$ is a hybrid flow of $\mathscr{H}^{\alpha}_{3D}$ with*

$$\dot{\varphi}_0(\tau_0) = -\frac{\alpha\varphi_0(\tau_0)}{m_{3D}(\theta_0(\tau_0))}, \qquad (6)$$

if and only if $\chi^{\mathscr{H}^{s}_{2D}} = (\Lambda, \mathcal{I}, \{\theta_i, \dot{\theta}_i\}_{i\in\Lambda})$ is a hybrid flow of $\mathscr{H}^{s}_{2D}$ and $\{(\varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda}$ satisfies:

$$\dot{\varphi}_i(t) = -\frac{\alpha\varphi_i(t)}{m_{3D}(\theta_i(t))}, \qquad (7)$$
$$\varphi_{i+1}(\tau_{i+1}) = \varphi_i(\tau_{i+1}).$$

To better understand the implications of Theorem 5, suppose that $\chi^{\mathscr{H}^{\alpha}_{3D}} = (\Lambda, \mathcal{I}, \{(\theta_i, \dot{\theta}_i, \varphi_i, \dot{\varphi}_i)\}_{i\in\Lambda})$ is a hybrid flow of $\mathscr{H}^{\alpha}_{3D}$. If this hybrid flow has an initial condition satisfying (6) with $\alpha > 0$ and the corresponding hybrid flow, $\chi^{\mathscr{H}^{s}_{2D}} = (\Lambda, \mathcal{I}, \{\theta_i, \dot{\theta}_i\}_{i\in\Lambda})$, of $\mathscr{H}^{s}_{2D}$ is a walking gait in 2D:

$$\Lambda = \mathbb{N}, \quad \lim_{i\to\infty} \tau_i = \infty, \quad \theta_i(\tau_i) = \theta_{i+1}(\tau_{i+1}),$$

then the result is walking in three-dimensions. This follows from the fact that θ and $\dot{\theta}$ will have the same behavior over time for the full-order system—the bipedal robot will walk. Moreover, since Theorem 5 implies that (7) holds, the walker stabilizes to the "upright" position. This is because the roll, φ, will tend to zero as time goes to infinity since (7) essentially defines a stable linear system $\dot{\varphi} = -\alpha\varphi$ ($m_{3D}(\theta_i(t)) > 0$ and $\alpha > 0$), which controls the behavior of φ when (6) is satisfied. This convergence can be seen in Fig. 3 along with a walking gait of the 3D walker.

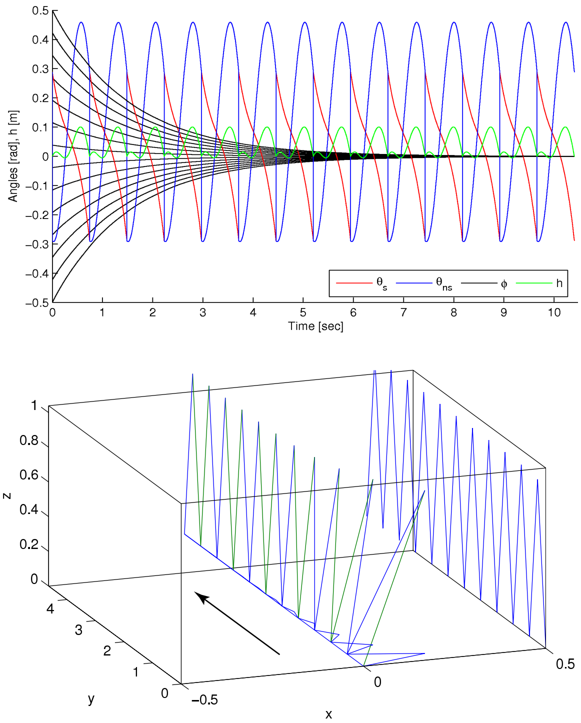

Fig. 3. θ_{ns}, θ_s and φ over time for different initial values of φ (top). A walking gait for the three-dimensional bipedal robot (bottom).

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