Comparing Impact Equations for Conservation of Angular Momentum to Perfectly Plastic Impacts: Application to a 2-D Point-footed Walker.

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Conservation of Angular Momentum

■ From Goswami et al:

```
Qbefore = {{-m a b, -m a b + (M L^2 + 2 m a L) Cos[θns[t] - θs[t]]}, {0, - m a b}};
Qafter = {{m b (b - L Cos[θns[t] - θs[t]]), m L (L - b Cos[θns[t] - θs[t]]) + m a^2 + M L^2},
   {m b^2, -m b L Cos[θns[t] - θs[t]]}};
QafterInv = Inverse[Qafter];
MatrixForm[H = FullSimplify[QafterInv.Qbefore]]
```

$$\begin{pmatrix} -\frac{a\,L\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]}{a^2\,m+L^2\,(m+M)-L^2\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2} & \frac{-a\,(a^2\,m+L^2\,(m+M))+L^2\,(2\,a\,m+L\,M)\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2}{b\,(a^2\,m+L^2\,(m+M)-L^2\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2)} \\ -\frac{a\,b\,m}{a^2\,m+L^2\,(m+M)-L^2\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2} & \frac{2\,L\,(a\,m+L\,M)\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]}{2\,a^2\,m+L^2\,(m+2\,M)-L^2\,m\,\text{Cos}[2\,(\theta\text{ns}[t]-\theta\text{s}[t])]} \end{pmatrix}$$

```
FullSimplify[H.{θns'[t], θs'[t]}]
```

$$\left\{-\frac{a\,b\,L\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{ns}'[t]+\left(a^3\,m+a\,L^2\,(m+M)-L^2\,(2\,a\,m+L\,M)\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2\right)\theta\text{s}'[t]}{b\,(a^2\,m+L^2\,(m+M)-L^2\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2)},\ \frac{2\,(-a\,b\,m\,\theta\text{ns}'[t]+L\,(a\,m+L\,M)\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{s}'[t])}{2\,a^2\,m+L^2\,(m+2\,M)-L^2\,m\,\text{Cos}[2\,(\theta\text{ns}[t]-\theta\text{s}[t])]}\right\}$$

```
angconsimpact = % /. {L → a + b} // FullSimplify
```

$$\left\{-\frac{(a\,b\,(a+b)\,m\,\text{Cos}[\theta ns[t]-\theta s[t]]\,\theta ns'[t]+(a^3\,m+a\,(a+b)^2\,(m+M)-(a+b)^2\,(2\,a\,m+(a+b)\,M)\,\text{Cos}[\theta ns[t]-\theta s[t]]^2)\,\theta s'[t])}{(b\,(a^2\,m+(a+b)^2\,(m+M)-(a+b)^2\,m\,\text{Cos}[\theta ns[t]-\theta s[t]]^2))},\ \frac{2\,(-a\,b\,m\,\theta ns'[t]+(a+b)\,(b\,M+a\,(m+M))\,\text{Cos}[\theta ns[t]-\theta s[t]]\,\theta s'[t])}{2\,a^2\,m+(a+b)^2\,(m+2\,M)-(a+b)^2\,m\,\text{Cos}[2\,(\theta ns[t]-\theta s[t])]}\right\}$$

Perfectly Plastic Impacts

Method obtained from Bullo et al:

■ Generalized M matrix

```
MatrixForm[
 Mmatrix = {{b^2 m, -b (a + b) m Cos[θns[t] - θs[t]], -b m Cos[θns[t]], 0, b m Sin[θns[t]]},
   {-b (a + b) m Cos[θns[t] - θs[t]], (2 a^2 + 2 a b + b^2) m + (a + b)^2 M,
    (b (m + M) + a (2 m + M)) Cos[θs[t]], 0, (-b (m + M) - a (2 m + M)) Sin[θs[t]]},
   {-b m Cos[θns[t]], (b (m + M) + a (2 m + M)) Cos[θs[t]], 2 m + M, 0, 0}, {0, 0, 0, 2 m + M, 0},
   {b m Sin[θns[t]], (-b (m + M) - a (2 m + M)) Sin[θs[t]], 0, 0, 2 m + M}}]
```

$$\begin{pmatrix} b^2\,m & -b\,(a+b)\,m\,\text{Cos}[\theta ns[t]-\theta s[t]] & -b\,m\,\text{Cos}[\theta ns[t]] \\ -b\,(a+b)\,m\,\text{Cos}[\theta ns[t]-\theta s[t]] & (2\,a^2+2\,a\,b+b^2)\,m+(a+b)^2\,M & (b\,(m+M)+a\,(2\,m+M))\,\text{Co} \\ -b\,m\,\text{Cos}[\theta ns[t]] & (b\,(m+M)+a\,(2\,m+M))\,\text{Cos}[\theta s[t]] & 2\,m+M \\ 0 & 0 & 0 \\ b\,m\,\text{Sin}[\theta ns[t]] & (-b\,(m+M)-a\,(2\,m+M))\,\text{Sin}[\theta s[t]] & 0 \end{pmatrix}$$

```
l = a + b;
Pos1 = l Sin[θs[t]] - l Sin[θns[t]] + x[t];
Pos2 = y[t];
Pos3 = l Cos[θs[t]] - l Cos[θns[t]] + z[t];
MatrixForm[F = FullSimplify[
   {{D[Pos1, θns[t]], D[Pos1, θs[t]], D[Pos1, x[t]], D[Pos1, y[t]], D[Pos1, z[t]]},
    {D[Pos2, θns[t]], D[Pos2, θs[t]], D[Pos2, x[t]], D[Pos2, y[t]], D[Pos2, z[t]]},
    {D[Pos3, θns[t]], D[Pos3, θs[t]], D[Pos3, x[t]], D[Pos3, y[t]], D[Pos3, z[t]]}}]]
```

$$\begin{pmatrix} -(a+b)\,\text{Cos}[\theta ns[t]] & (a+b)\,\text{Cos}[\theta s[t]] & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ (a+b)\,\text{Sin}[\theta ns[t]] & -(a+b)\,\text{Sin}[\theta s[t]] & 0 & 0 & 1 \end{pmatrix}$$

```
MatrixForm[nFT = Join[-Transpose[F], {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}]]
```

$$\begin{pmatrix} (a+b)\,\text{Cos}[\theta\text{ns}[t]] & 0 & -(a+b)\,\text{Sin}[\theta\text{ns}[t]] \\ -(a+b)\,\text{Cos}[\theta\text{s}[t]] & 0 & (a+b)\,\text{Sin}[\theta\text{s}[t]] \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
MatrixForm[FullMatrix = Transpose[Join[Transpose[Join[Mmatrix, F]], Transpose[nFT]]]]
```

$$\begin{pmatrix} b^2\,m & -b\,(a+b)\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]] & -b\,m\,\text{Cos}[\theta\text{ns}[t]] \\ -b\,(a+b)\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]] & (2\,a^2+2\,a\,b+b^2)\,m+(a+b)^2\,M & (b\,(m+M)+a\,(2\,m+M))\,\text{Co} \\ -b\,m\,\text{Cos}[\theta\text{ns}[t]] & (b\,(m+M)+a\,(2\,m+M))\,\text{Cos}[\theta\text{s}[t]] & 2\,m+M \\ 0 & 0 & 0 \\ b\,m\,\text{Sin}[\theta\text{ns}[t]] & (-b\,(m+M)-a\,(2\,m+M))\,\text{Sin}[\theta\text{s}[t]] & 0 \\ -(a+b)\,\text{Cos}[\theta\text{ns}[t]] & (a+b)\,\text{Cos}[\theta\text{s}[t]] & 1 \\ 0 & 0 & 0 \\ (a+b)\,\text{Sin}[\theta\text{ns}[t]] & -(a+b)\,\text{Sin}[\theta\text{s}[t]] & 0 \end{pmatrix}$$

```
MatrixForm[DMatrix = Join[Mmatrix.{θns'[t], θs'[t], 0, 0, 0}, {0, 0, 0}]]
```

$$\begin{pmatrix} b^2\,m\,\theta\text{ns}'[t]-b\,(a+b)\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{s}'[t] \\ -b\,(a+b)\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{ns}'[t]+((2\,a^2+2\,a\,b+b^2)\,m+(a+b)^2\,M)\,\theta\text{s}'[t] \\ -b\,m\,\text{Cos}[\theta\text{ns}[t]]\,\theta\text{ns}'[t]+(b\,(m+M)+a\,(2\,m+M))\,\text{Cos}[\theta\text{s}[t]]\,\theta\text{s}'[t] \\ 0 \\ b\,m\,\text{Sin}[\theta\text{ns}[t]]\,\theta\text{ns}'[t]+(-b\,(m+M)-a\,(2\,m+M))\,\text{Sin}[\theta\text{s}[t]]\,\theta\text{s}'[t] \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
K = Inverse[FullMatrix].DMatrix;

plasticimpact = {FullSimplify[K[[2]]], FullSimplify[K[[1]]]}
```

$$\left\{-\frac{2\,a\,b\,(a+b)\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{ns}'[t]+(2\,a^3\,m+(a-b)\,(a+b)^2\,M-(a+b)^2\,(2\,a\,m+(a+b)\,M)\,\text{Cos}[2\,(\theta\text{ns}[t]-\theta\text{s}[t])])\,\theta\text{s}'[t]}{(b\,((3\,a^2+2\,a\,b+b^2)\,m+2\,(a+b)^2\,M-(a+b)^2\,m\,\text{Cos}[2\,(\theta\text{ns}[t]-\theta\text{s}[t])]))},\ \frac{2\,(-a\,b\,m\,\theta\text{ns}'[t]+(a+b)\,(b\,M+a\,(m+M))\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{s}'[t])}{(3\,a^2+2\,a\,b+b^2)\,m+2\,(a+b)^2\,M-(a+b)^2\,m\,\text{Cos}[2\,(\theta\text{ns}[t]-\theta\text{s}[t])]}\right\}$$

■ Compared to the angular conservation impact equations...

```
angconsimpact
```

$$\left\{-\frac{a\,b\,(a+b)\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{ns}'[t]+(a^3\,m+a\,(a+b)^2\,(m+M)-(a+b)^2\,(2\,a\,m+(a+b)\,M)\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2)\,\theta\text{s}'[t]}{(b\,(a^2\,m+(a+b)^2\,(m+M)-(a+b)^2\,m\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]^2))},\ \frac{2\,(-a\,b\,m\,\theta\text{ns}'[t]+(a+b)\,(b\,M+a\,(m+M))\,\text{Cos}[\theta\text{ns}[t]-\theta\text{s}[t]]\,\theta\text{s}'[t])}{2\,a^2\,m+(a+b)^2\,(m+2\,M)-(a+b)^2\,m\,\text{Cos}[2\,(\theta\text{ns}[t]-\theta\text{s}[t])]}\right\}$$

■ The two methods ARE equivalent:

```
plasticimpact == angconsimpact // FullSimplify
```

```
True
```