Outline

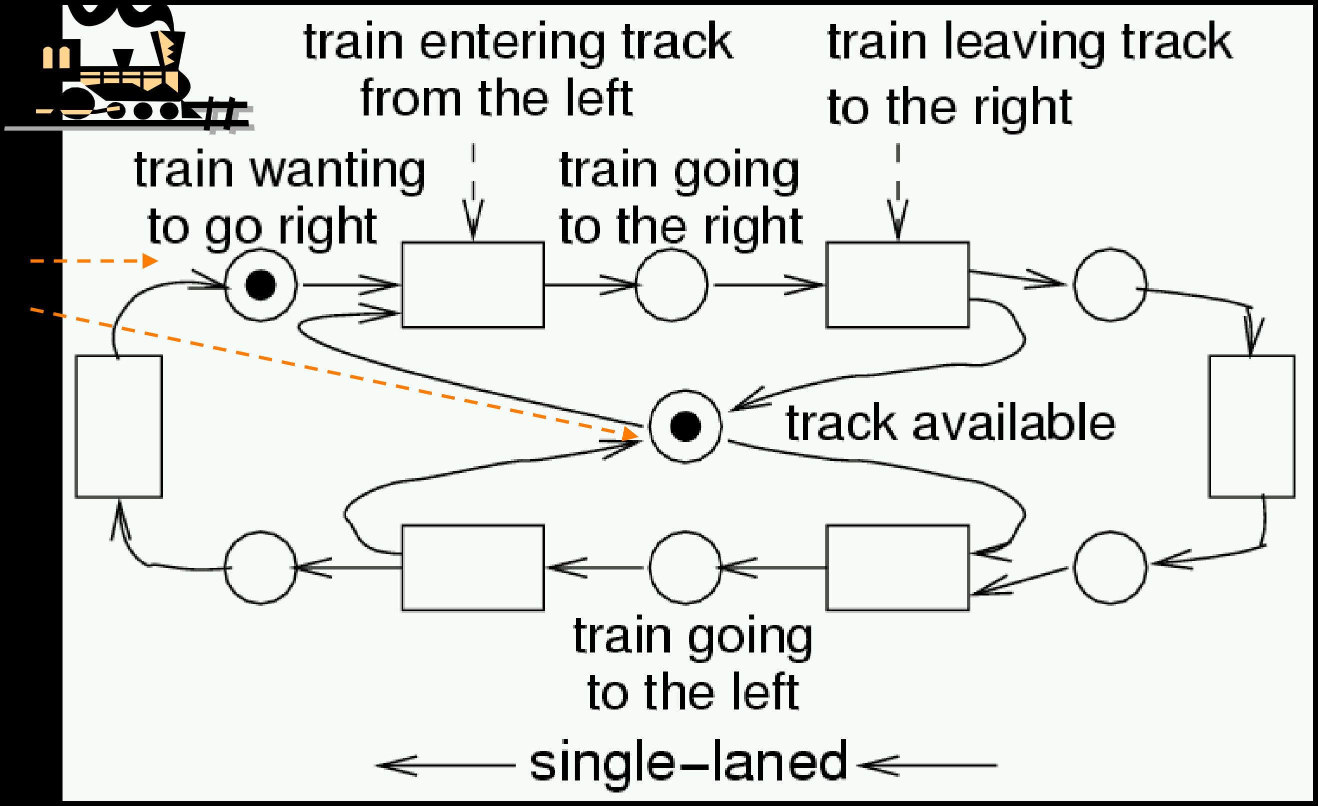
- **Petri nets**
  - **Introduction**
  - **Examples**
  - **Properties**
  - **Analysis techniques**

Petri Nets (PNs)

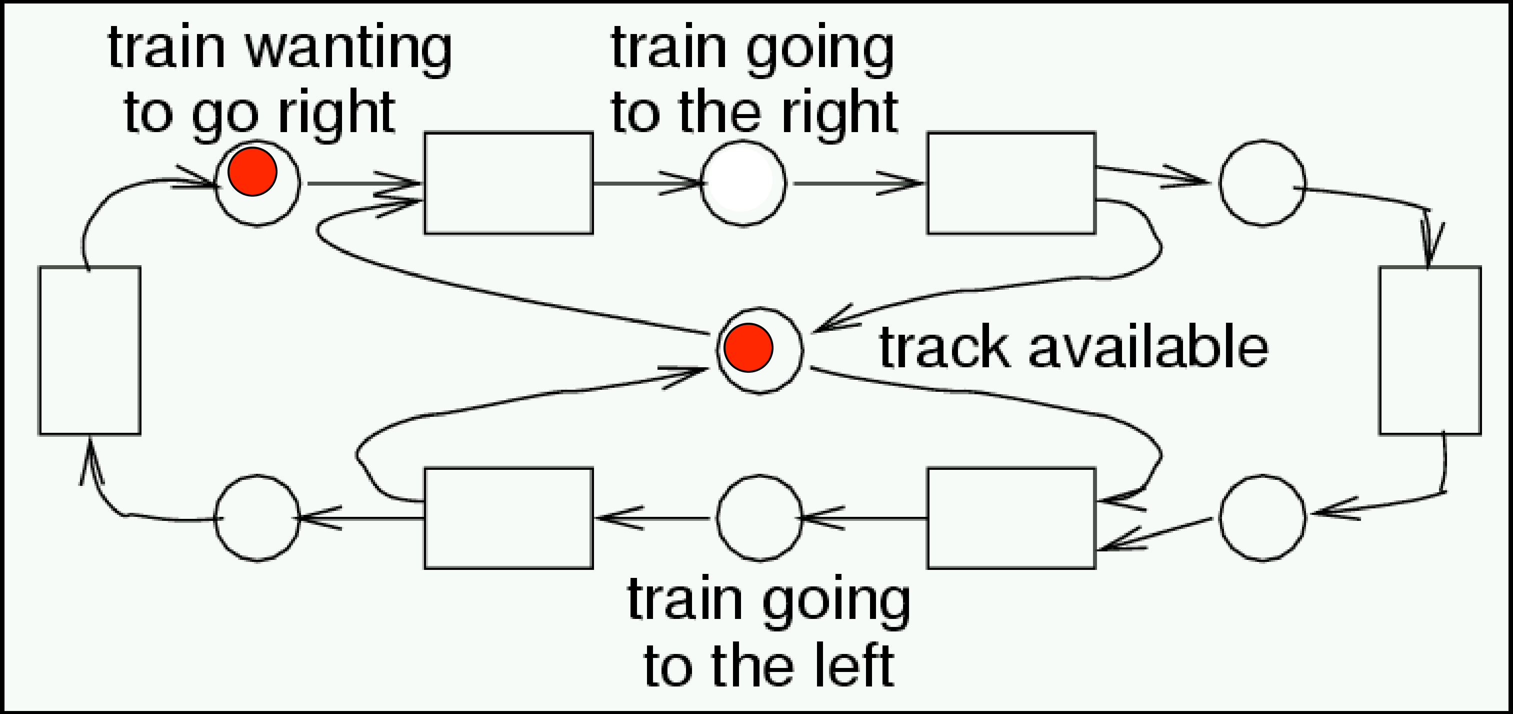
- **Model introduced by C.A. Petri in 1962**
  - **Ph.D. Thesis: “Communication with Automata”**
- **Applications: distributed computing, manufacturing, control, communication networks, transportation…**
- **PNs describe explicitly and graphically:**
  - **sequencing/causality**
  - **conflict/non-deterministic choice**
  - **concurrency**
- **Basic PN model**
  - **Asynchronous model (partial ordering)**
  - **Main drawback: no hierarchy**

Example:
Synchronization at single track rail segment

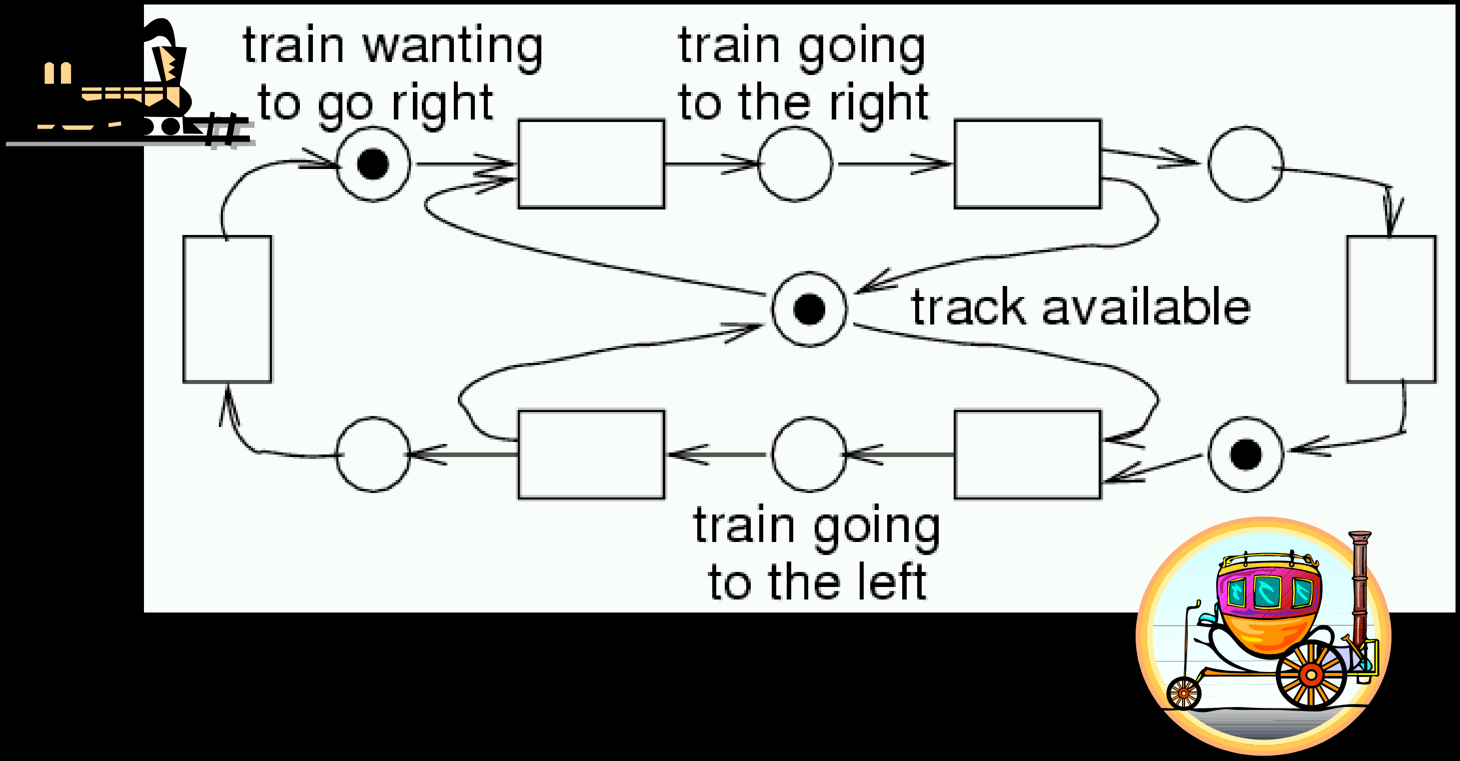
- „Preconditions“



Playing the „token game“

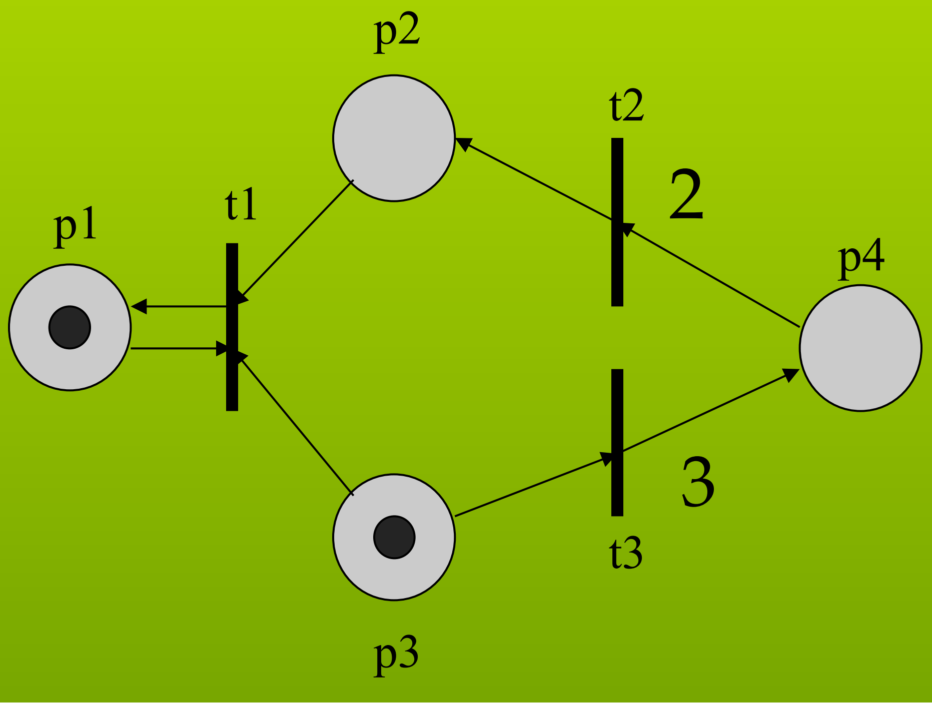


Conflict for resource „track“



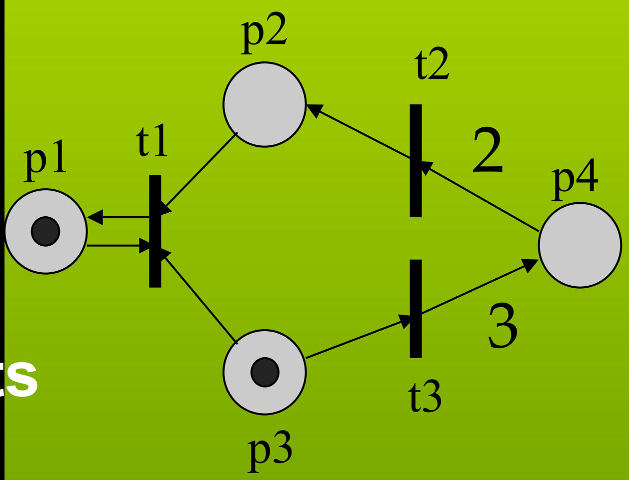
Petri Net Graph

- **Bipartite weighted directed graph:**
  - **Places: circles**
  - **Transitions: bars or boxes**
  - **Arcs: arrows labeled with weights**
- **Tokens: black dots**



Petri Net

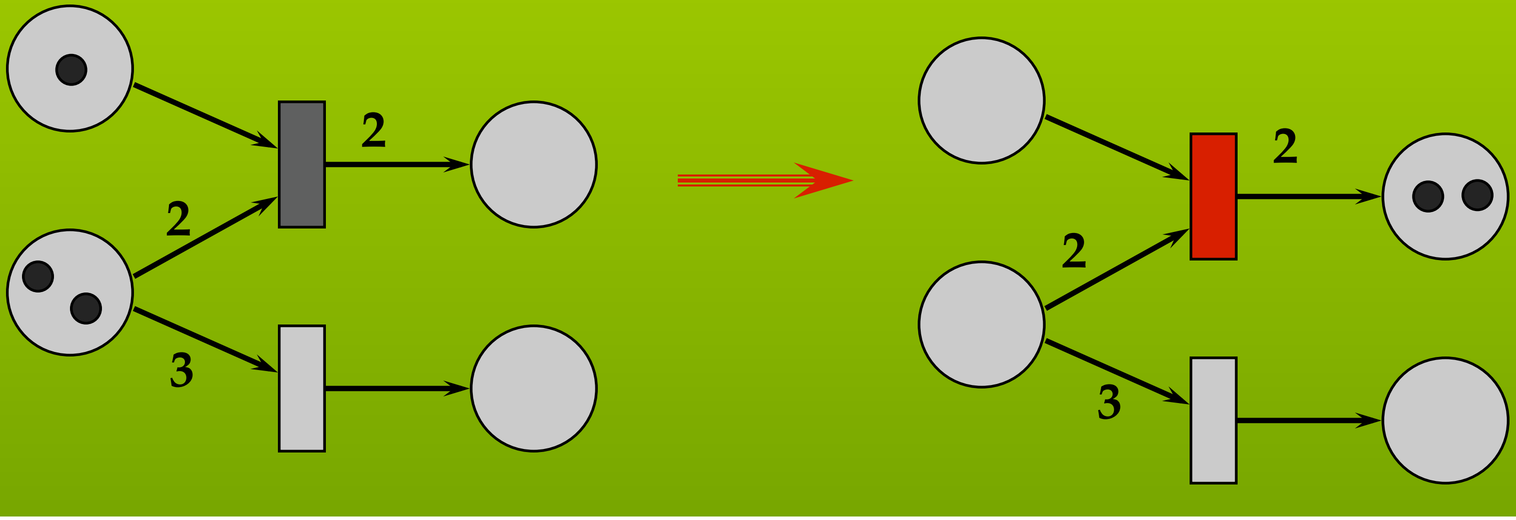
- A PN (N,M_0) is a Petri Net Graph N
  - **places**: represent distributed state by holding tokens
    - marking (state) M is an n-vector (m_1,m_2,m_3…), where m_i is the non-negative number of tokens in place p_i.
    - initial marking (M_0) is initial state
  - **transitions**: represent actions/events
    - enabled transition: enough tokens in predecessors
    - firing transition: modifies marking
- …and an initial marking M_0.



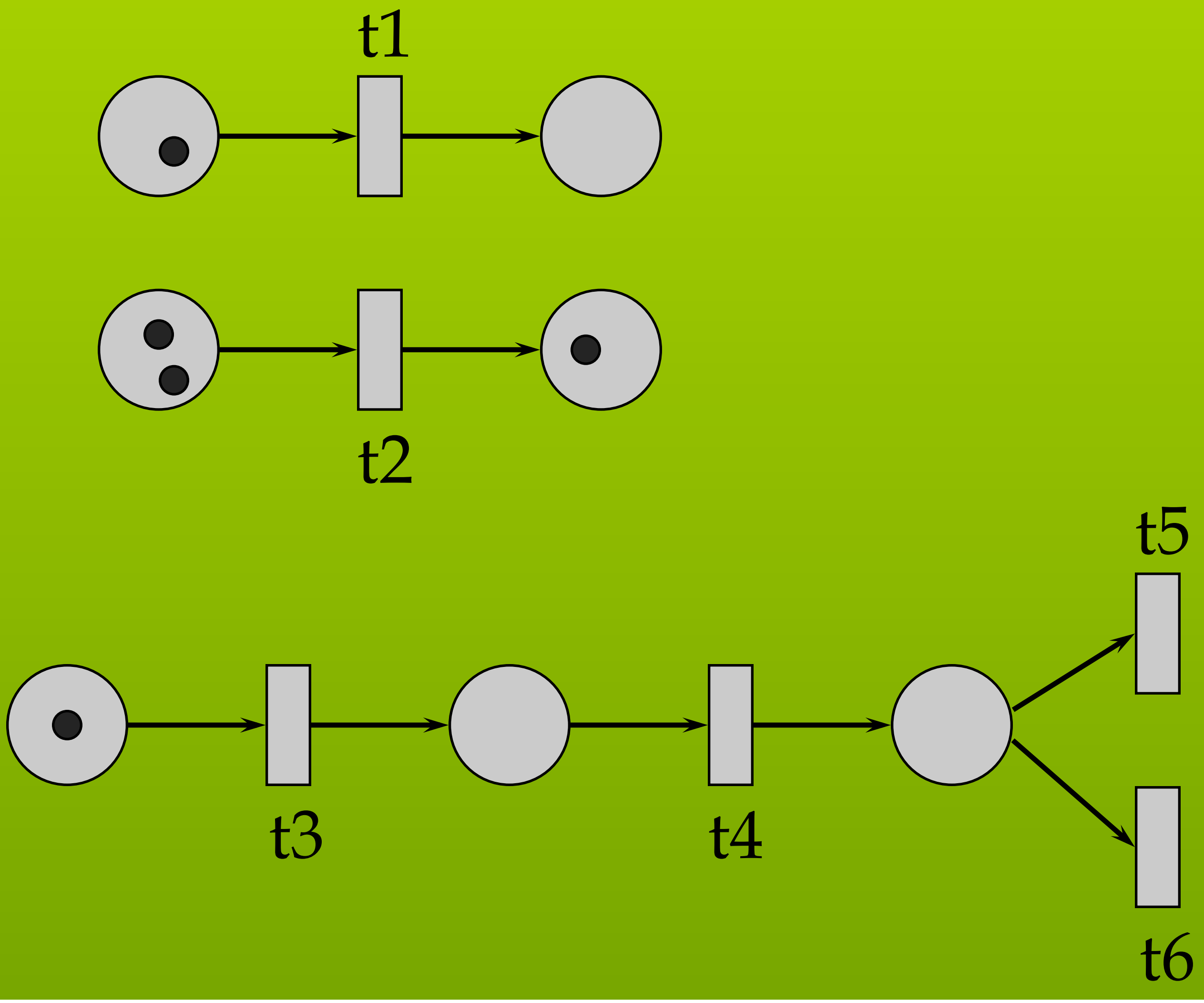
Places/Transitions: conditions/events

Transition firing rule

- A marking is changed according to the following rules:
  - A transition is enabled if there are enough tokens in each input place
  - An enabled transition may or may not fire
  - The firing of a transition modifies marking by consuming tokens from the input places and producing tokens in the output places



Concurrency, causality, choice



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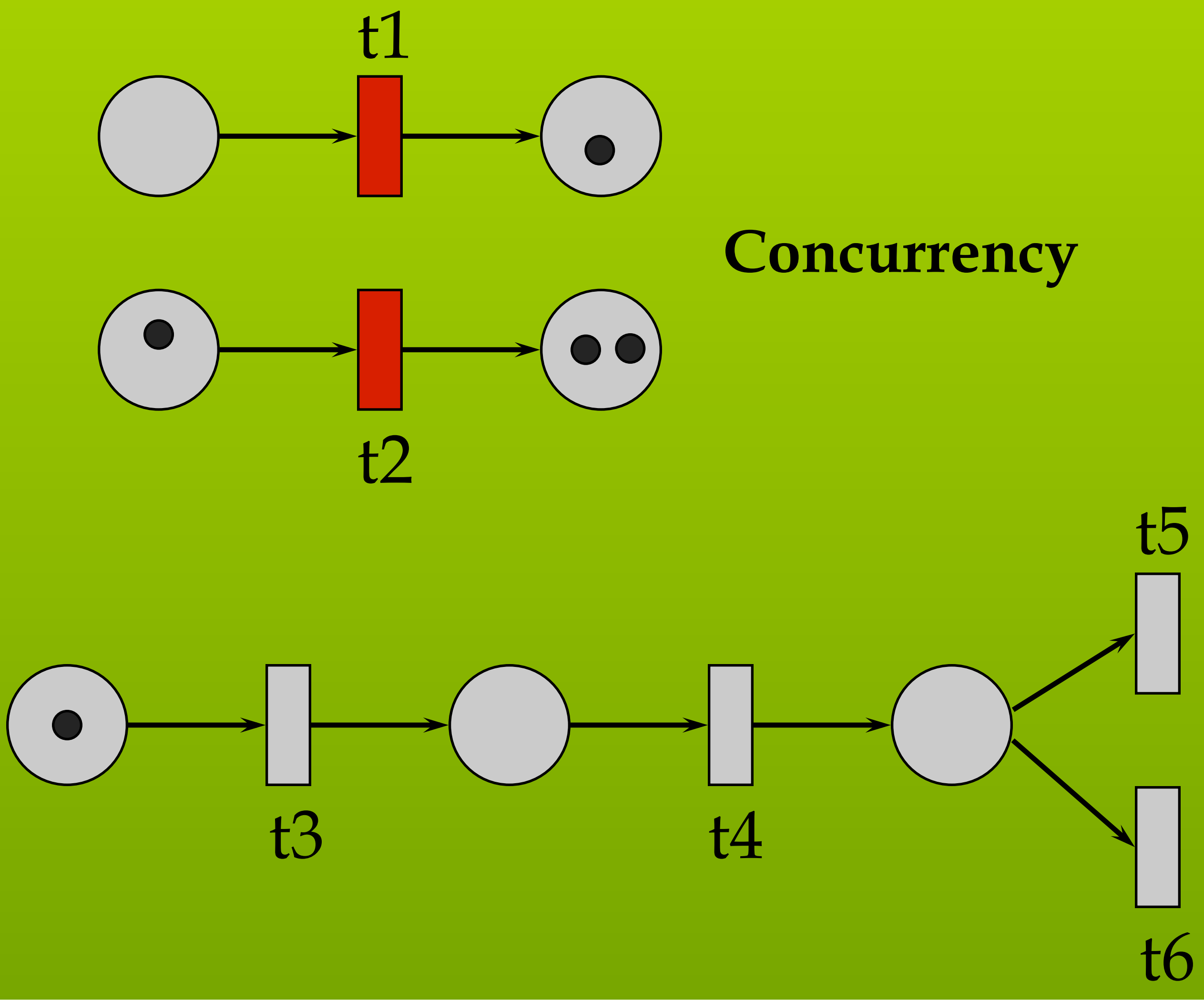
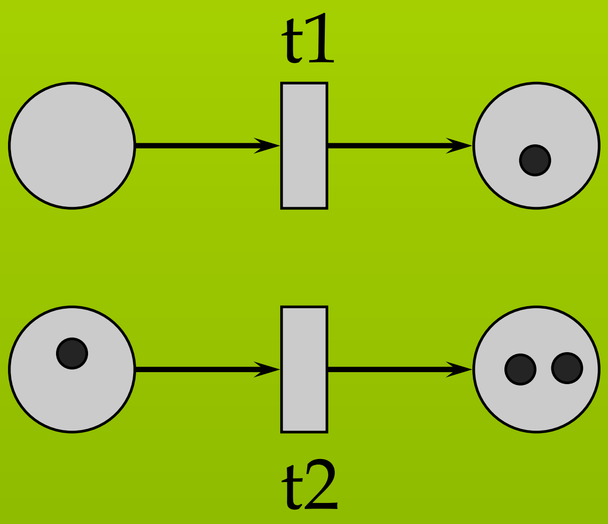
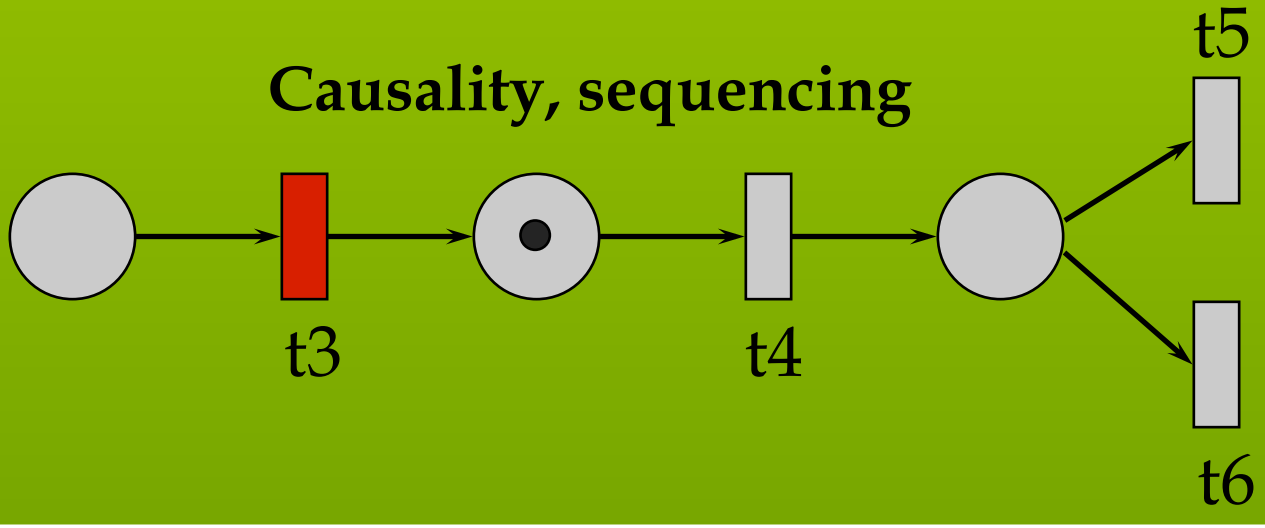


Concurrency

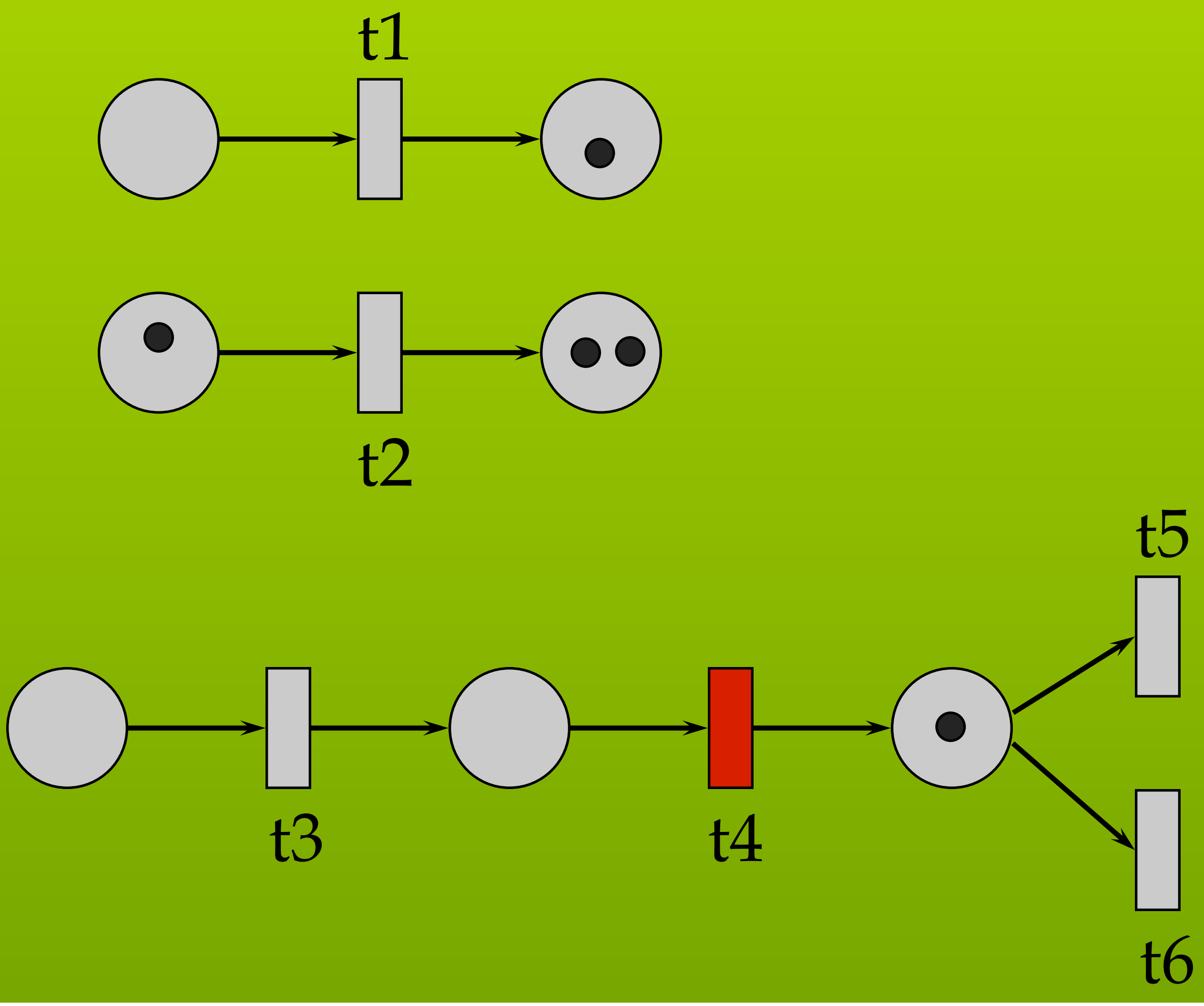
Concurrency, causality, choice



Causality, sequencing

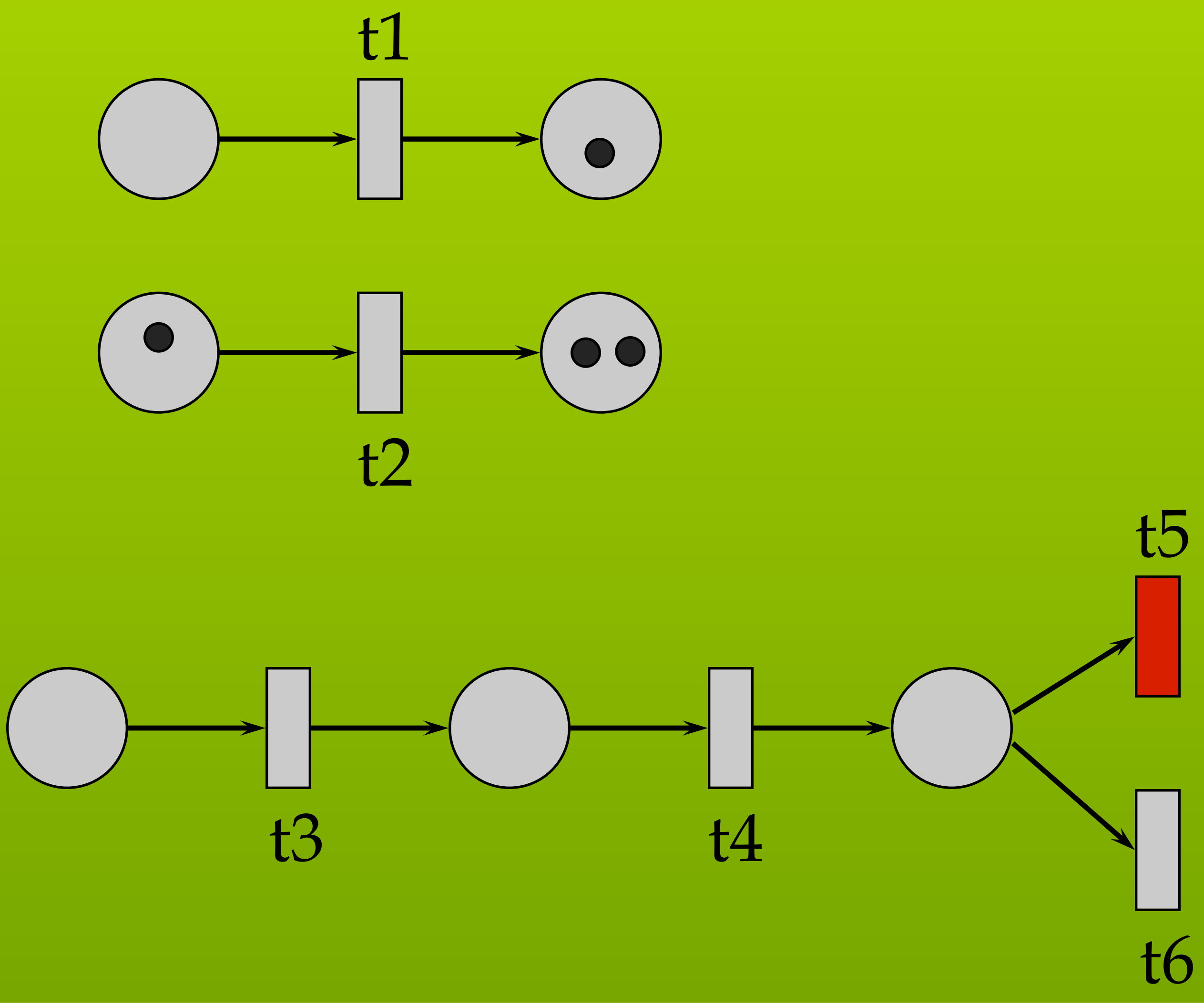


Concurrency, causality, choice



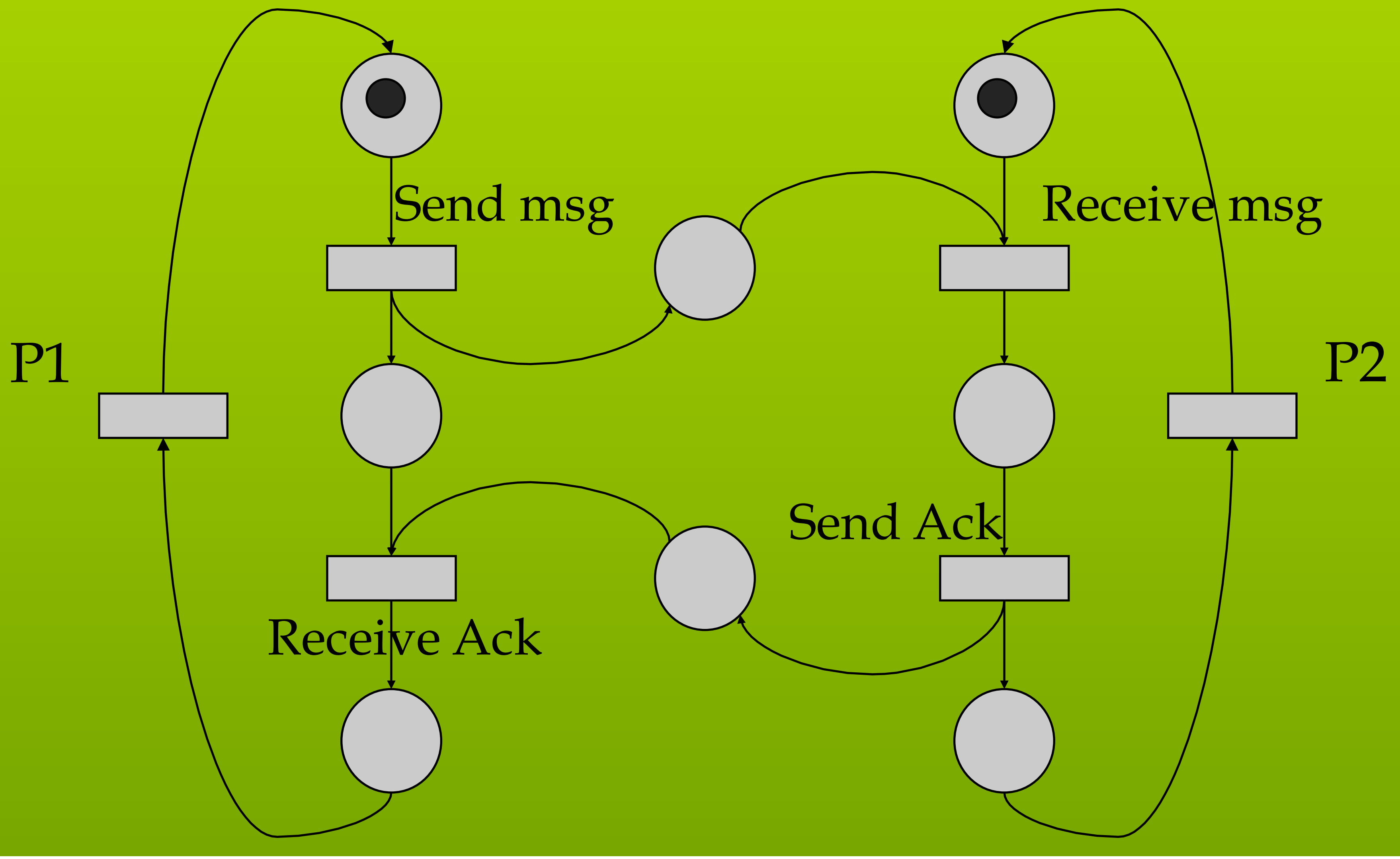
Choice, conflict

Concurrency, causality, choice

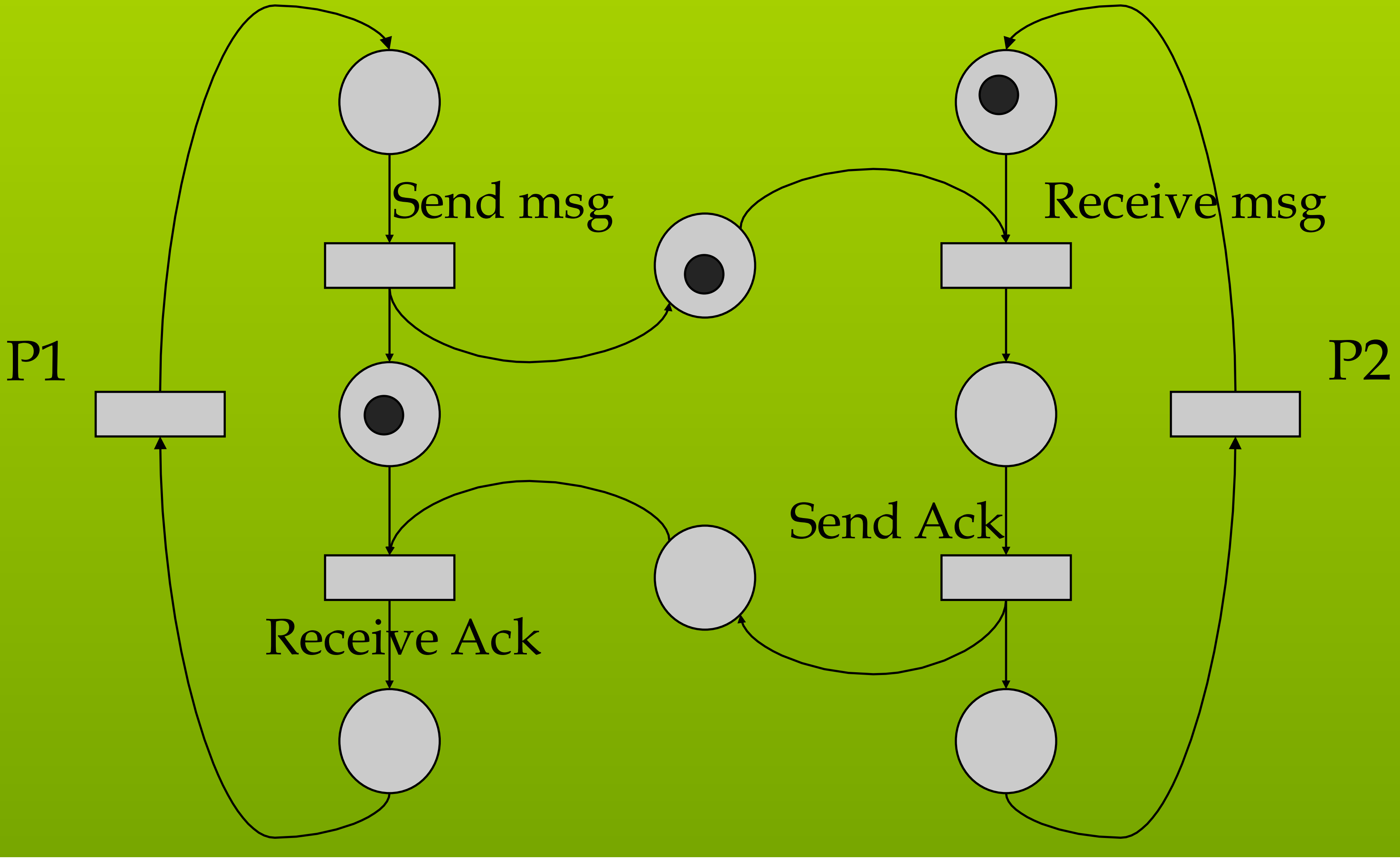


Choice, conflict

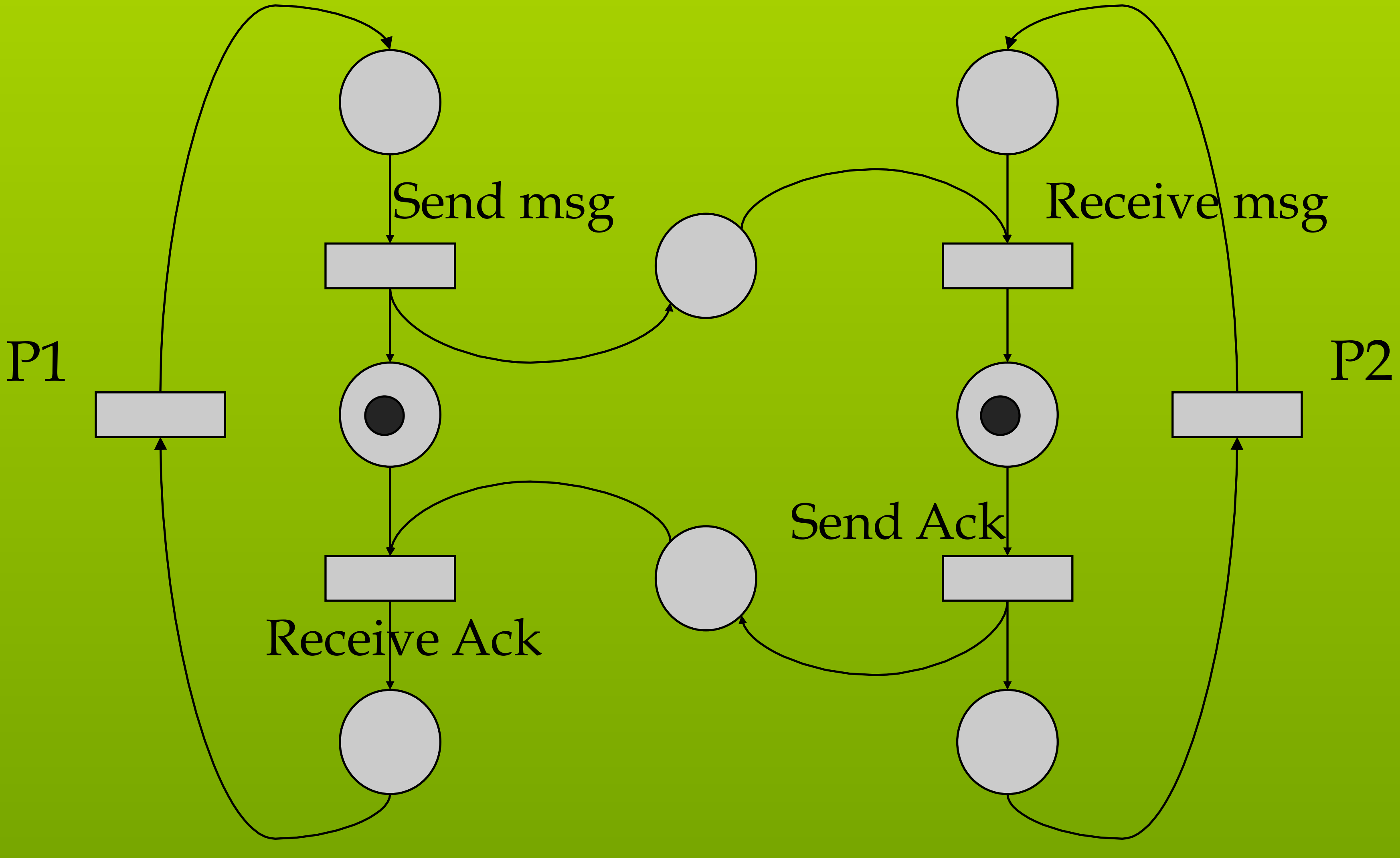
Communication Protocol



Communication Protocol



Communication Protocol



Send msg

Receive msg

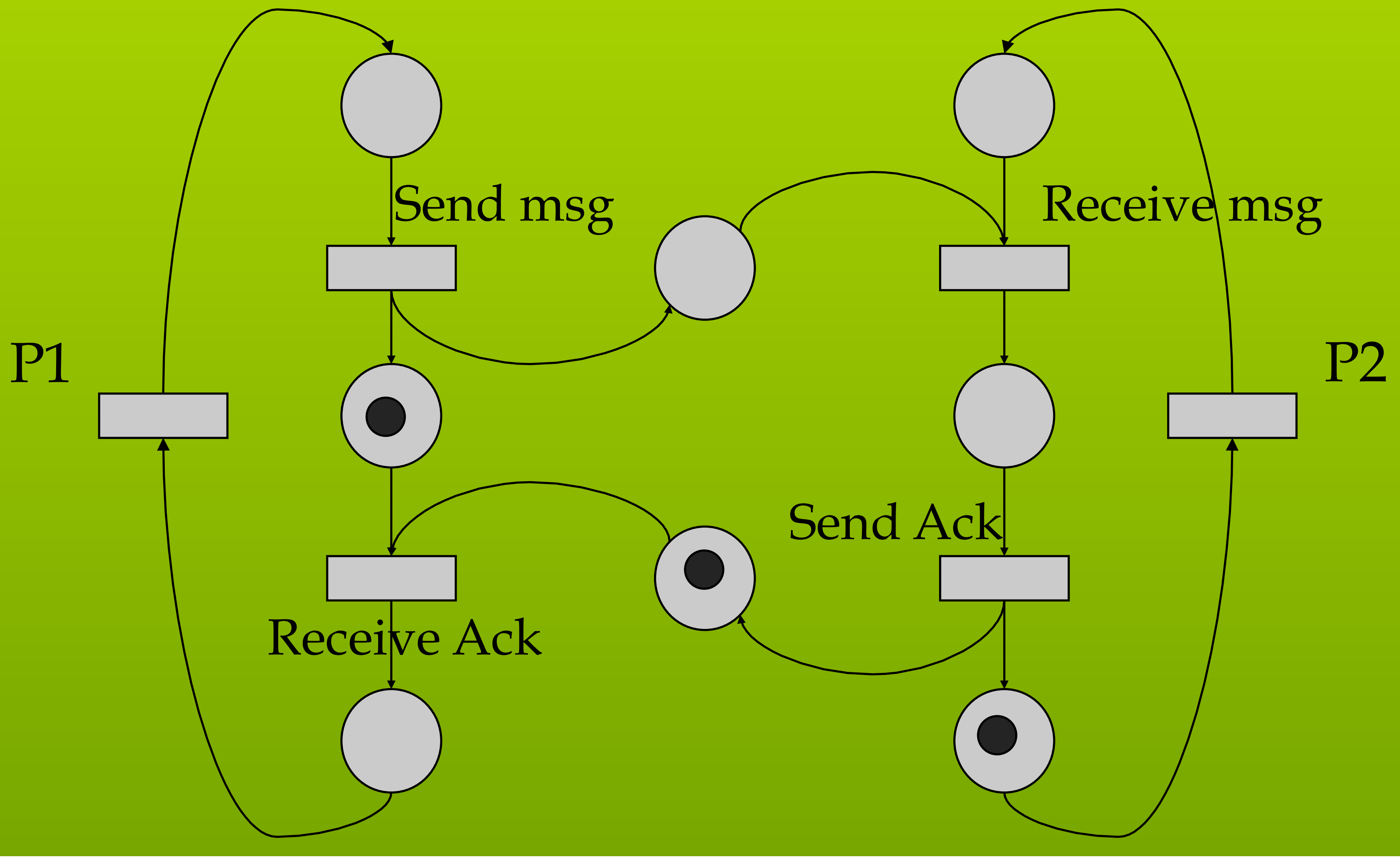
P1

P2

Send Ack

Receive Ack

Communication Protocol



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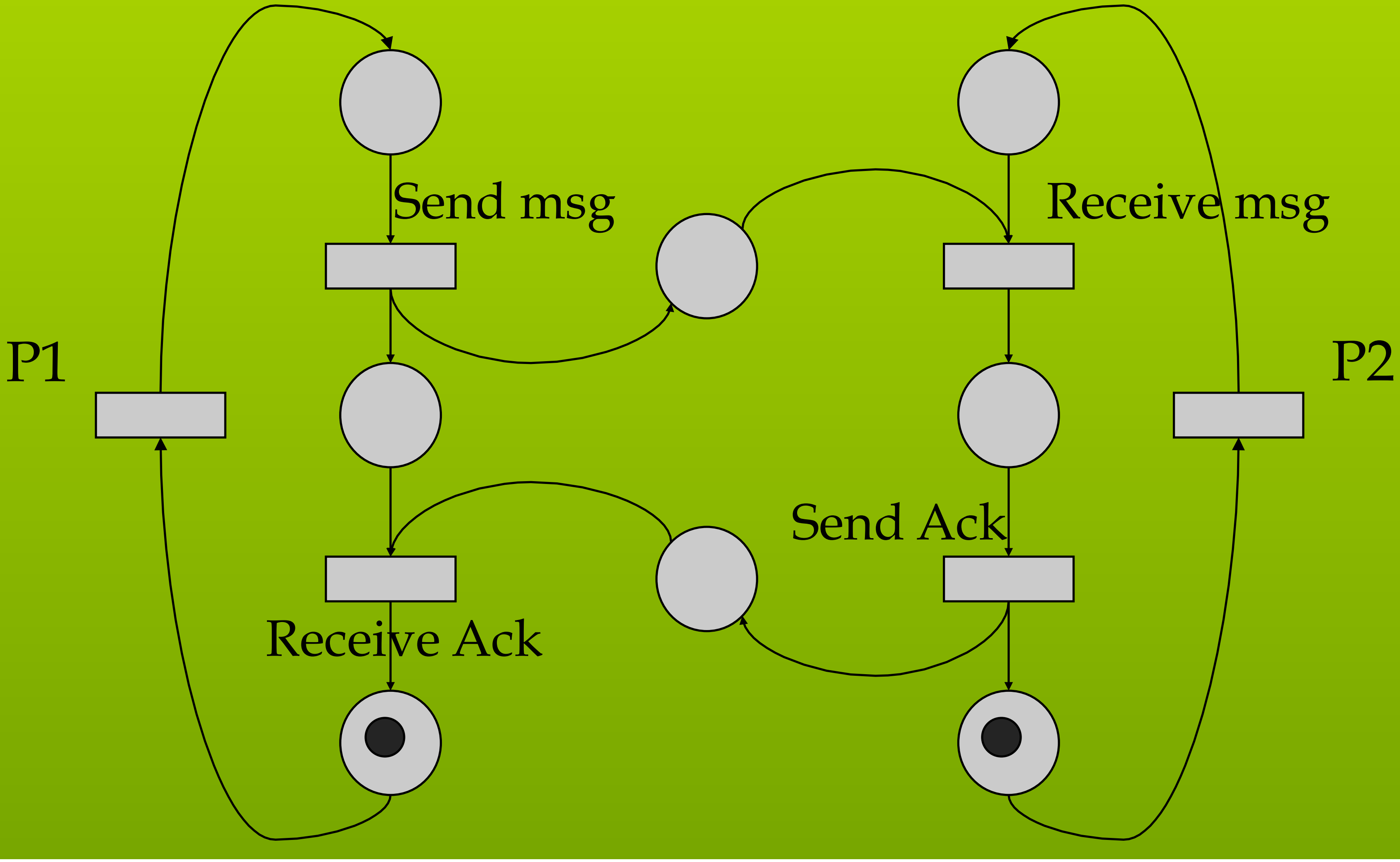

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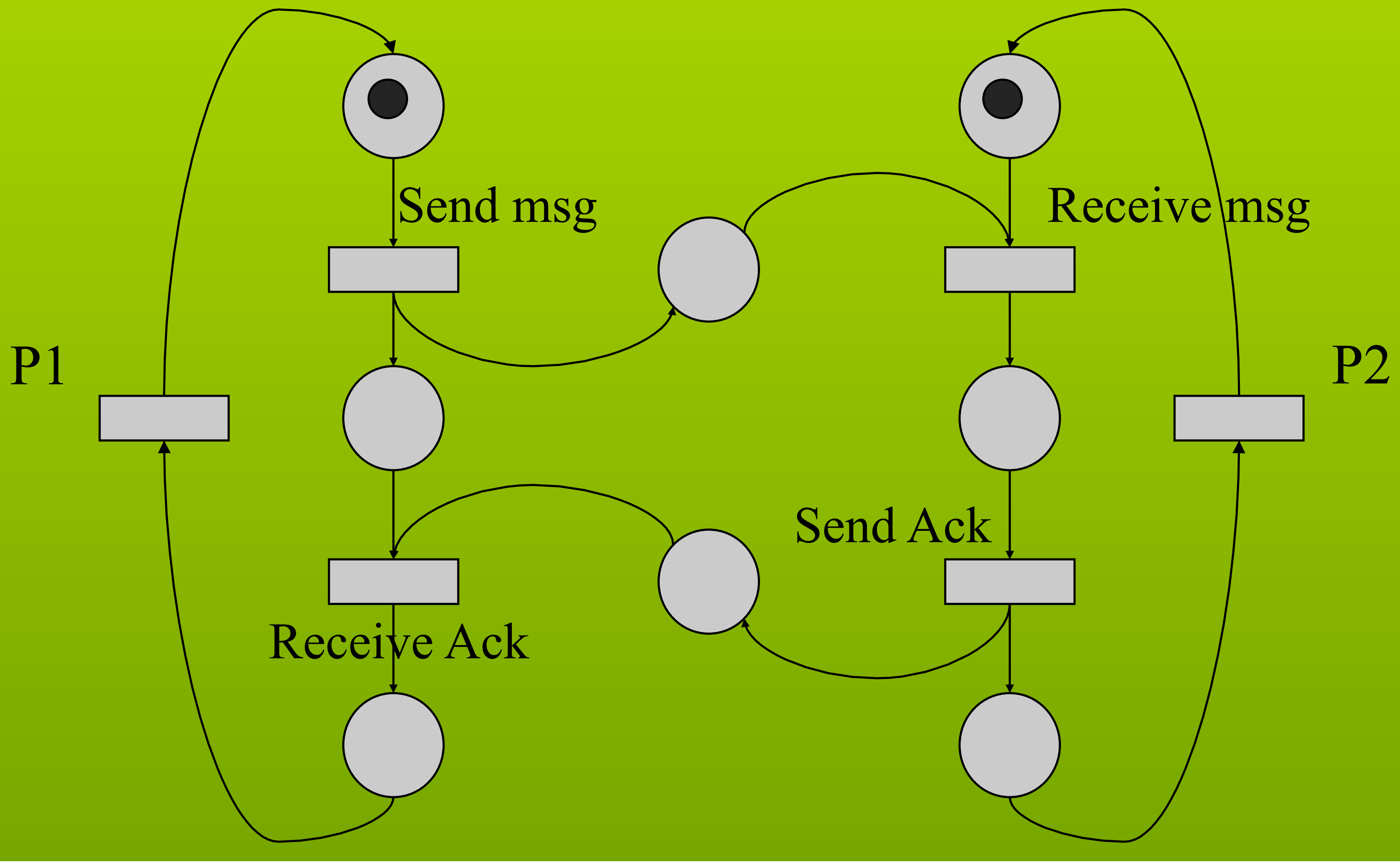
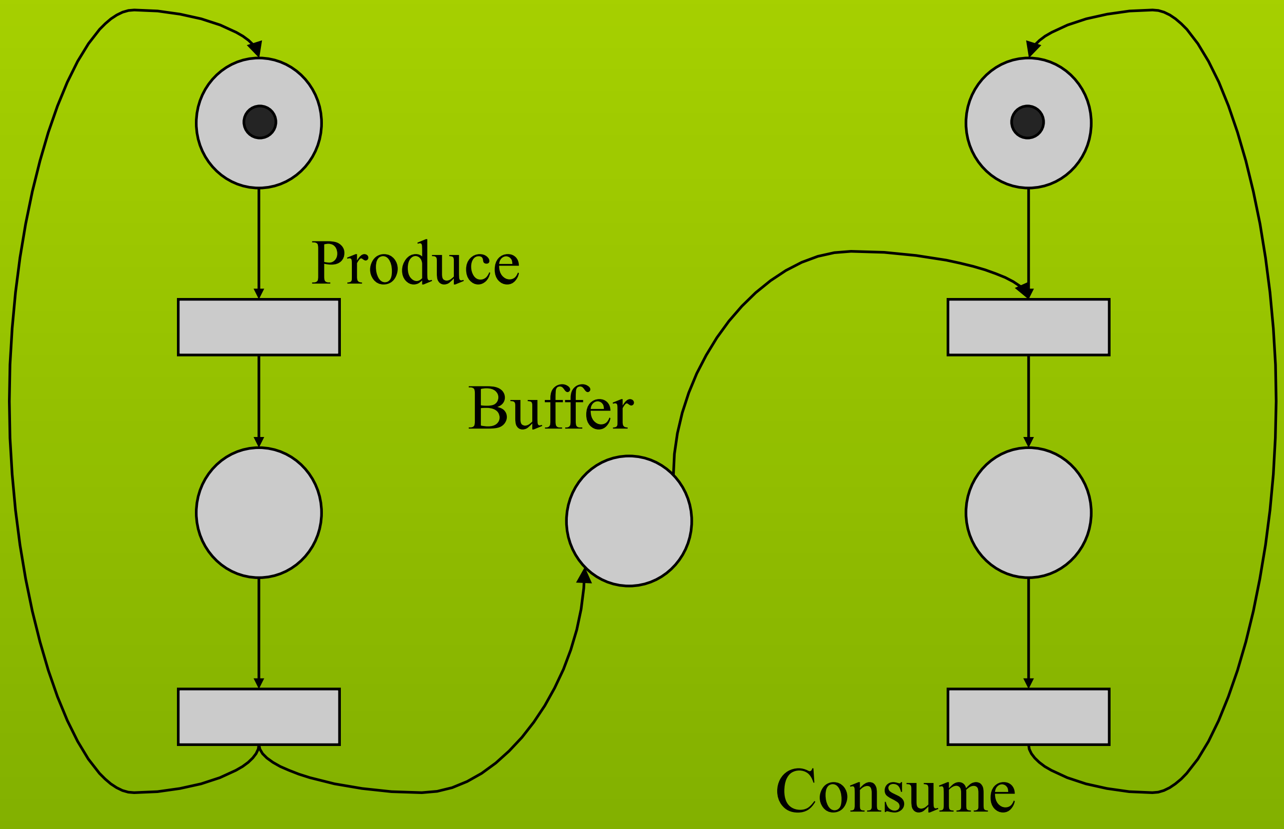


Producer-Consumer Problem



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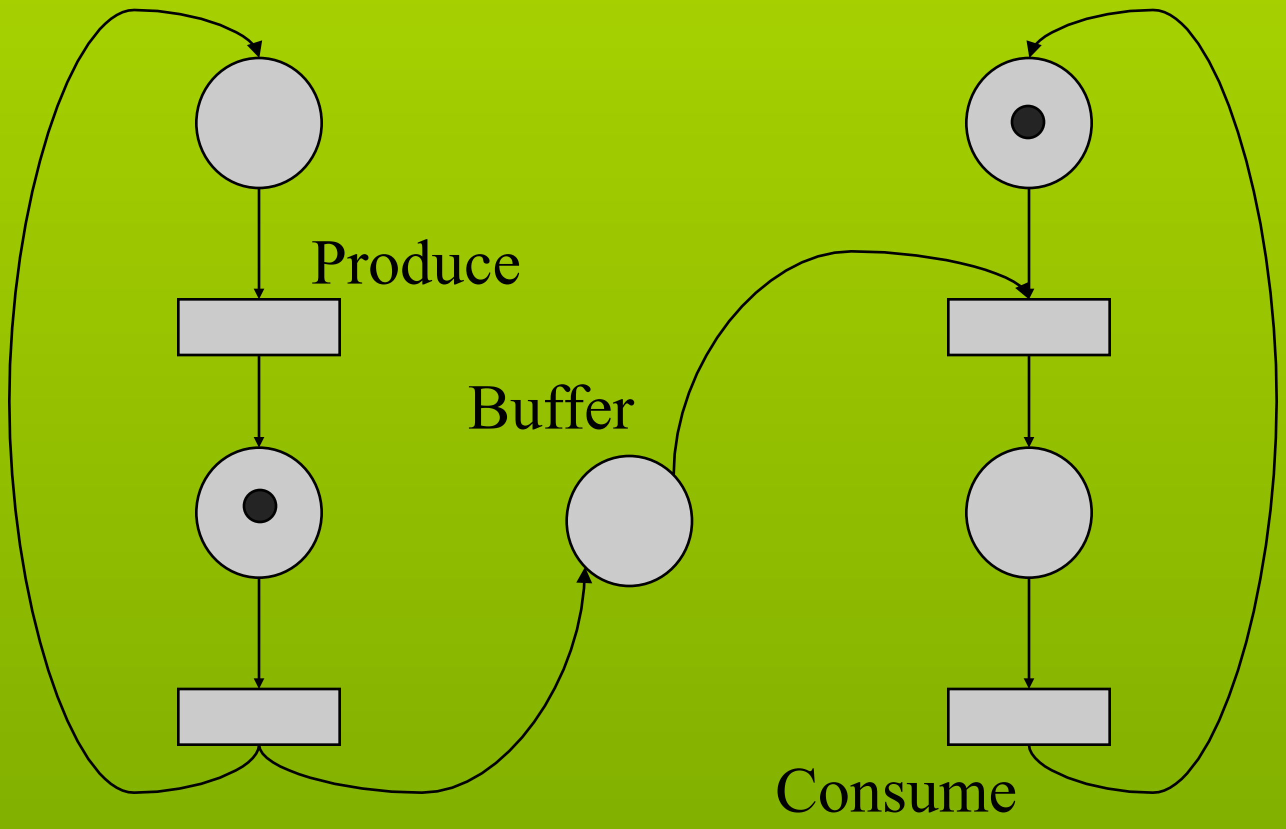

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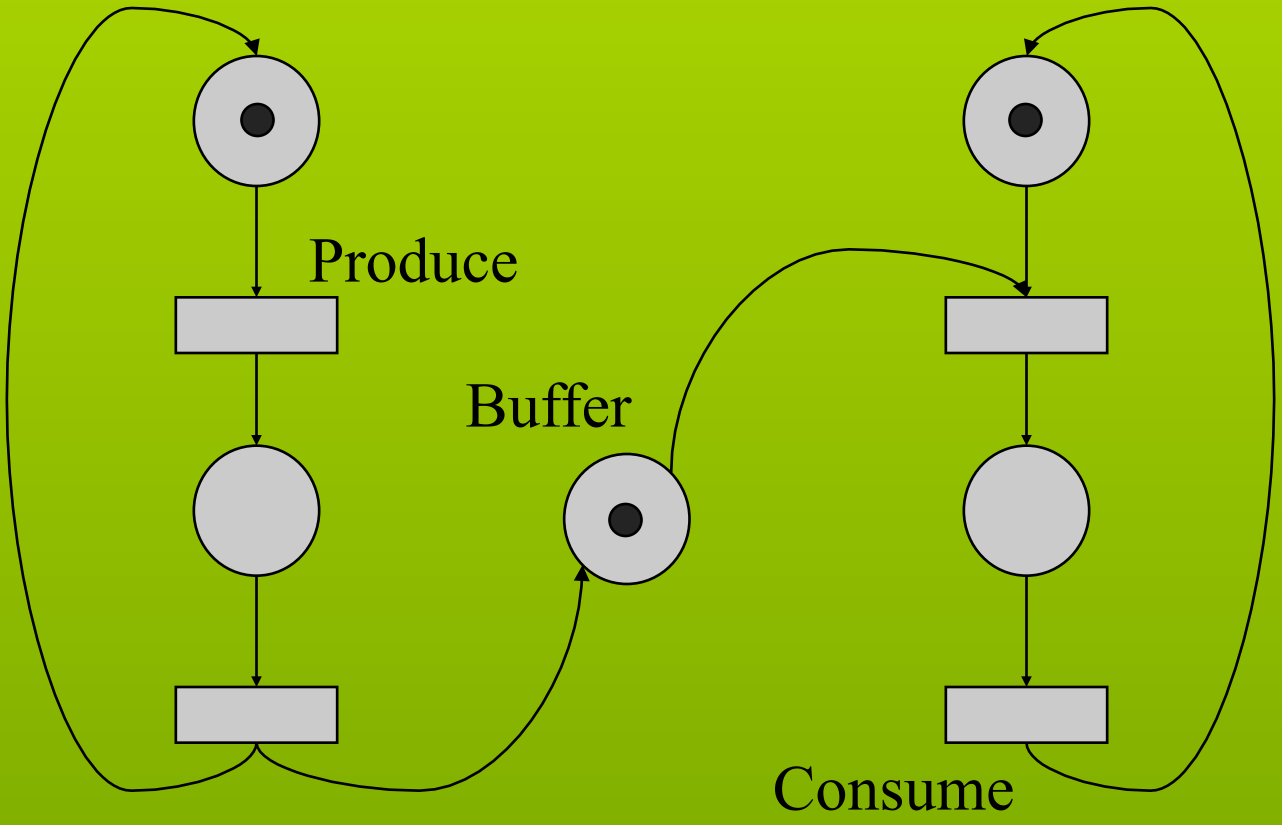

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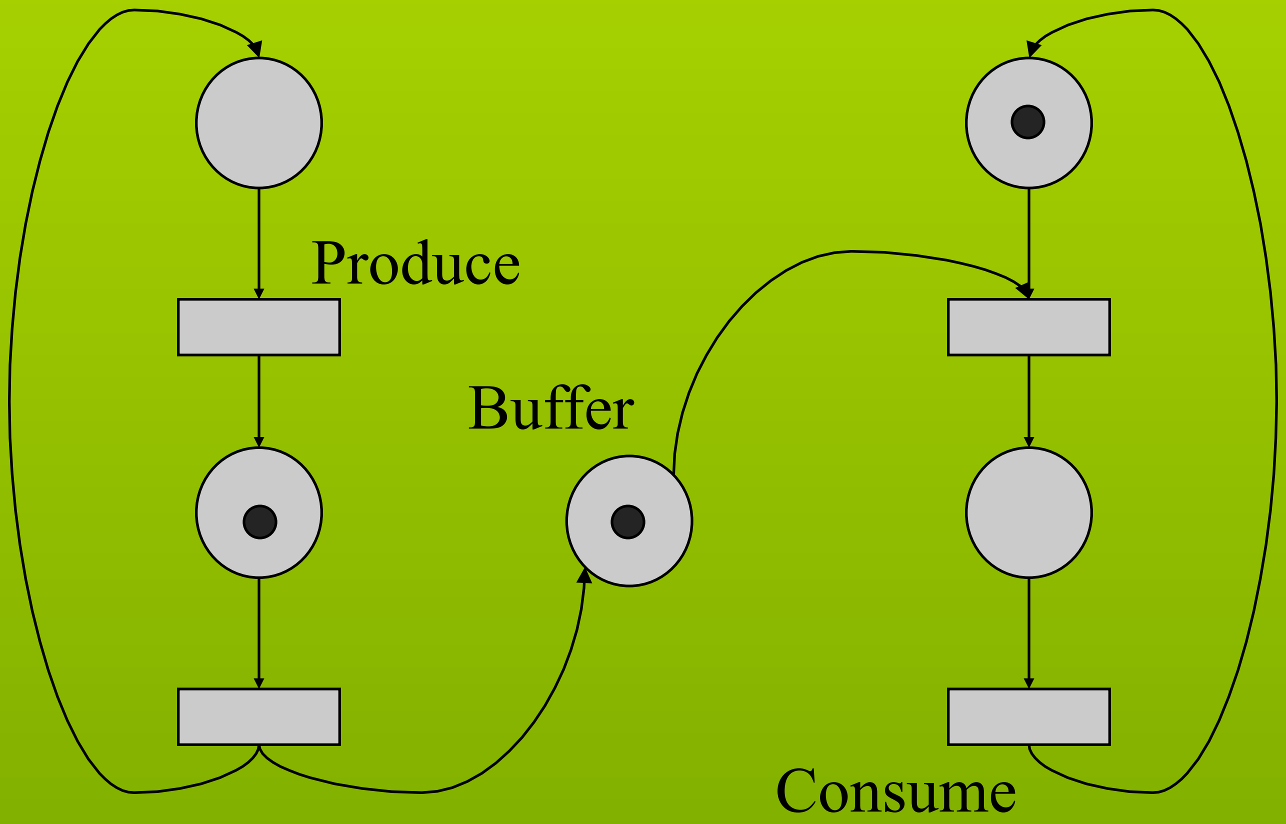

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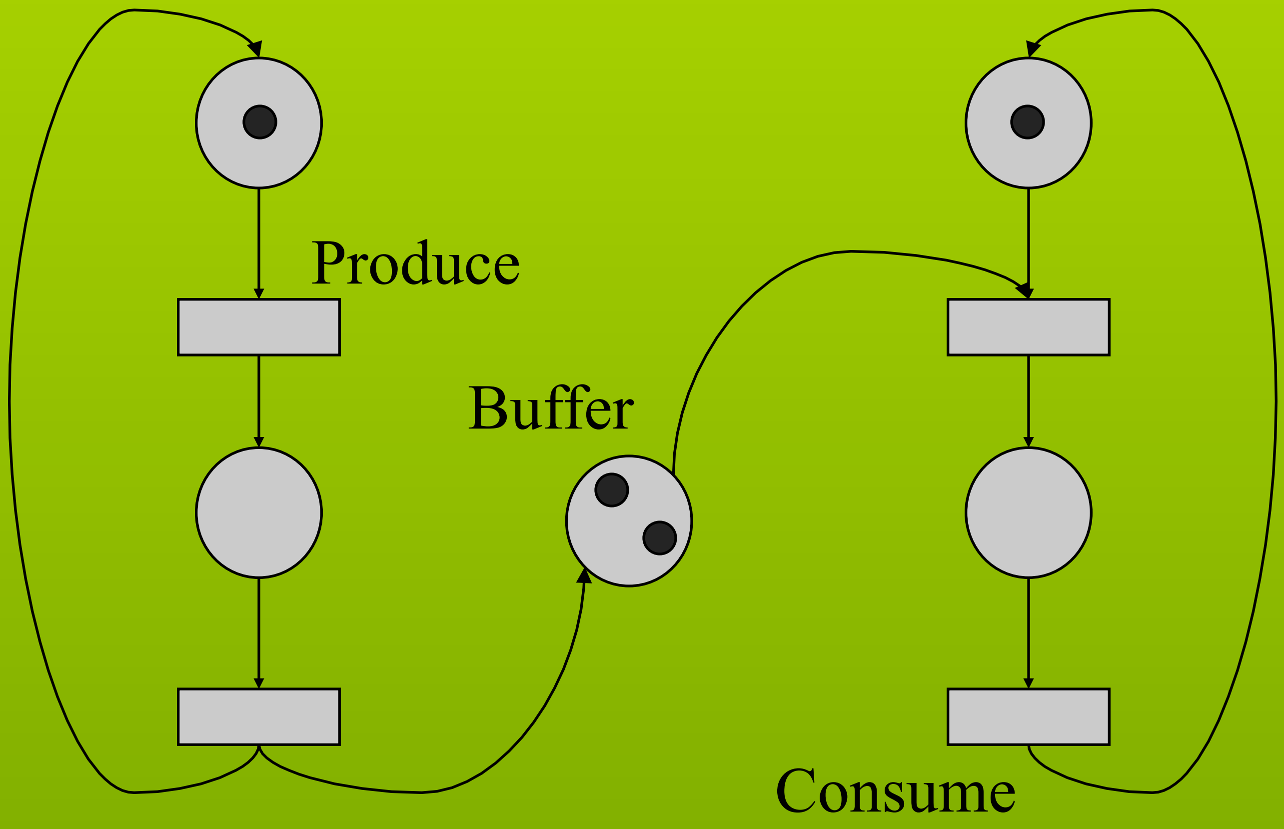

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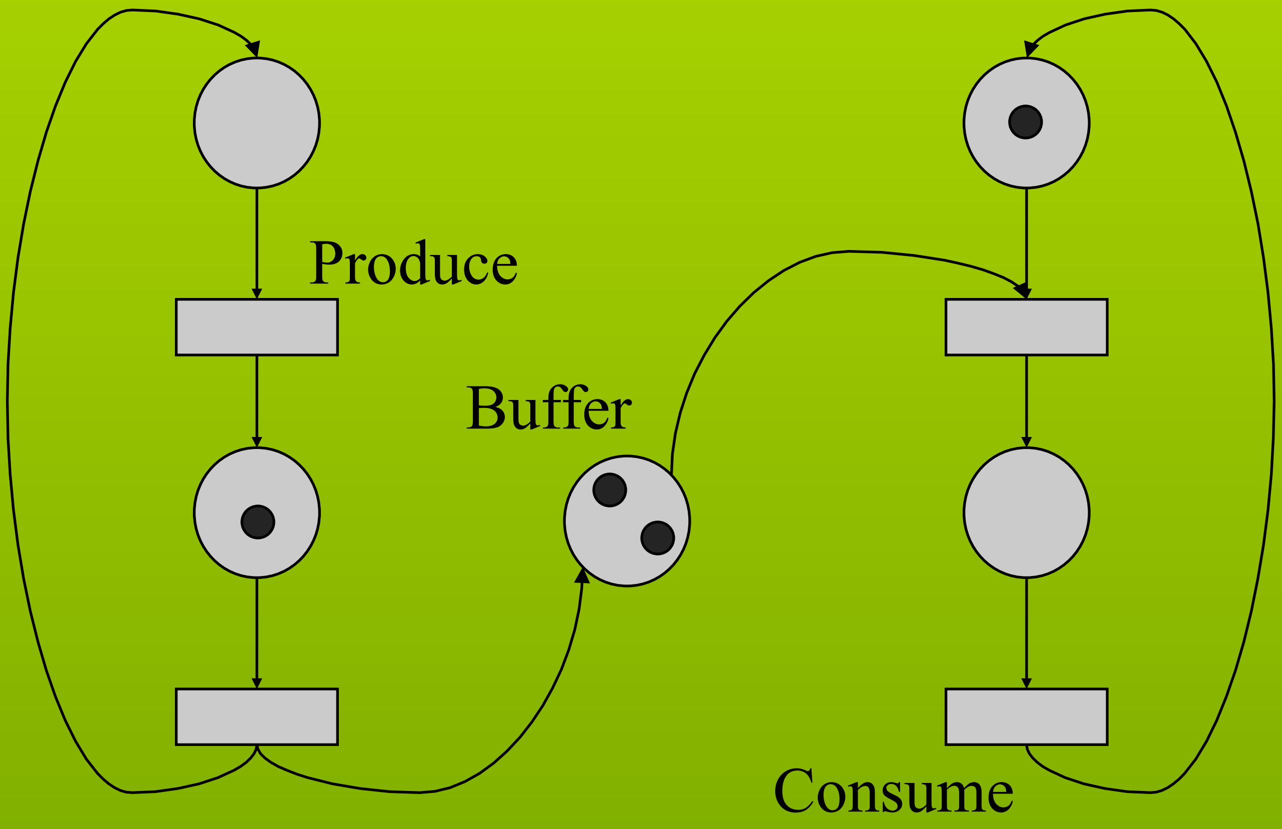

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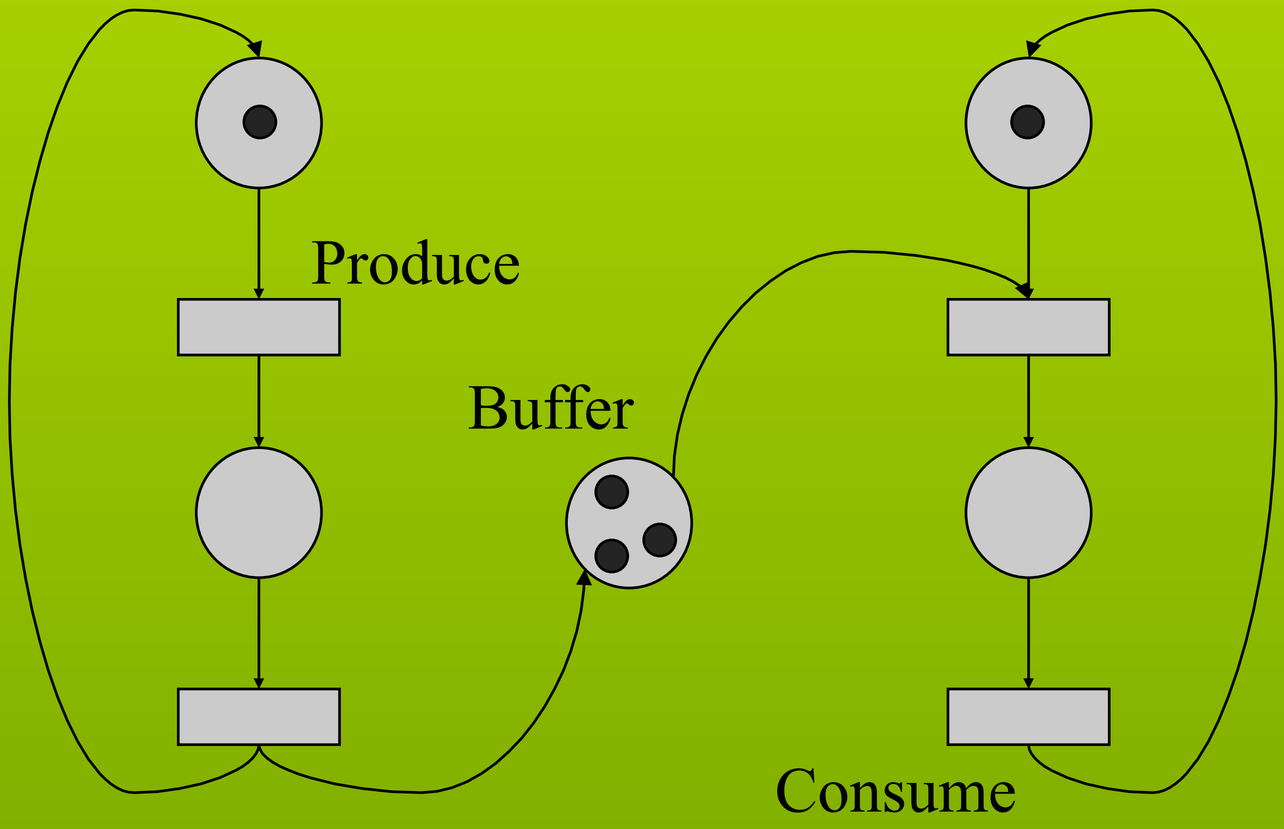

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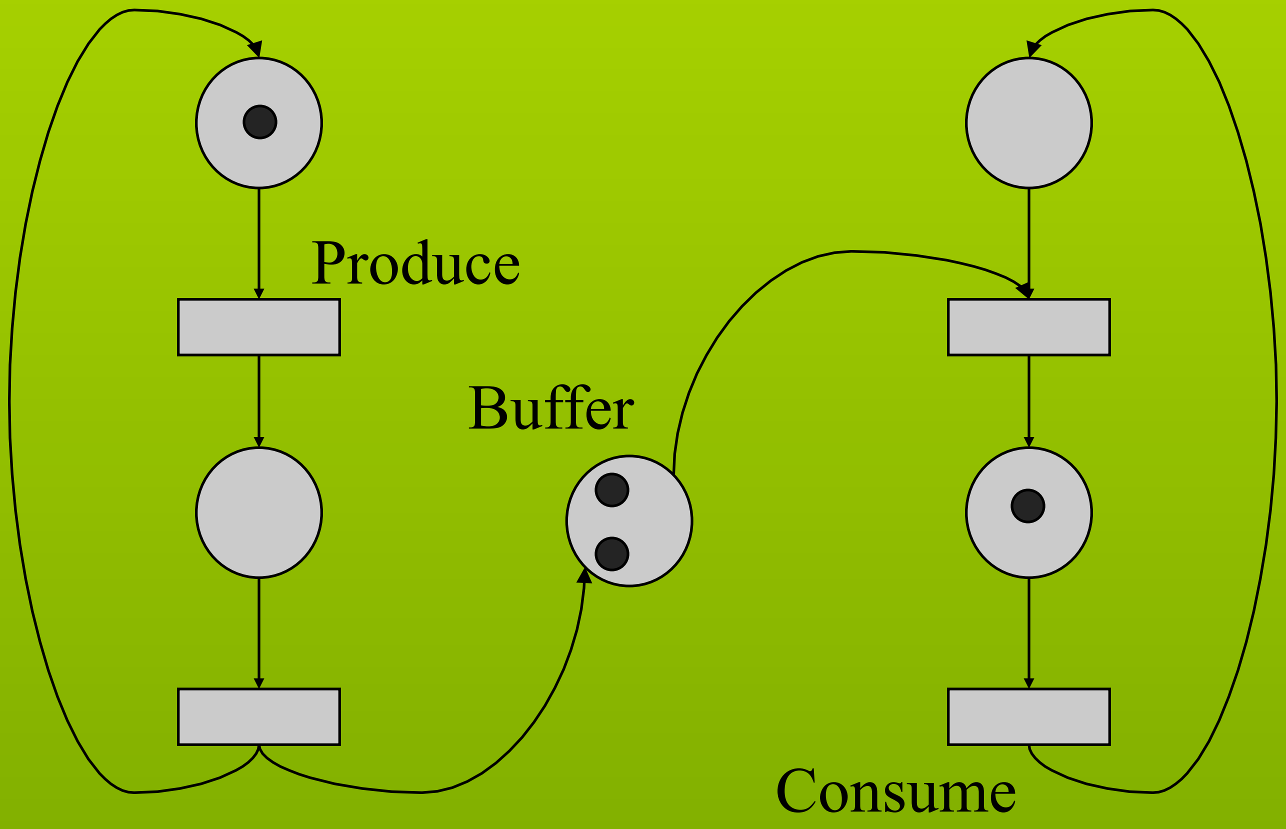

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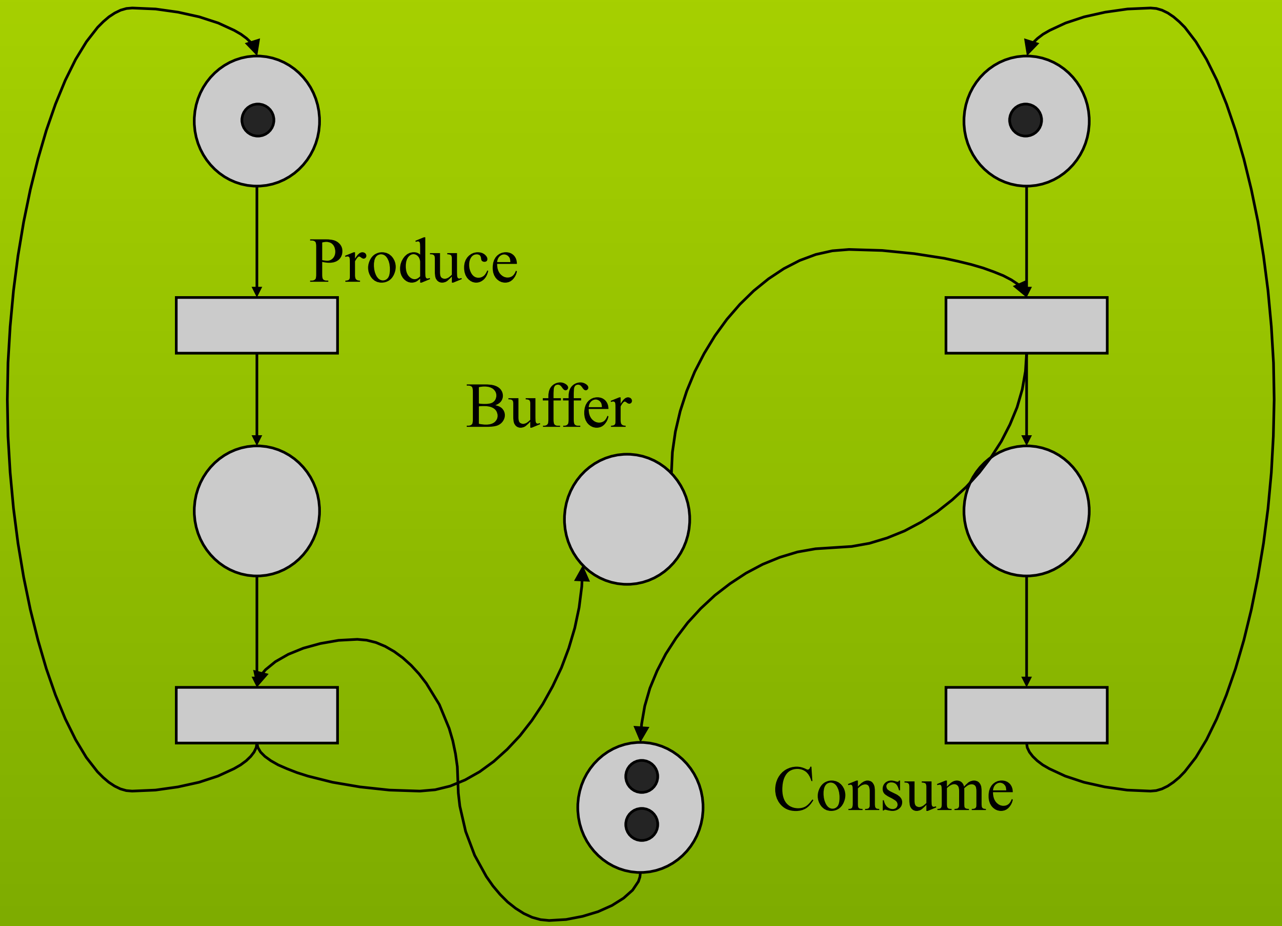

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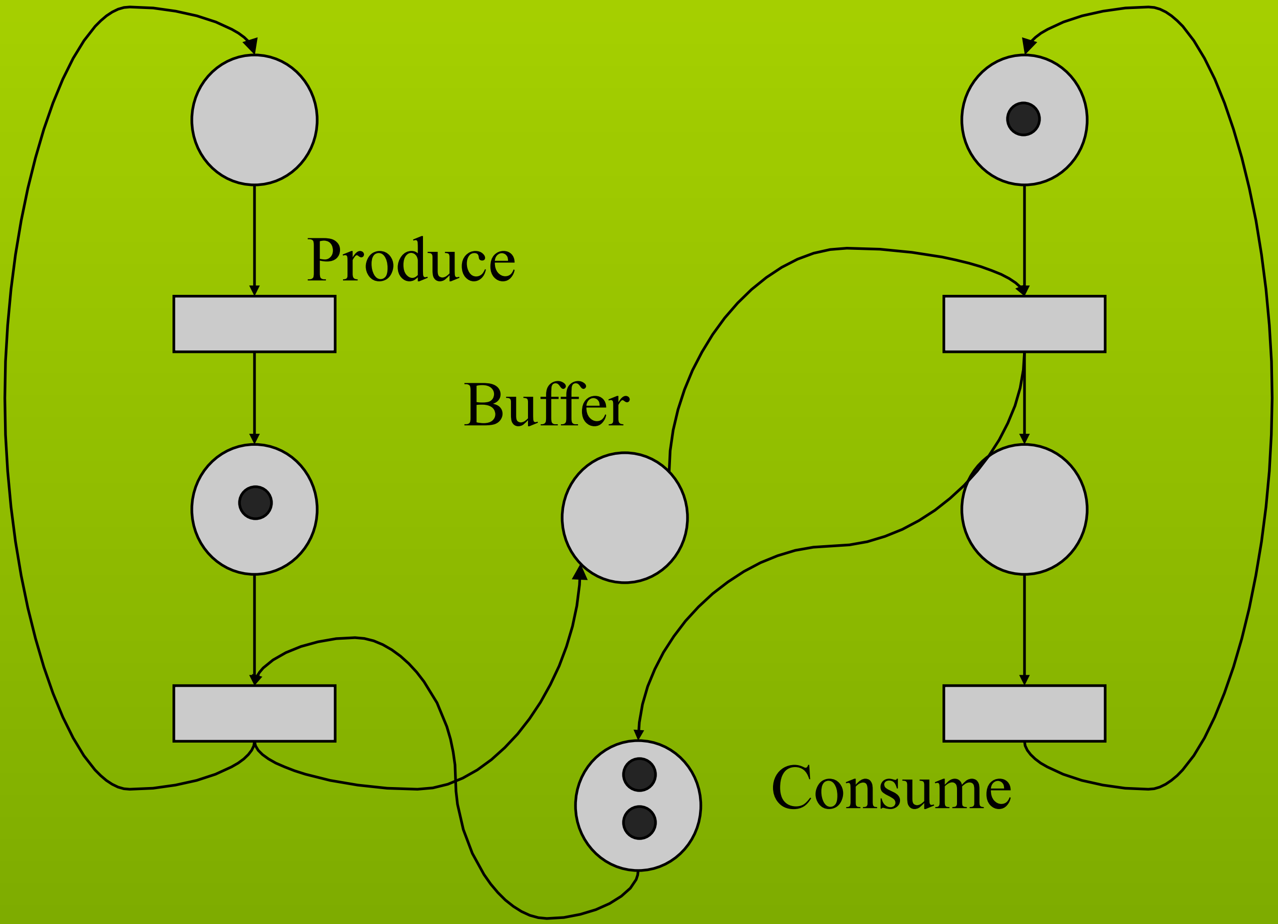

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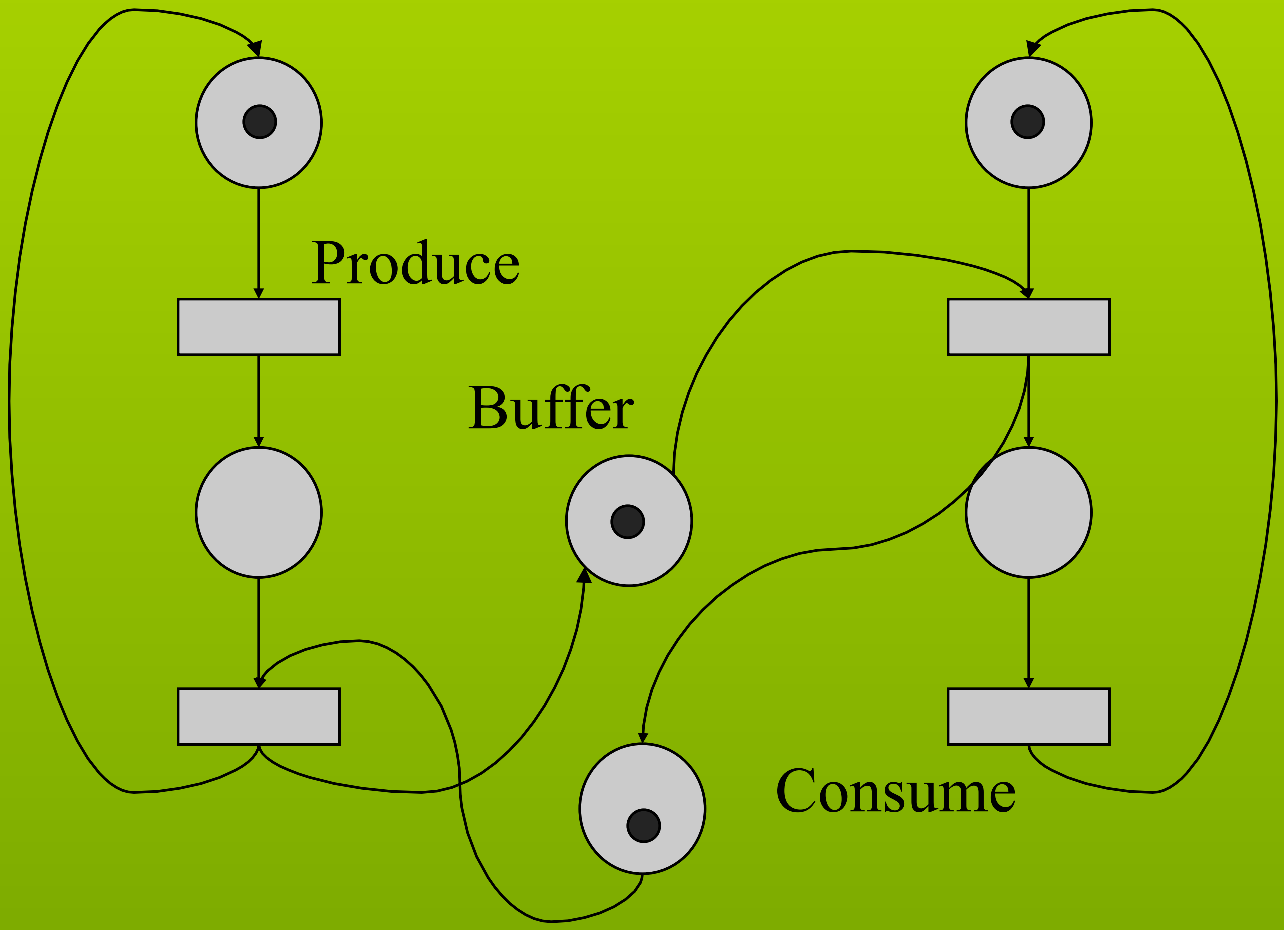

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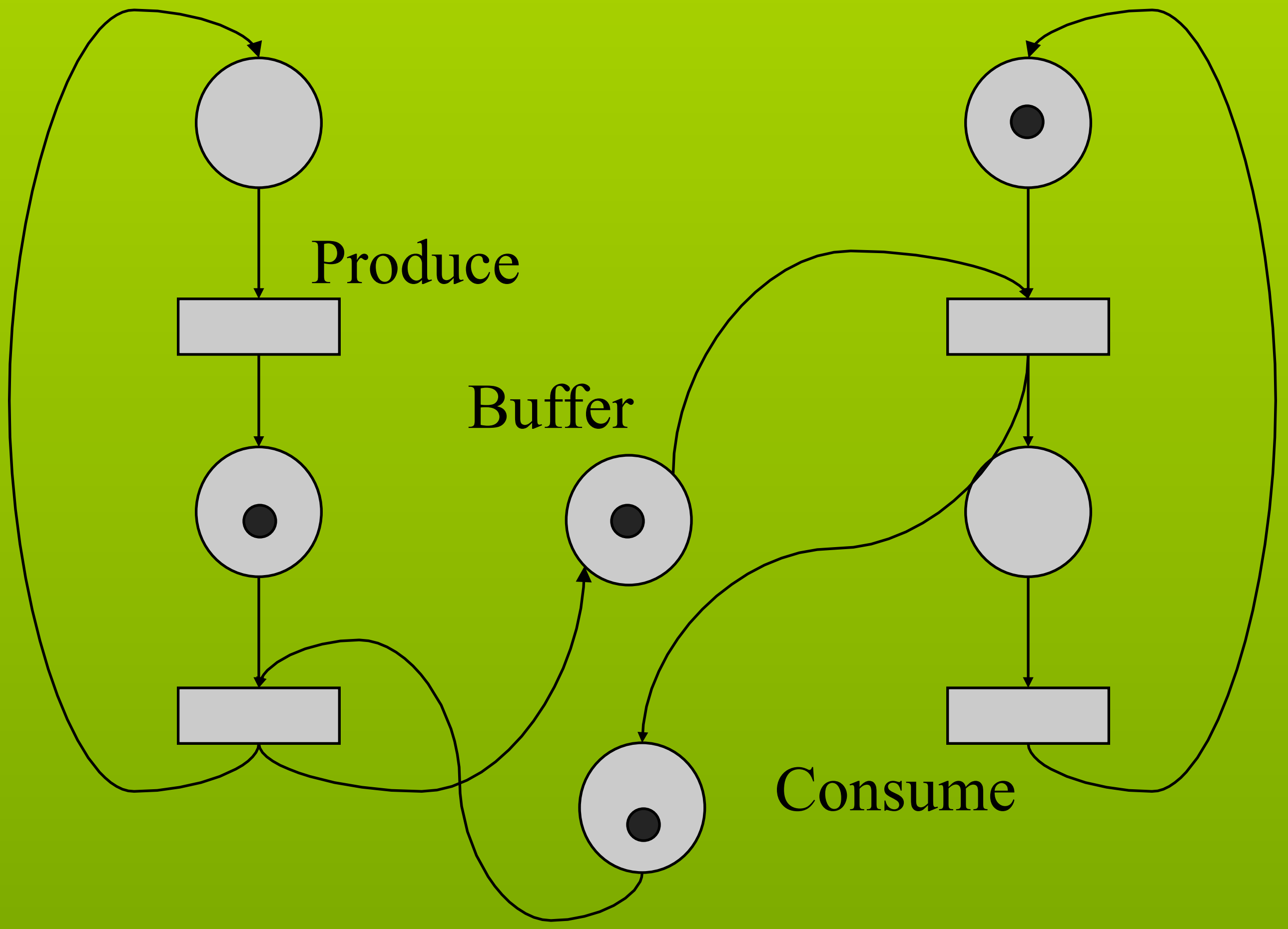

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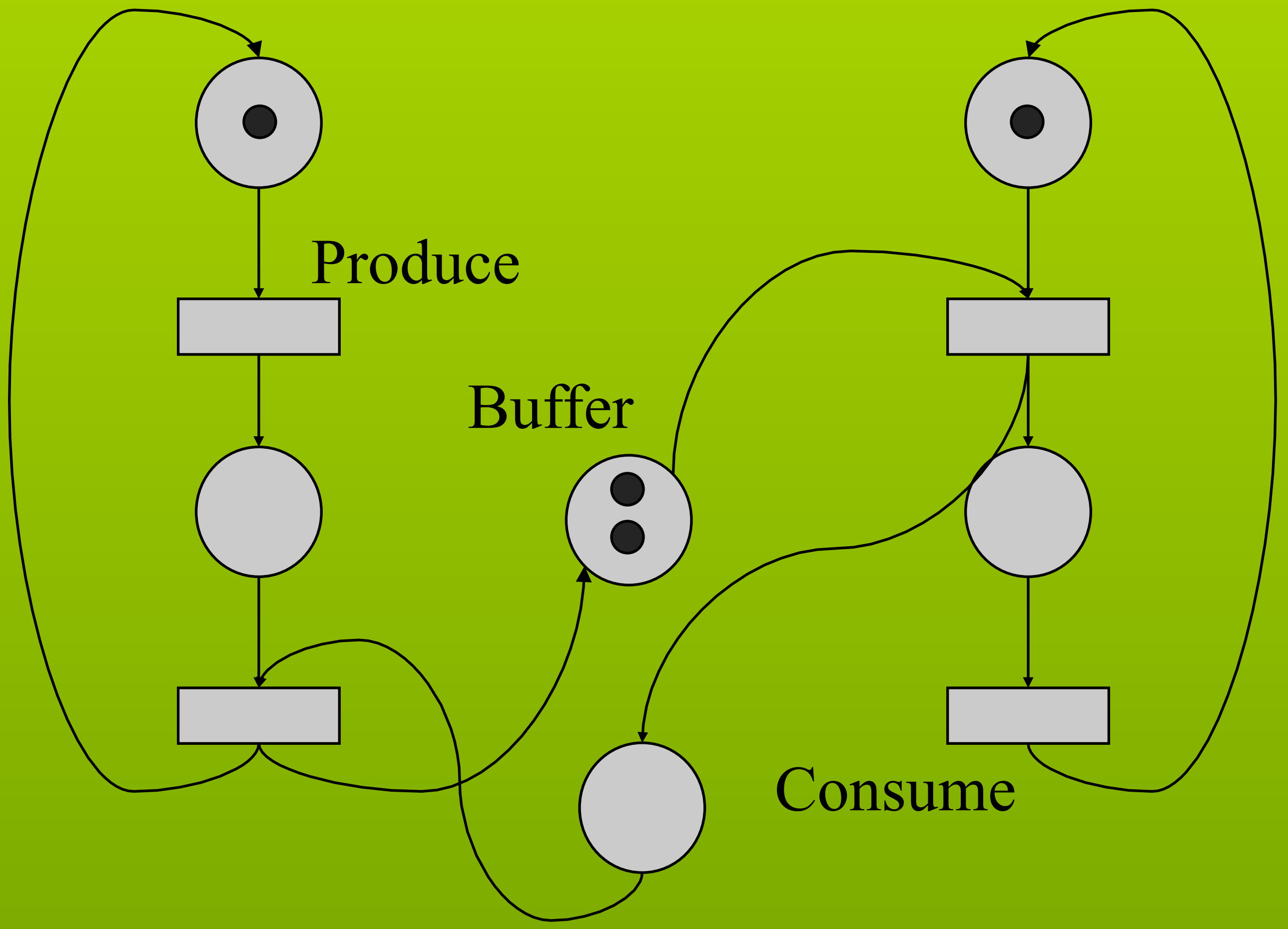

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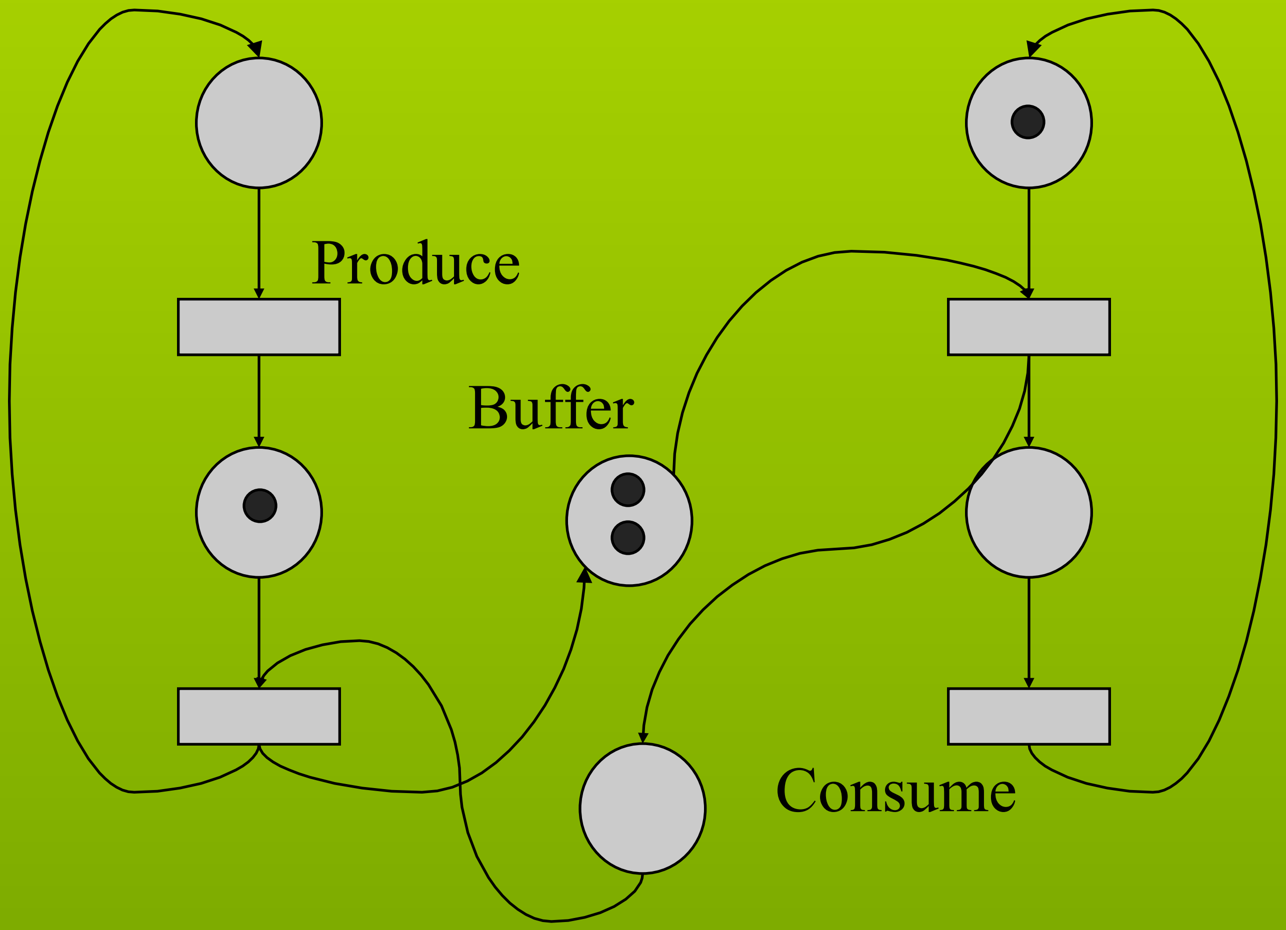
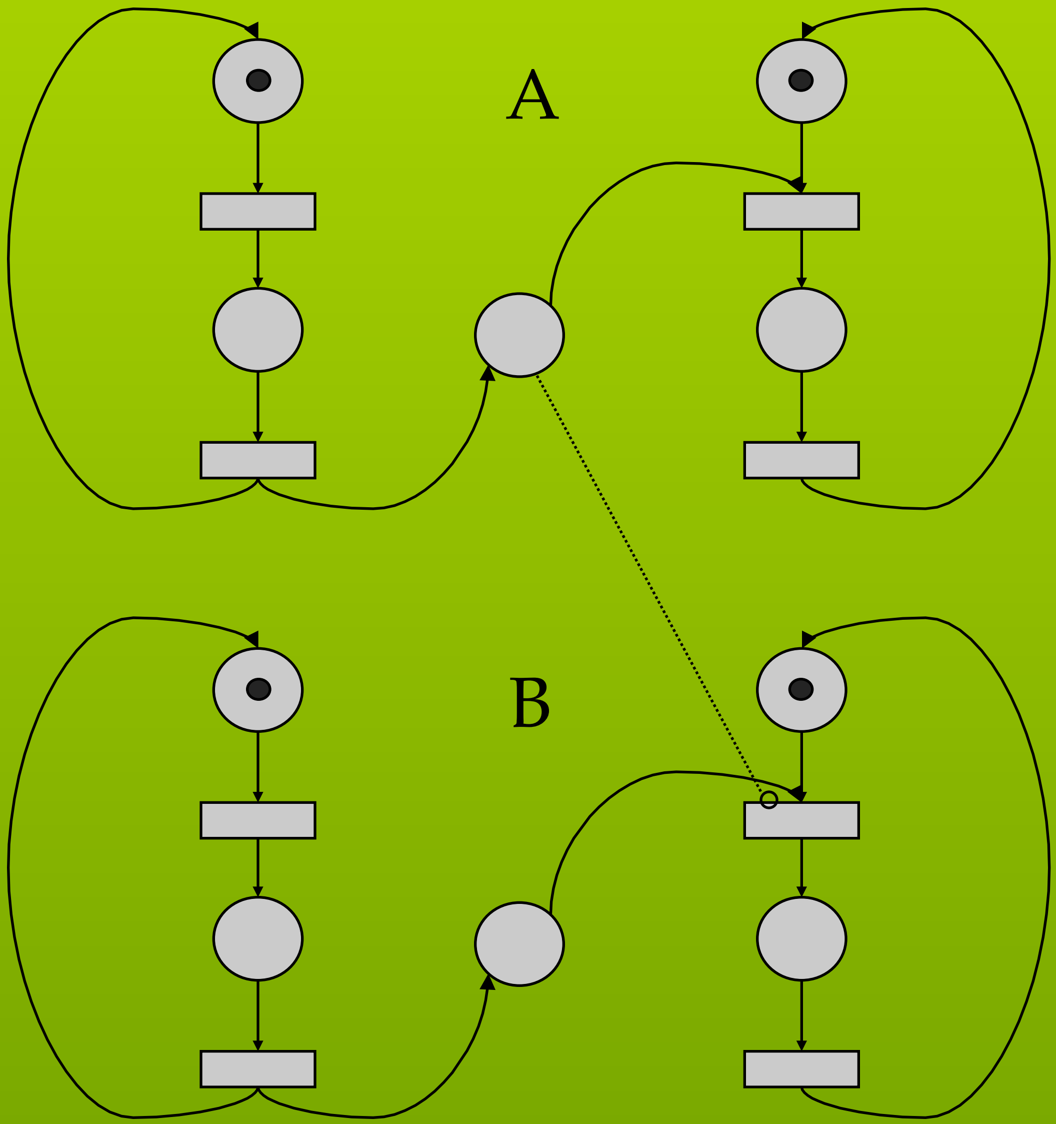


Producer-Consumer with priority

Consumer B can consume only if buffer A is empty

Inhibitor arcs

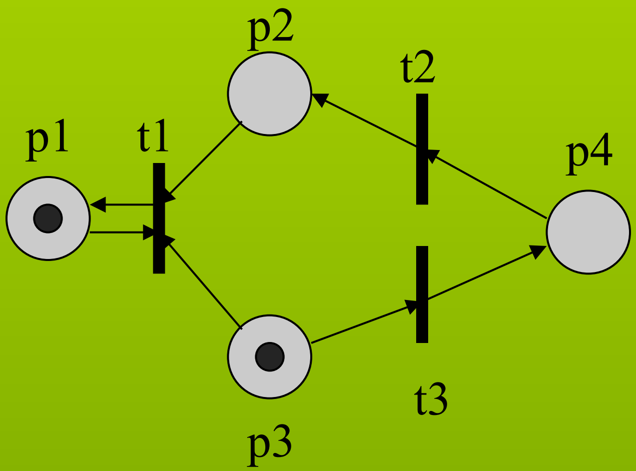


PN properties

- **Behavioral: depend on the initial marking (most interesting)**
  - **Reachability**
  - **Boundedness**
  - **Schedulability**
  - **Liveness**
  - **Conservation**
- **Structural: do not depend on the initial marking (often too restrictive)**
  - **Consistency**
  - **Structural boundedness**

Reachability

- **Marking M is reachable from marking M_0 if there exists a sequence of firings $\sigma = M_0\ t_1\ M_1\ t_2\ M_2 \ldots M$ that transforms M_0 to M.**
- **The reachability problem is decidable.**

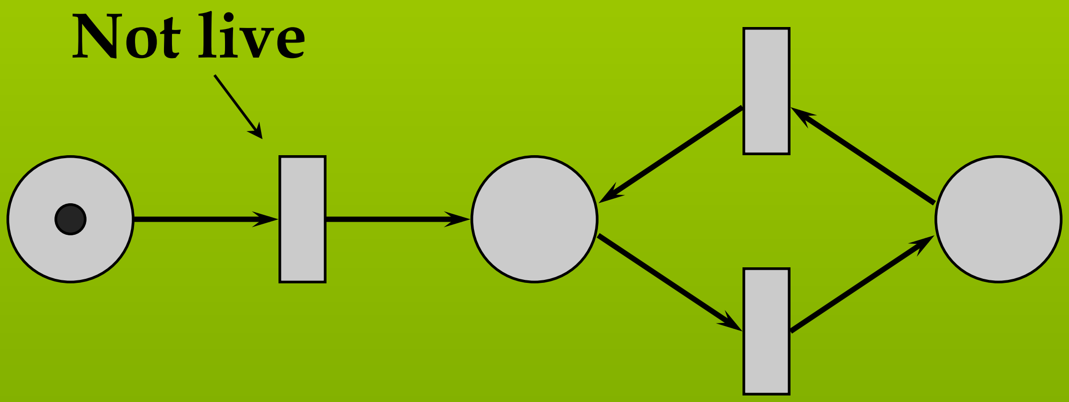


$M_0 = (1,0,1,0)$
$M = (1,1,0,0)$

$$\begin{tikzcd}[row sep=large] M_0 = (1,0,1,0) \arrow[d, "t3"] \\ M_1 = (1,0,0,1) \arrow[d, "t2"] \\ M = (1,1,0,0) \end{tikzcd}$$

Liveness

- **Liveness**: from any marking any transition can become fireable
    - Liveness implies deadlock freedom, not viceversa



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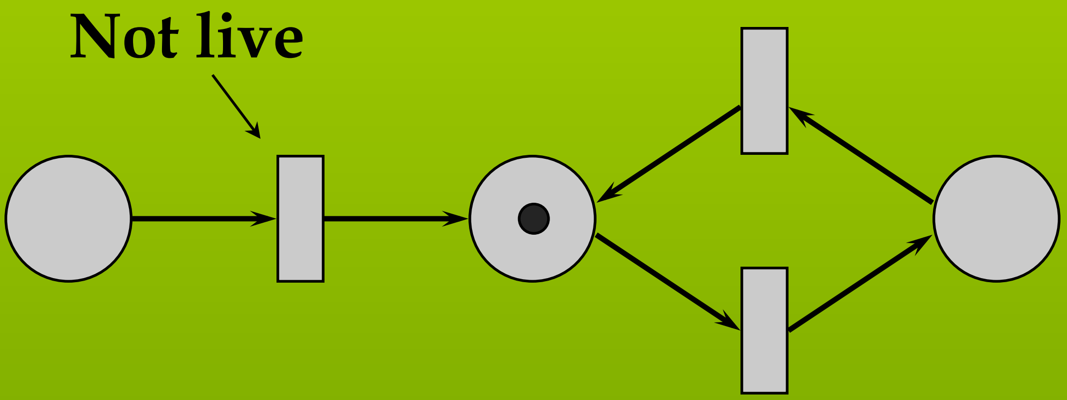

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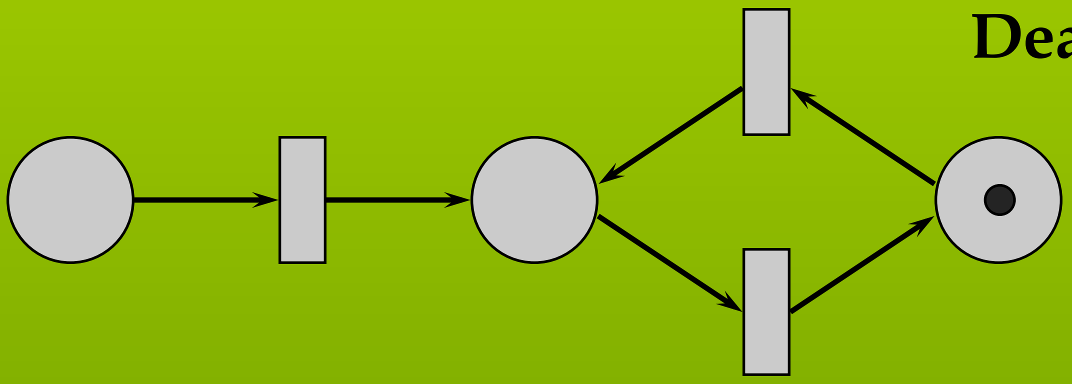

Deadlock-free

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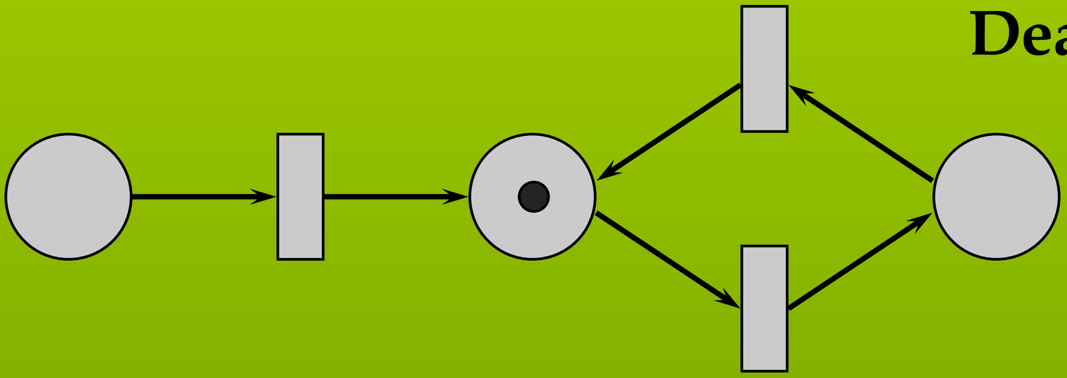
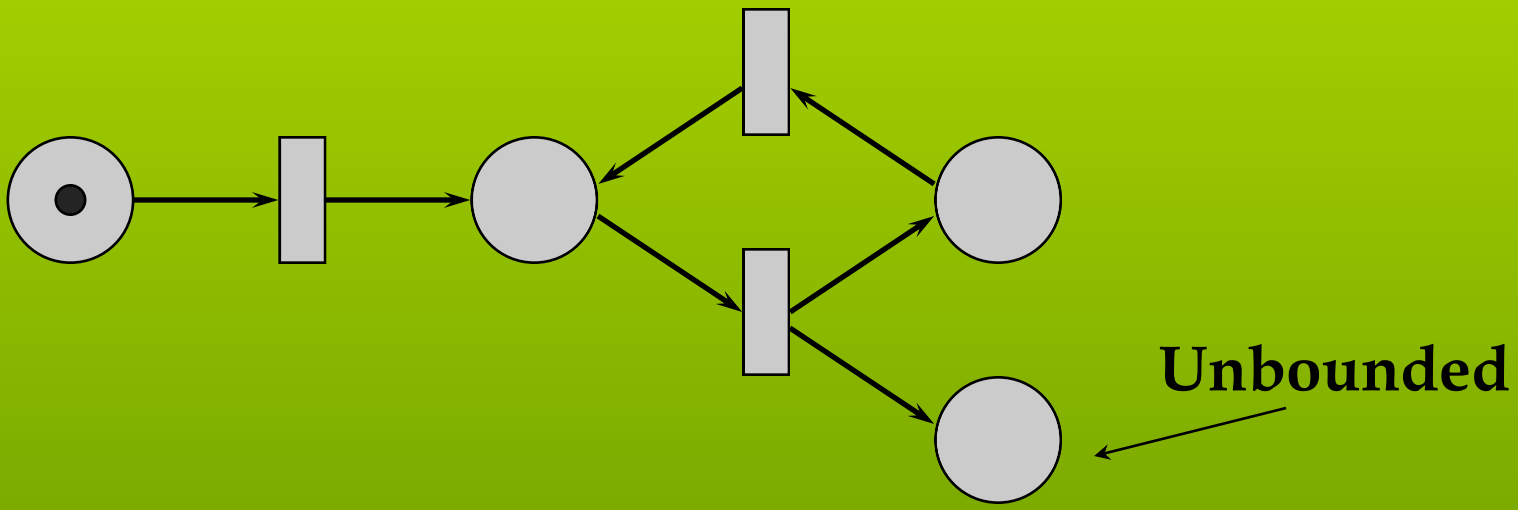


Deadlock-free

Boundedness

- **Boundedness**: the number of tokens in any place cannot grow indefinitely
  - (1-bounded also called *safe*)
  - Application: places represent buffers and registers (check there is no overflow)



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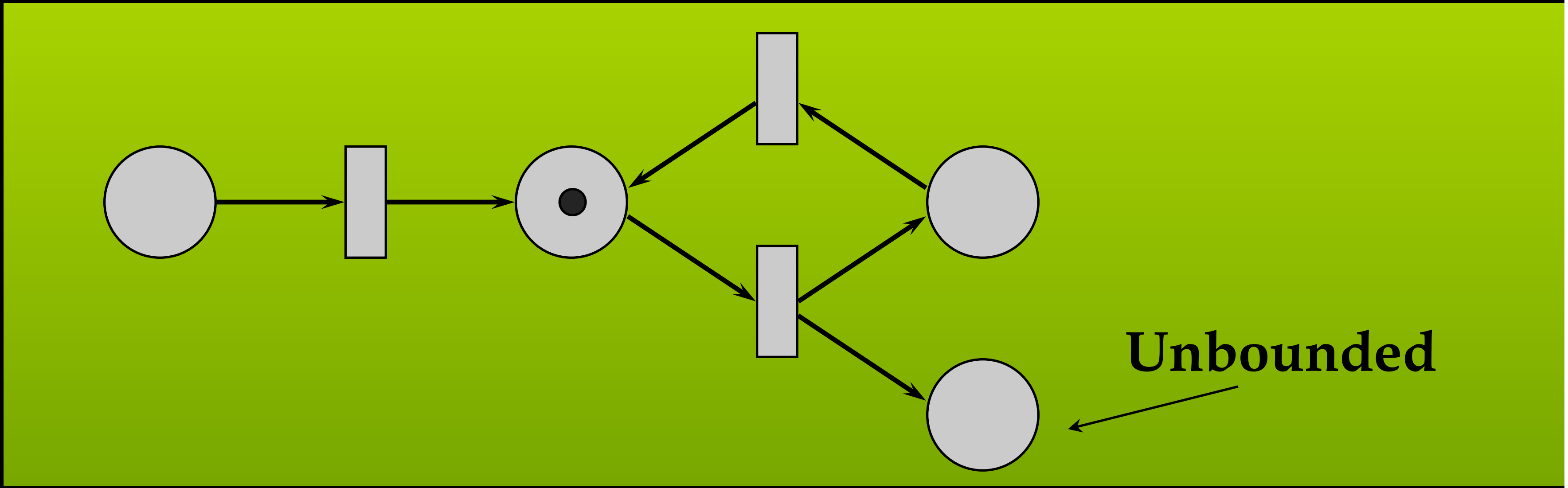

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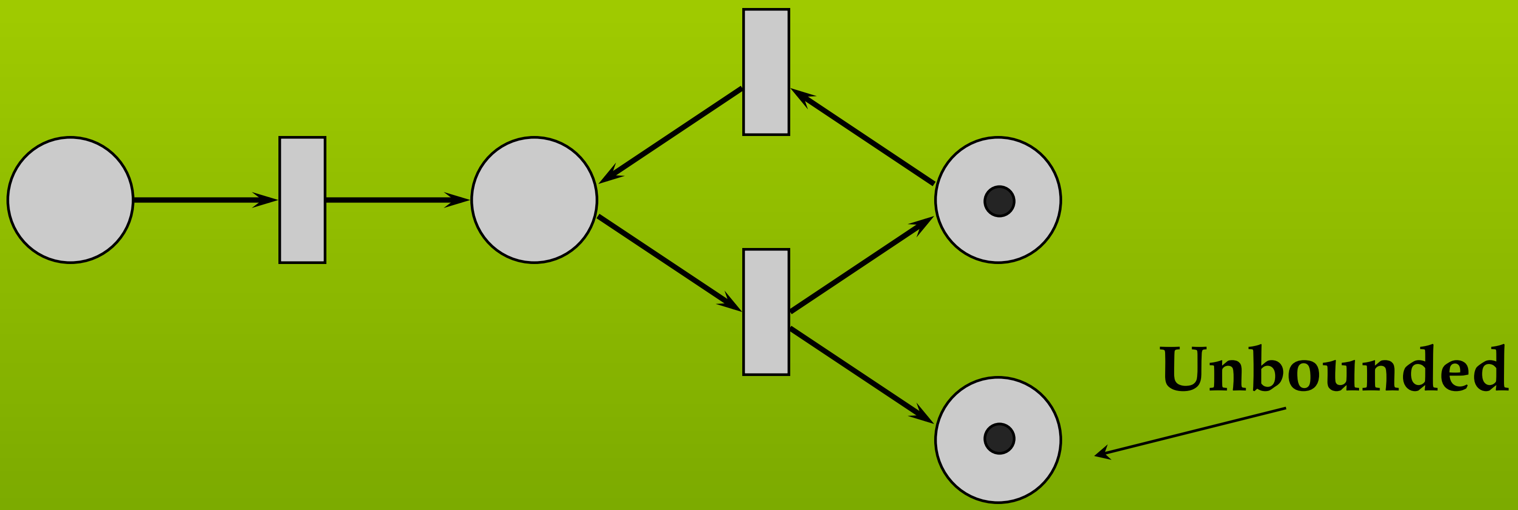

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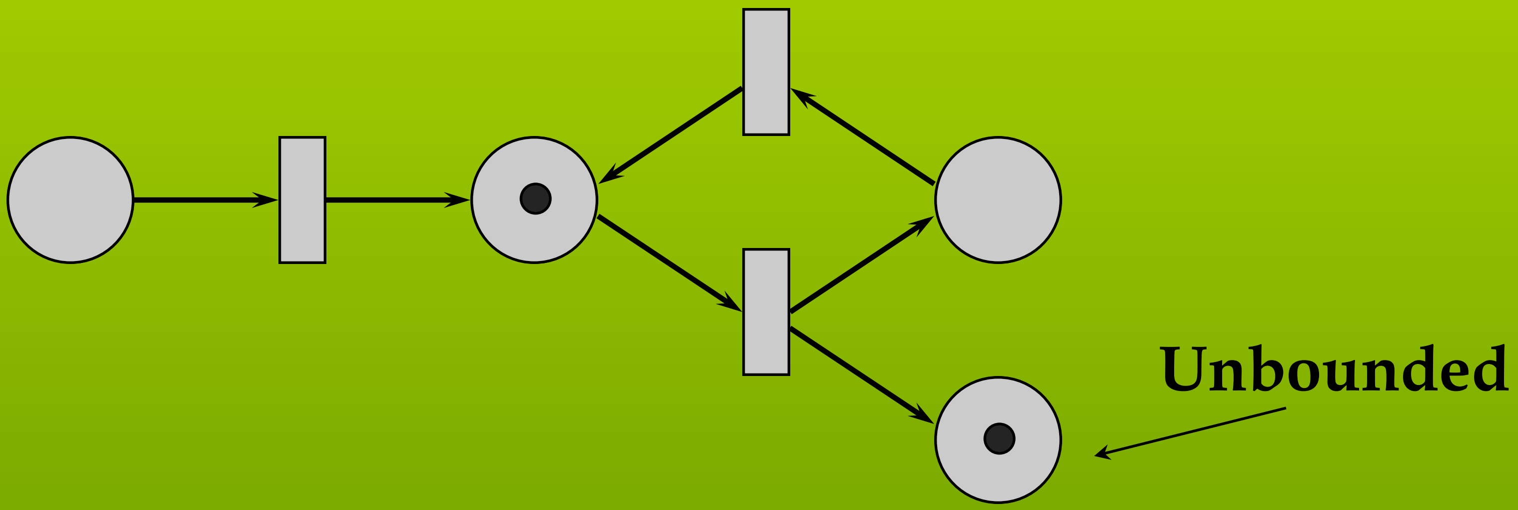

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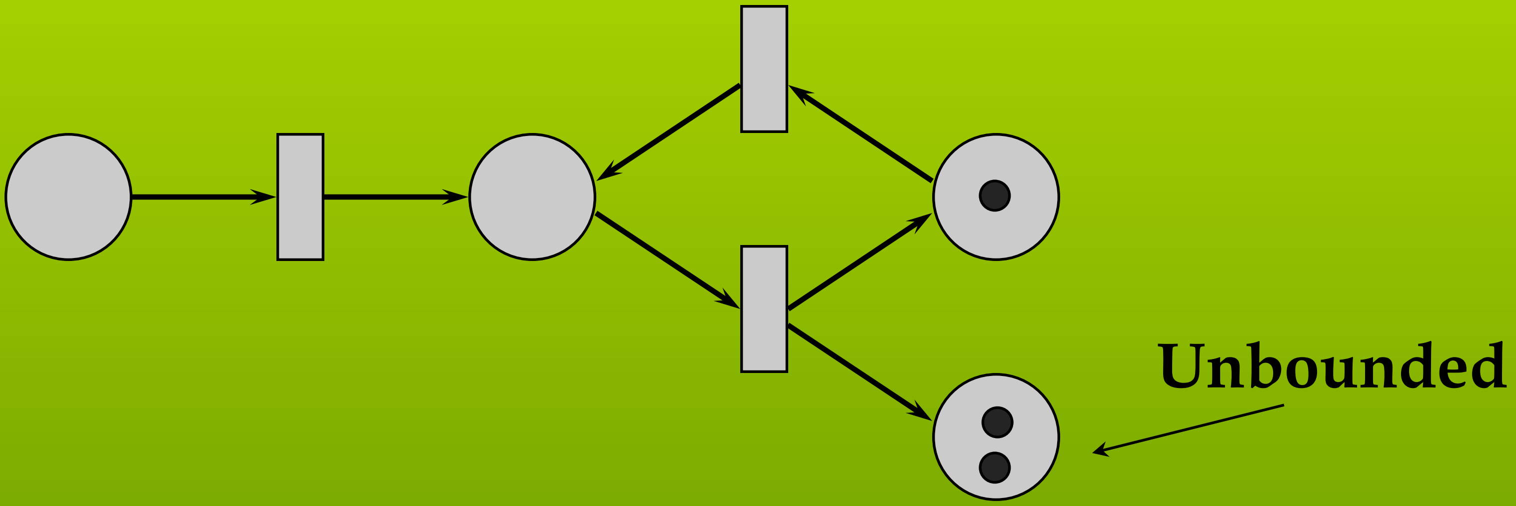
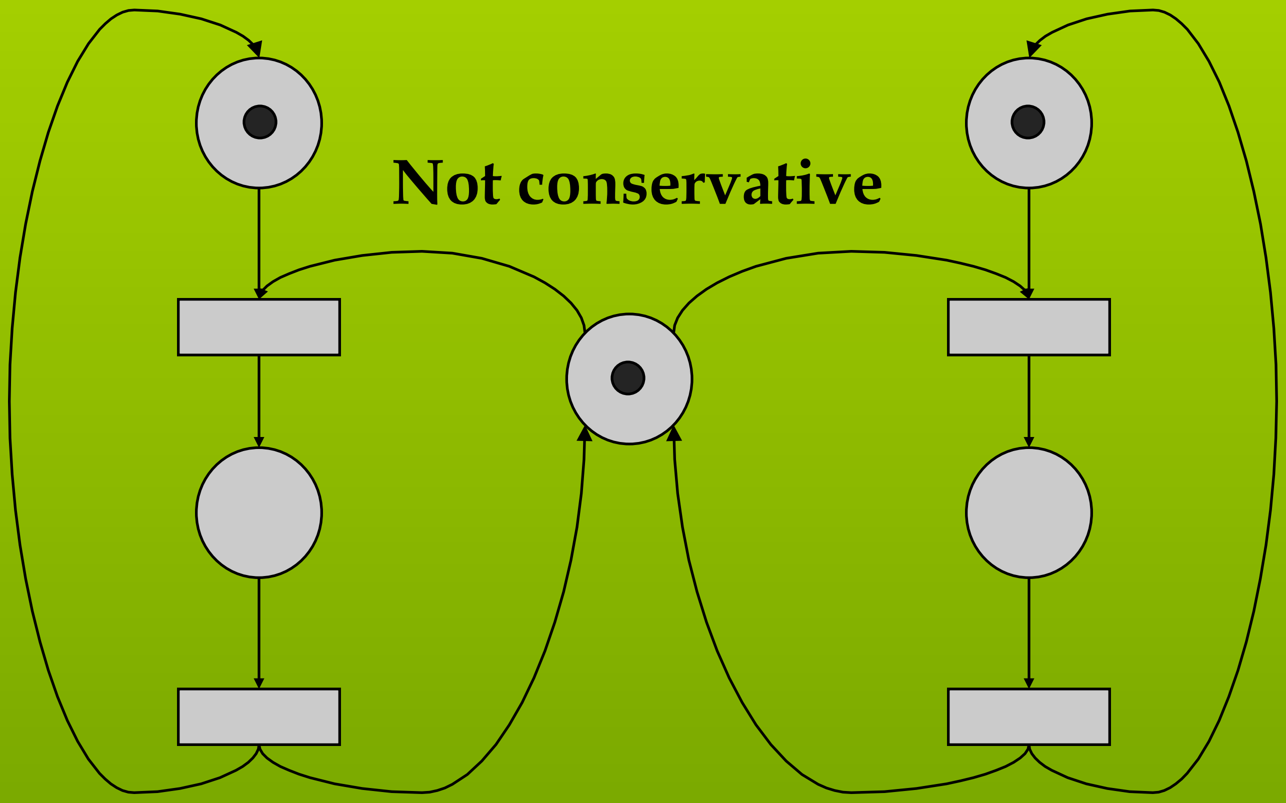


Conservation

- **Conservation**: the total number of tokens in the net is constant



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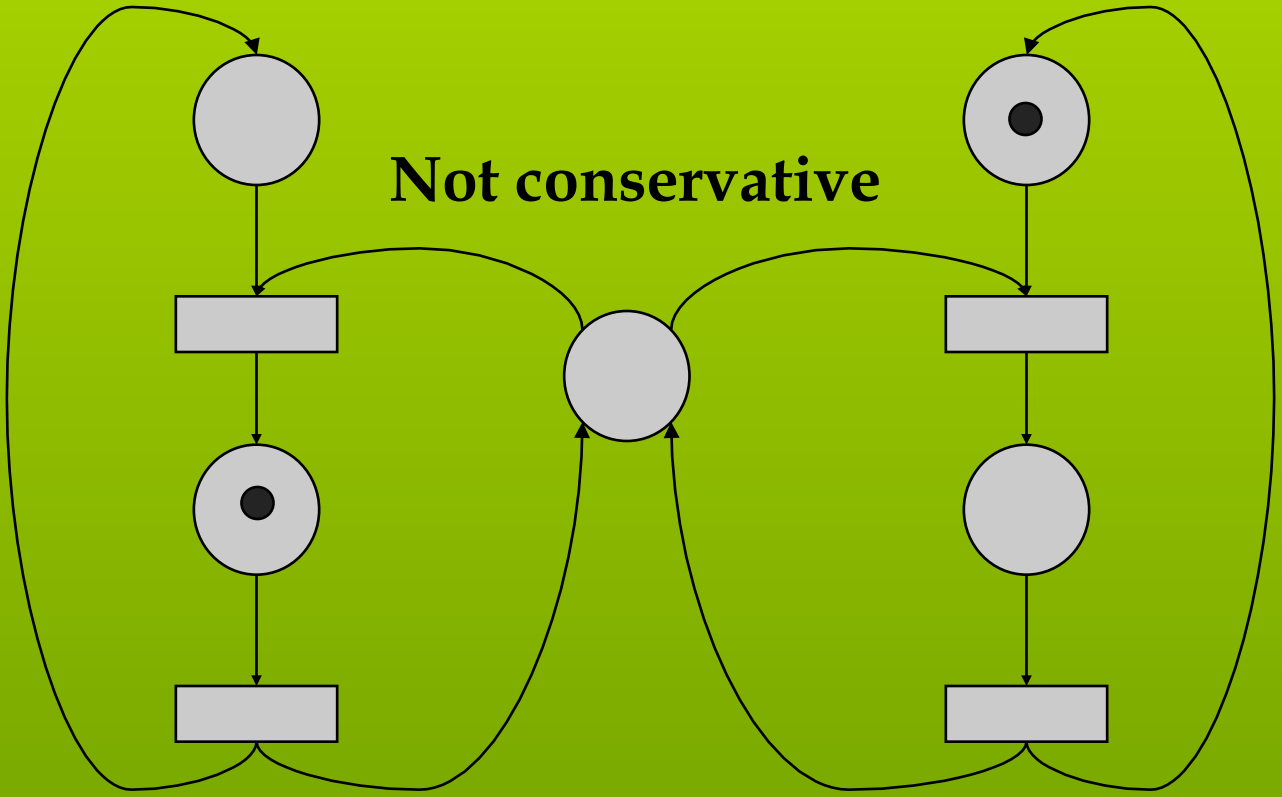

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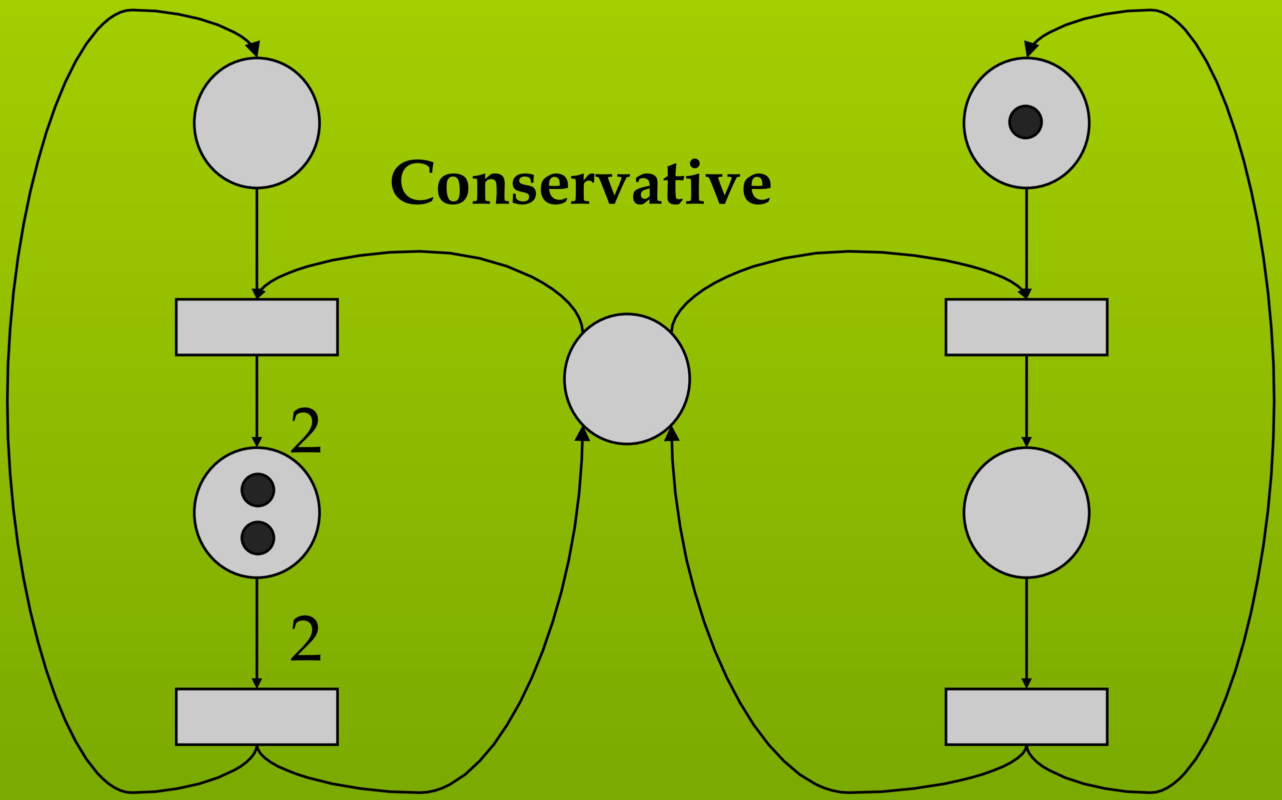
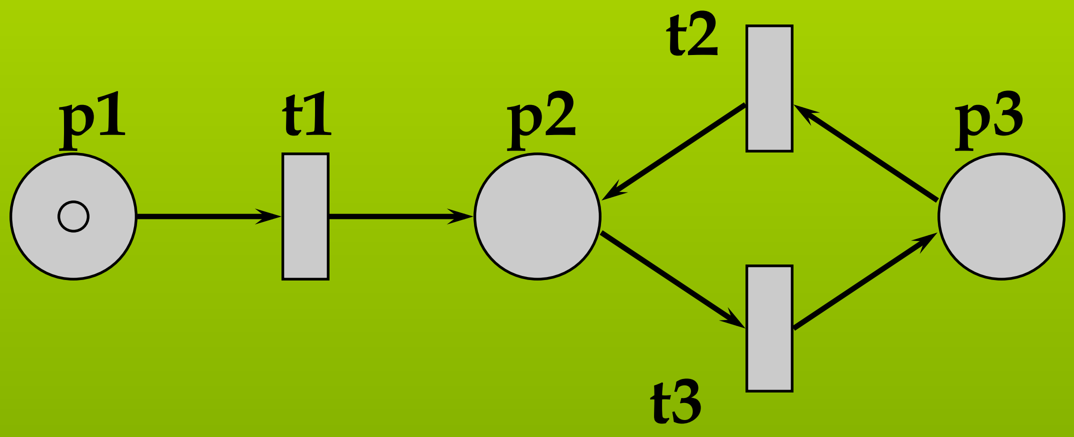


Analysis techniques

- **Structural analysis techniques**
  - **Incidence matrix**
  - **T- and S- Invariants**
- **State Space Analysis techniques**
  - **Coverability Tree**
  - **Reachability Graph**

Incidence Matrix



$$A = \begin{array}{ccc|c} t1 & t2 & t3 & \\ -1 & 0 & 0 & p1 \\ 1 & 1 & -1 & p2 \\ 0 & -1 & 1 & p3 \end{array}$$

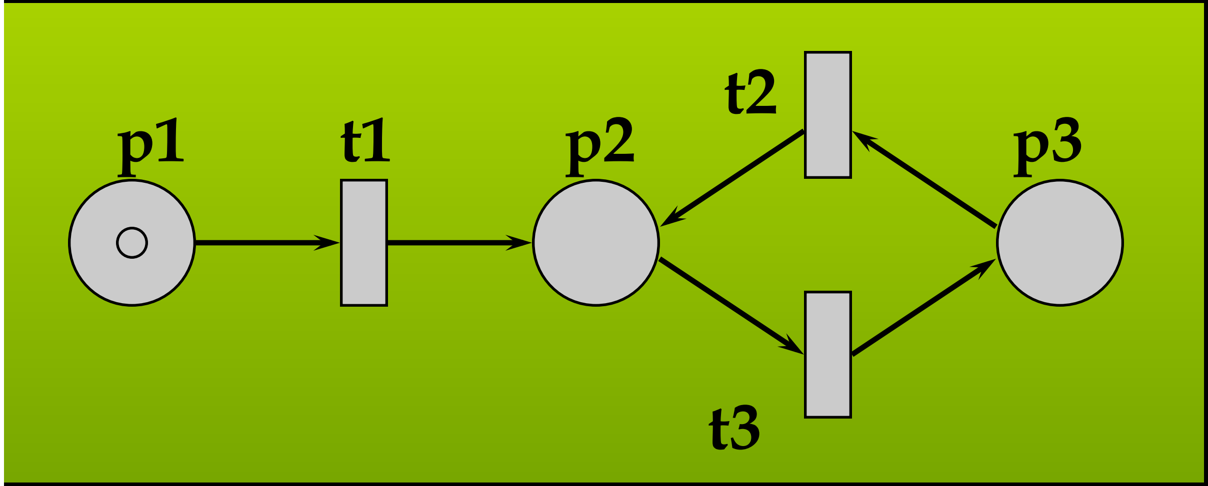
- **Necessary condition for marking M to be reachable from initial marking M_0:**

there exists firing vector v s.t.:

$$M = M_0 + A v$$

State equations

- E.g. reachability of $M = |0\ 0\ 1|^T$ from $M_0 = |1\ 0\ 0|^T$

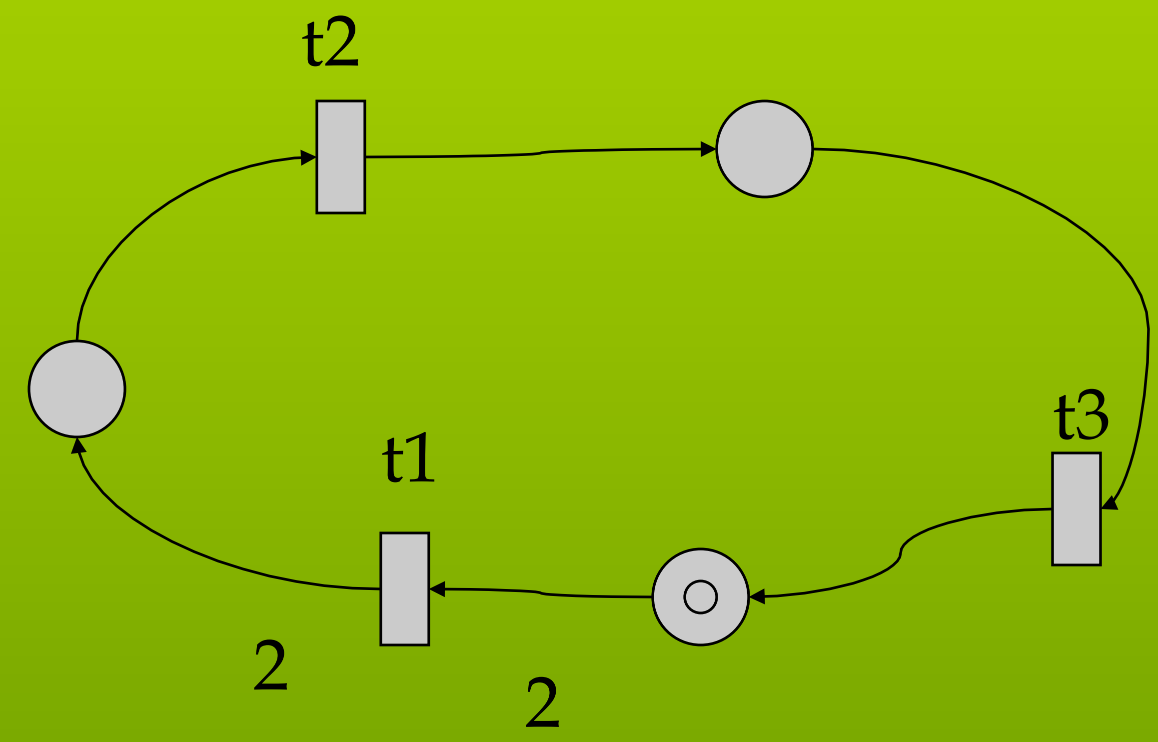


$$A = \begin{array}{ccc} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{array}$$

$$v_1 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \qquad \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

but also $v_2 = |\ 1\ 1\ 2\ |^T$ or any $v_k = |\ 1\ (k)\ (k+1)\ |^T$

Necessary Condition only



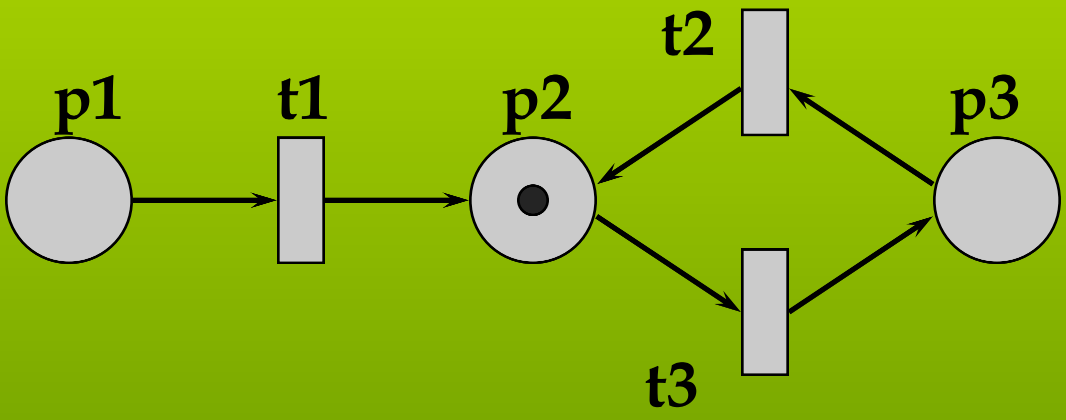
Deadlock!!

State equations and invariants

- **Solutions of $Ax = 0$ (in $M = M_0 + Ax$, $M = M_0$)**

  T-invariants

  - **sequences of transitions that (if fireable) bring back to original marking**
  - **periodic schedule in SDF**
  - **e.g. $x = | 0\ 1\ 1 |^T$**



$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

Application of T-invariants

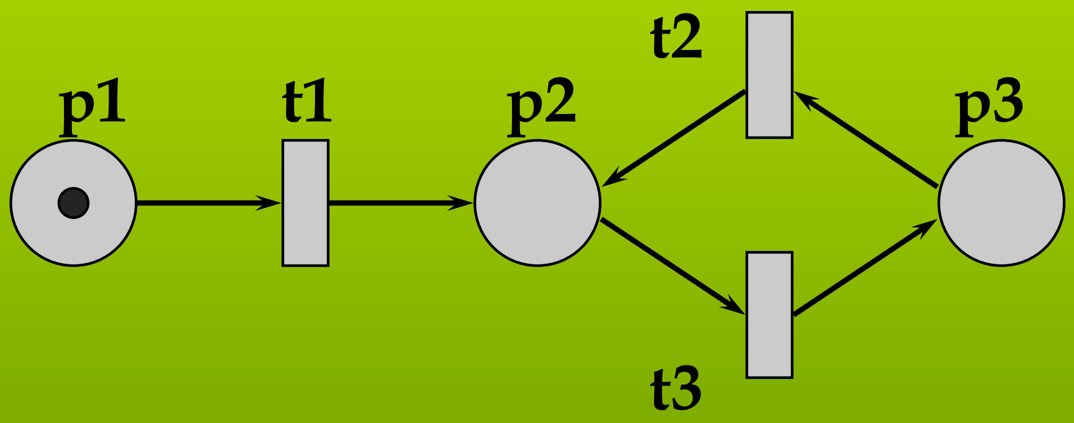
- **Scheduling**
  - **Cyclic schedules: need to return to the initial state**

State equations and invariants

- **Solutions of yA = 0**

  S-invariants

  - **sets of places whose weighted total token count does not change after the firing of any transition (y M = y M')**
  - **e.g. y =| 1 1 1 |^T**



$$A^T = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

Application of S-invariants

- **Structural Boundedness: bounded for any finite initial marking M_0**
- **Existence of a positive S-invariant is CS for structural boundedness**
  - **initial marking is finite**
  - **weighted token count does not change**

Summary of algebraic methods

- **Extremely efficient**

  (polynomial in the size of the net)
- **Generally provide only necessary or sufficient information**
- **Excellent for ruling out some deadlocks or otherwise dangerous conditions**
- **Can be used to infer structural boundedness**

Coverability Tree

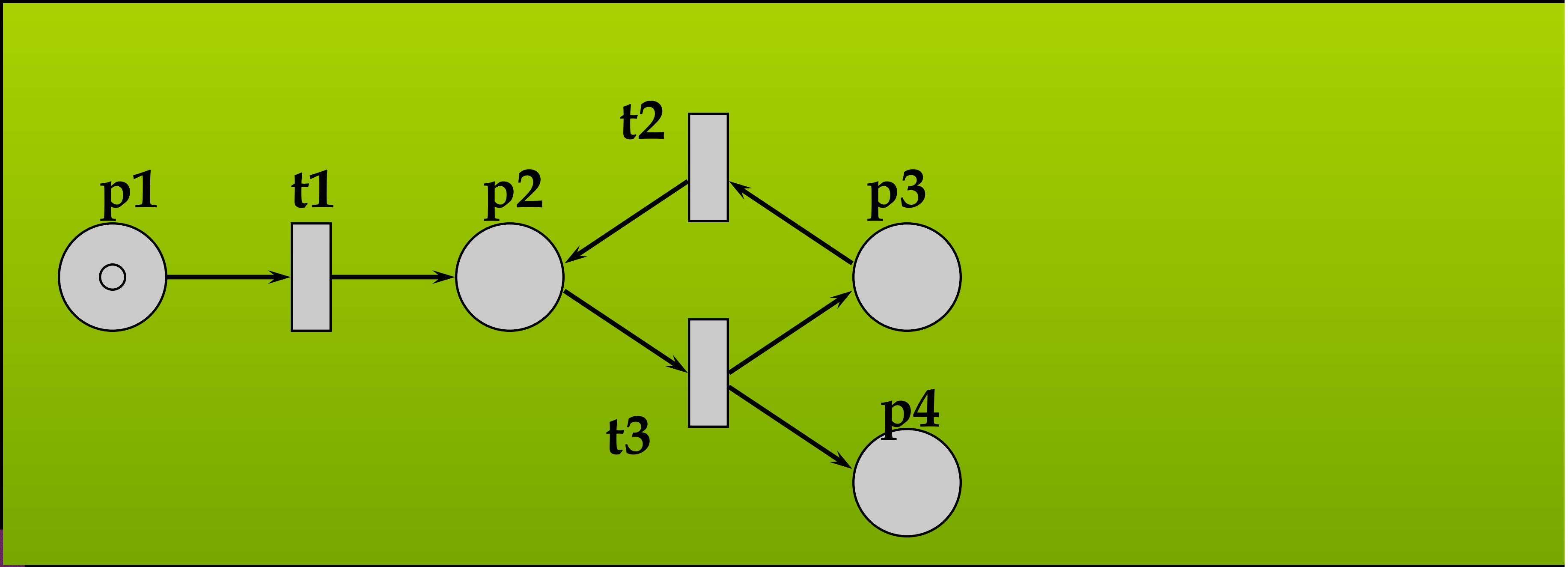
- **Build a (finite) tree representation of the markings**

Karp-Miller algorithm

- **Label initial marking M0 as the root of the tree and tag it as *new***
- **While new markings exist do:**
  - **select a new marking M**
  - **if M is identical to a marking on the path from the root to M, then tag M as *old* and go to another new marking**
  - **if no transitions are enabled at M, tag M *dead-end***
  - **while there exist enabled transitions at M do:**
    - **obtain the marking M' that results from firing t at M**
    - **on the path from the root to M if there exists a marking M'' such that M'(p)>=M''(p) for each place p and M' is different from M'', then replace M'(p) by ω for each p such that M'(p) >M''(p)**
    - **introduce M' as a node, draw an arc with label t from M to M' and tag M' as *new*.**

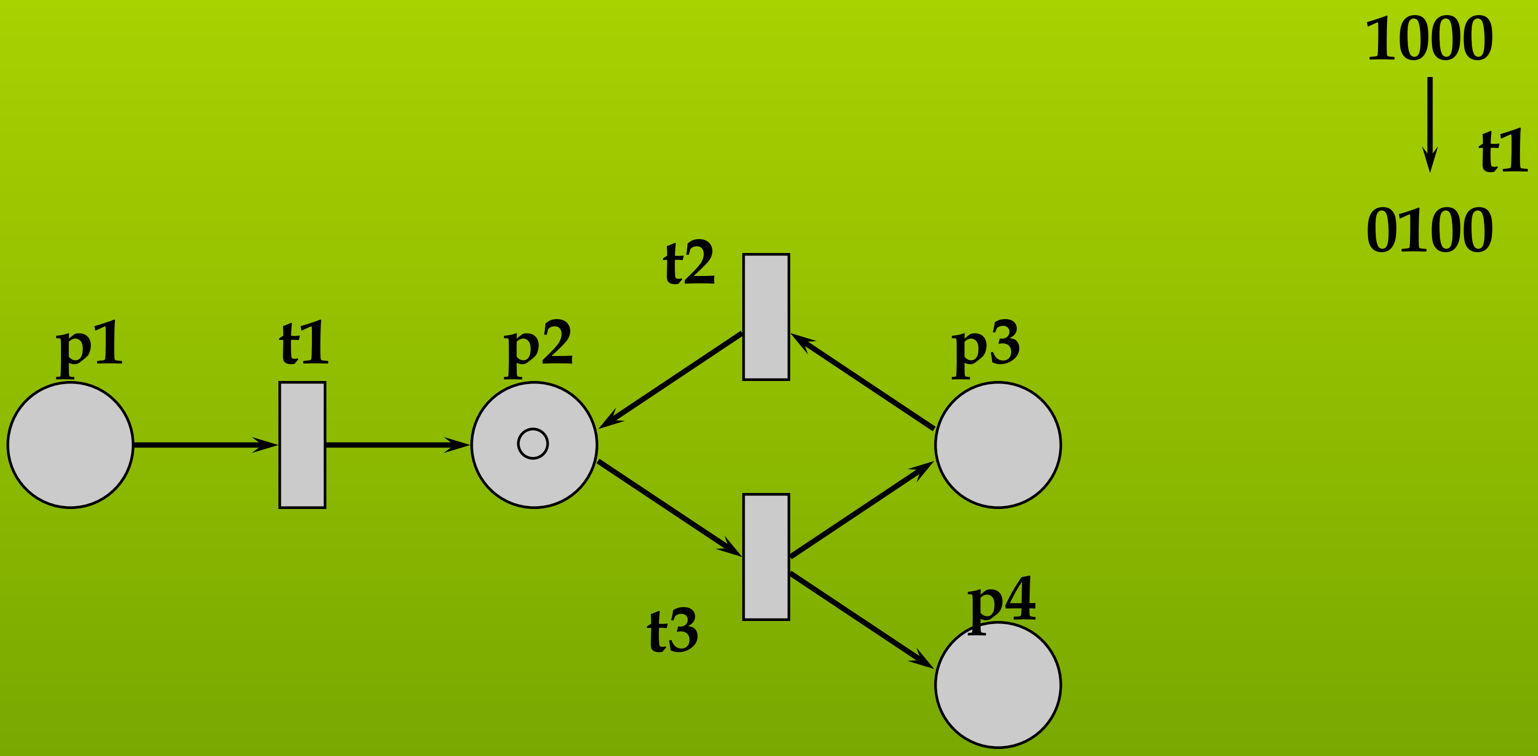
Coverability Tree

- **Boundedness is decidable with *coverability tree***



Coverability Tree

- **Boundedness is decidable with *coverability tree***



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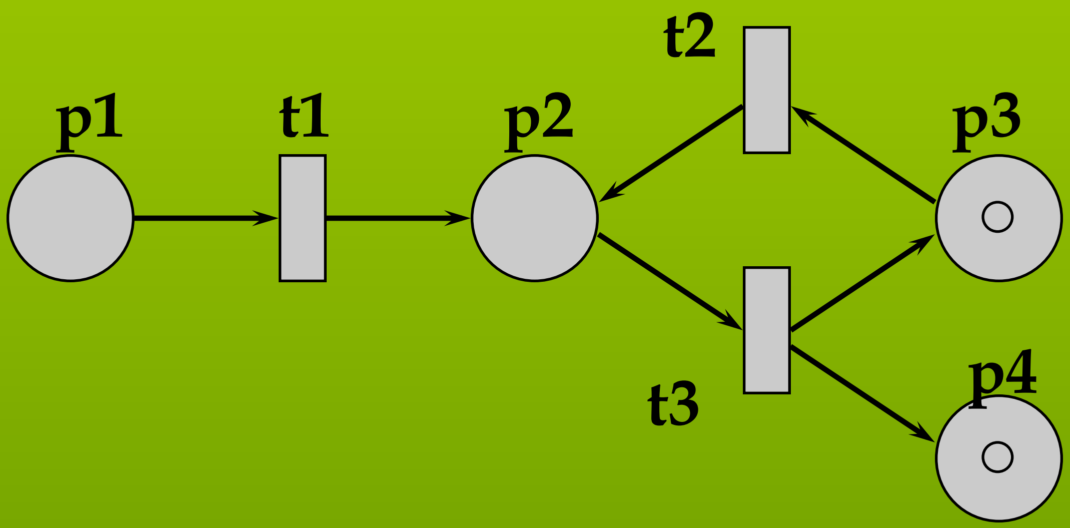

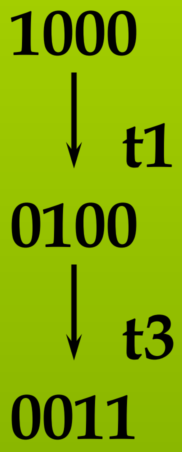

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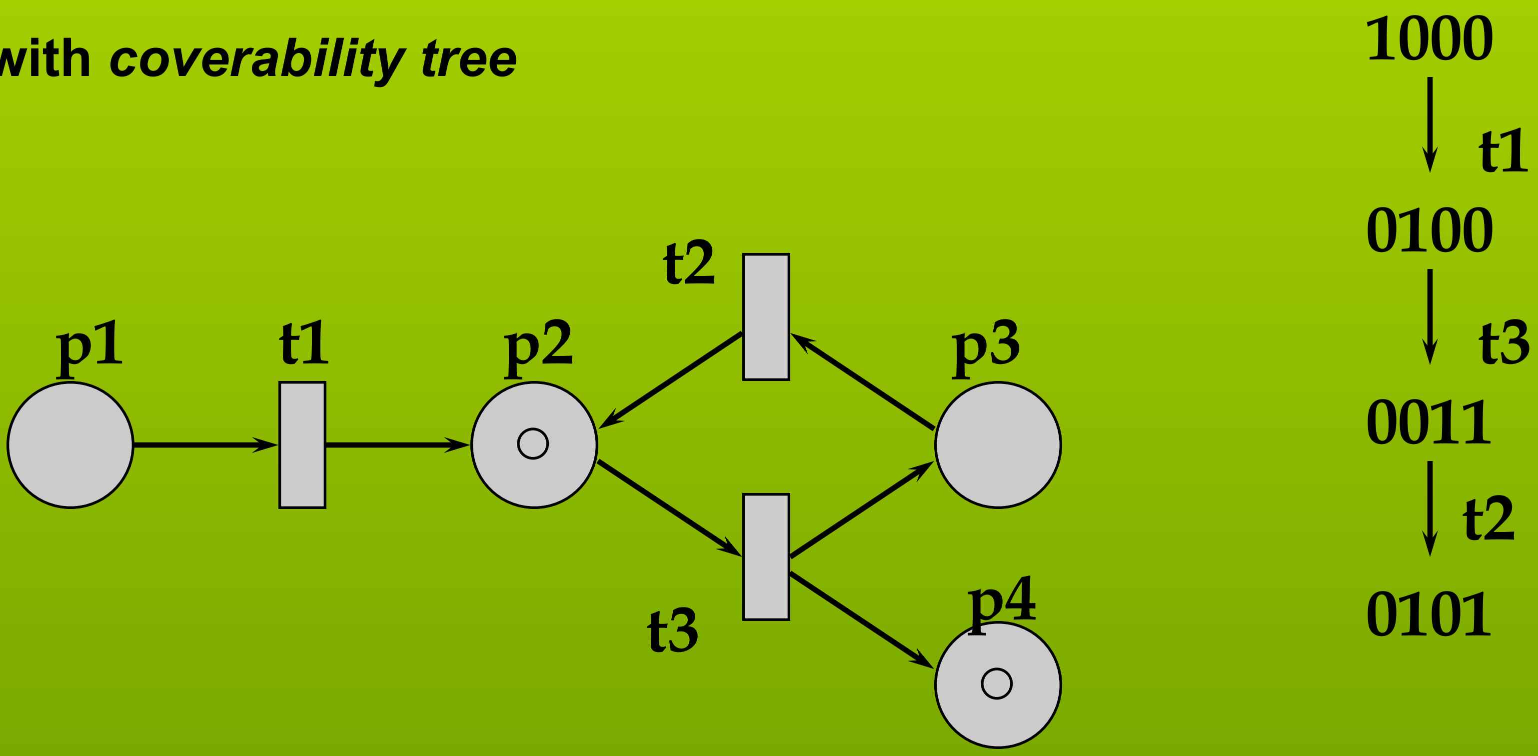

1000

↓ t1

0100

↓ t3

0011

↓ t2

0101

Coverability Tree

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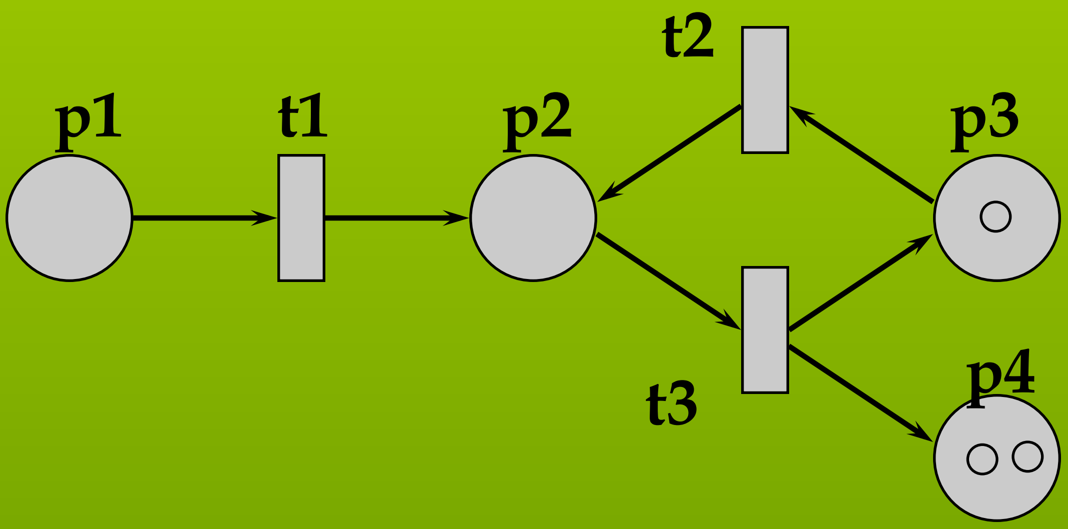
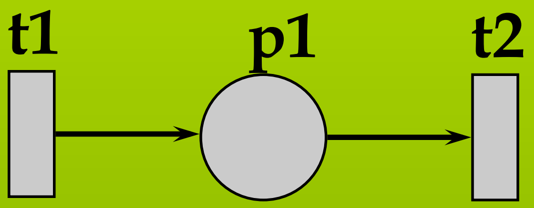
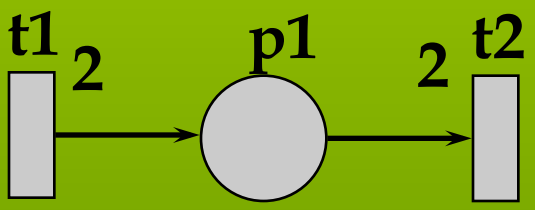


1000
↓ t1
0100
↓ t3
0011
↓ t2
010ω

Coverability Tree

- **Is (1) reachable from (0)?**





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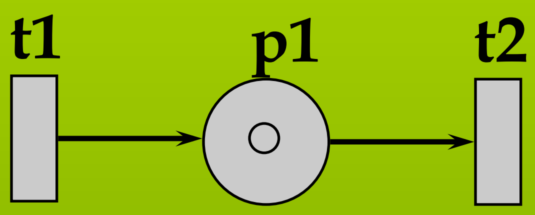
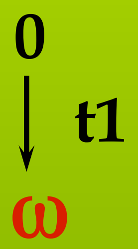
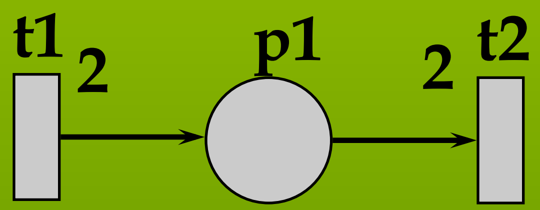




$(0) \longrightarrow (1) \longrightarrow (2) \ldots$



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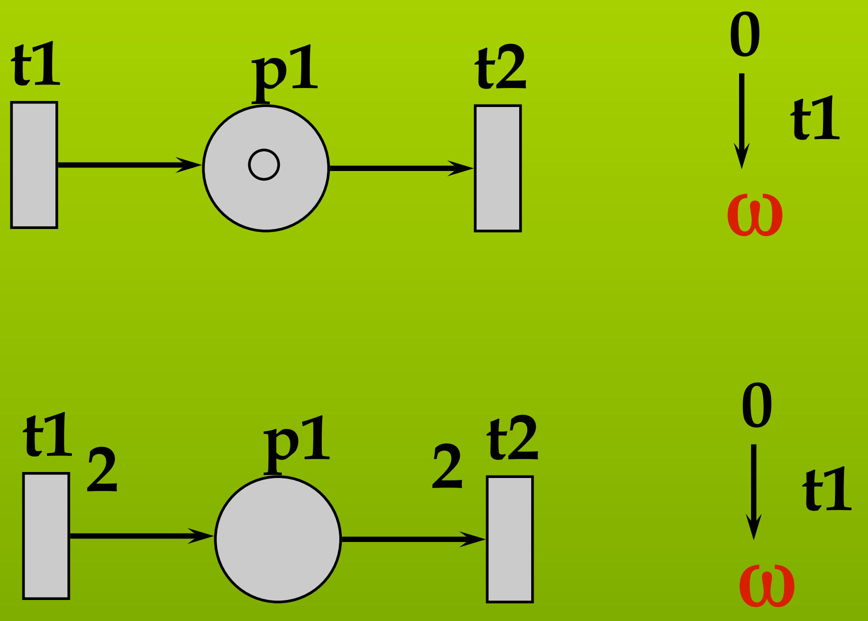

$(0) \Longrightarrow (1) \Longrightarrow (2) \ldots$

$(0) \longrightarrow (2) \longrightarrow (0) \ldots$

Coverability Tree

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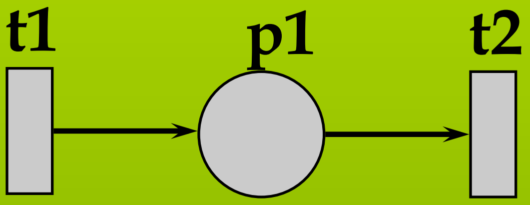
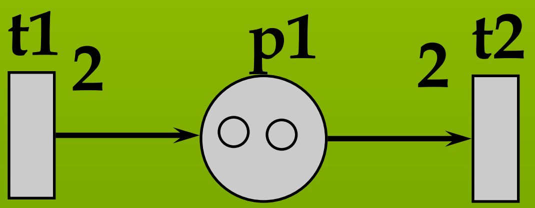


$$\begin{tikzcd} 0 \arrow[d, "t1"] \\ \omega \end{tikzcd}$$

$(0) \longrightarrow (1) \longrightarrow (2)\ldots$

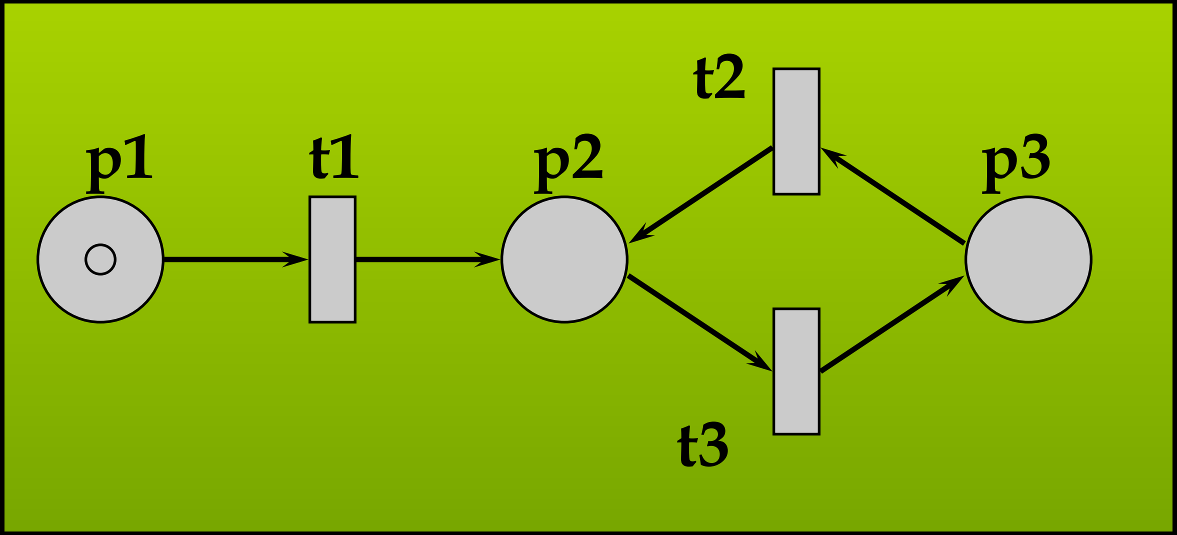


$$\begin{tikzcd} 0 \arrow[d, "t1"] \\ \omega \end{tikzcd}$$

$(0) \longrightarrow (2) \longrightarrow (0)\ldots$

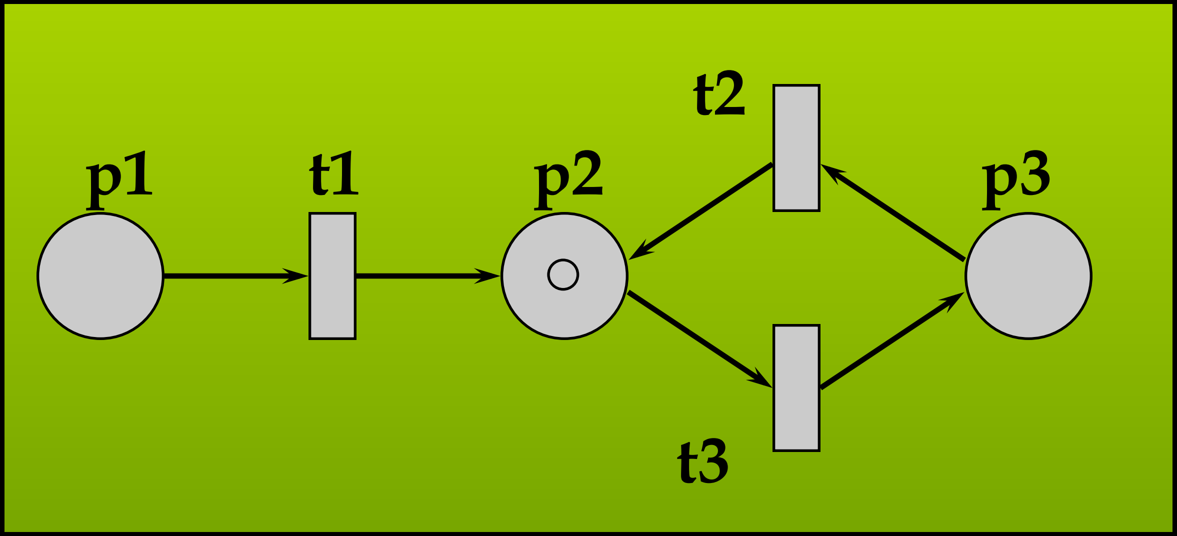
- Cannot solve the reachability problem

Reachability graph



- **For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings**

Reachability graph



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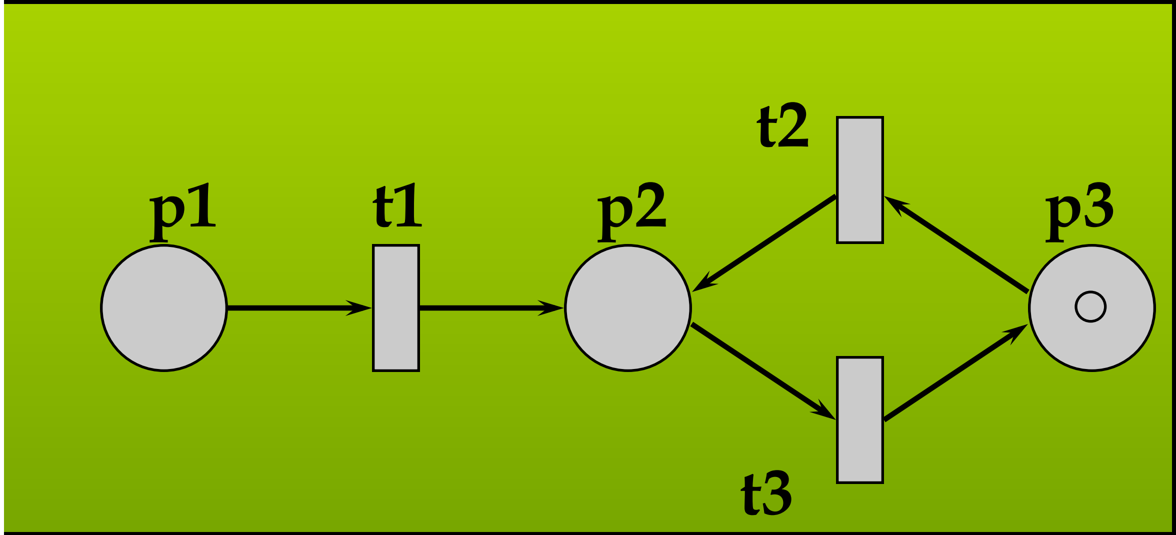
Reachability graph



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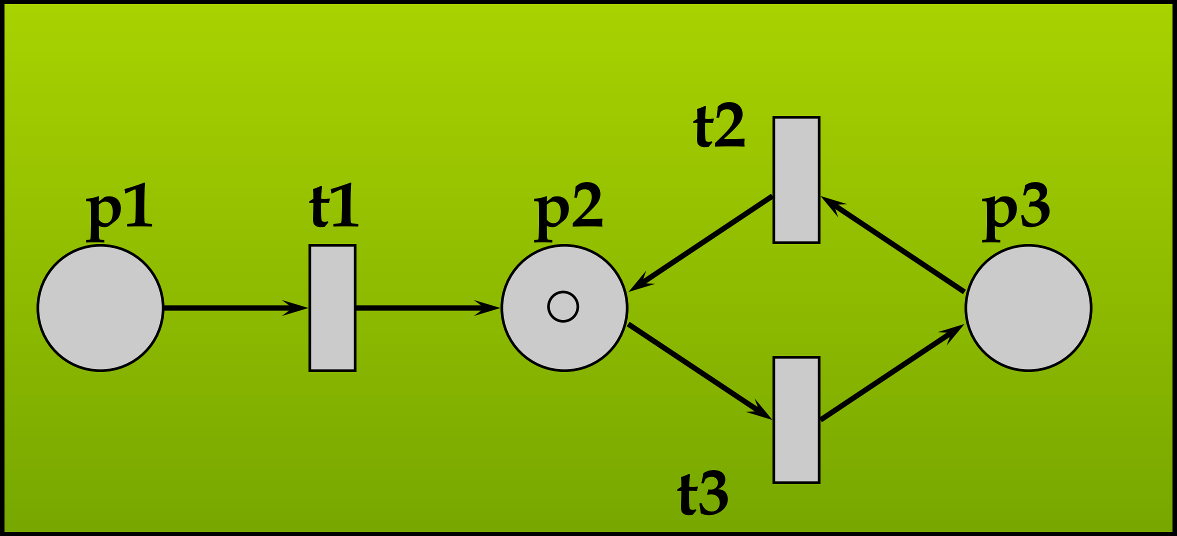
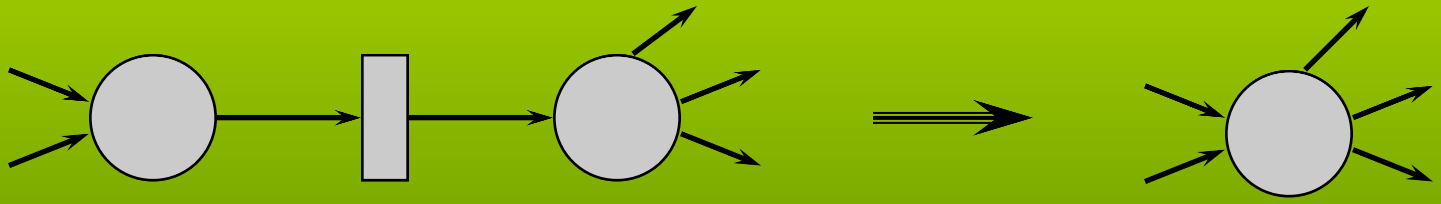


- **For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings**

Subclasses of Petri nets

- Reachability analysis is too expensive
- State equations give only partial information
- Some properties are preserved by reduction rules

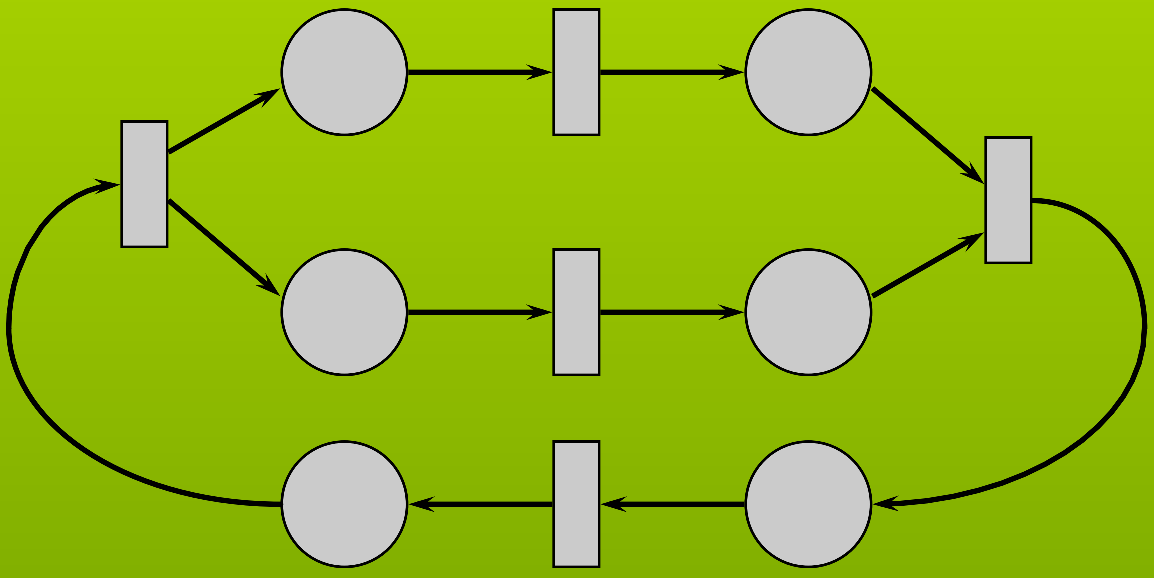
  e.g. for liveness and safeness



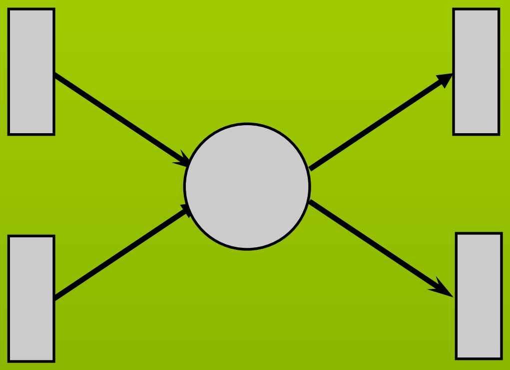
- Even reduction rules only work in some cases
- Must restrict class in order to prove stronger results

Marked Graphs

- **Every place has at most 1 predecessor and 1 successor transition**
- **Models only causality and concurrency (no conflict)**



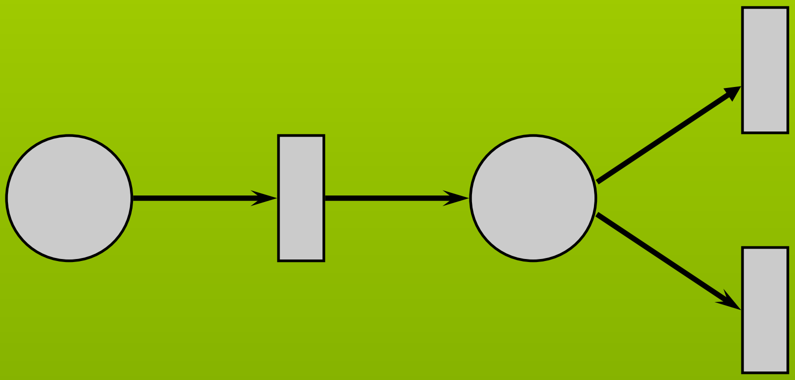
YES



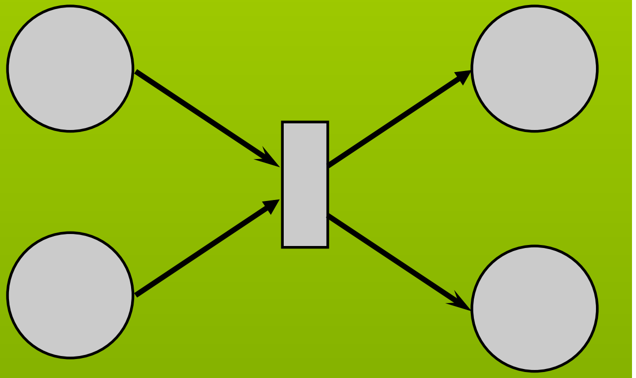
NO

State Machines

- **Every transition has at most 1 predecessor and 1 successor place**
- **Models only causality and conflict**
  - **(no concurrency, no synchronization of parallel activities)**

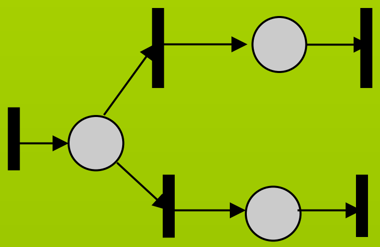


YES



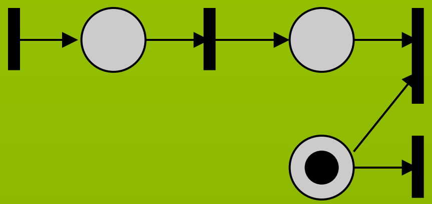
NO

Free-Choice Petri Nets (FCPN)

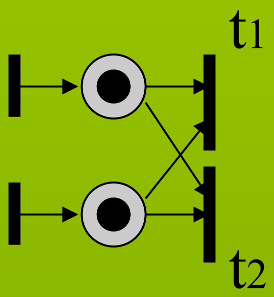


Free-Choice (FC)

every transition after choice has exactly 1 predecessor place



Confusion (not-Free-Choice)



Extended Free-Choice

Free-Choice: the outcome of a choice depends on the value of a token (abstracted non-deterministically) rather than on its arrival time.

Free-Choice nets

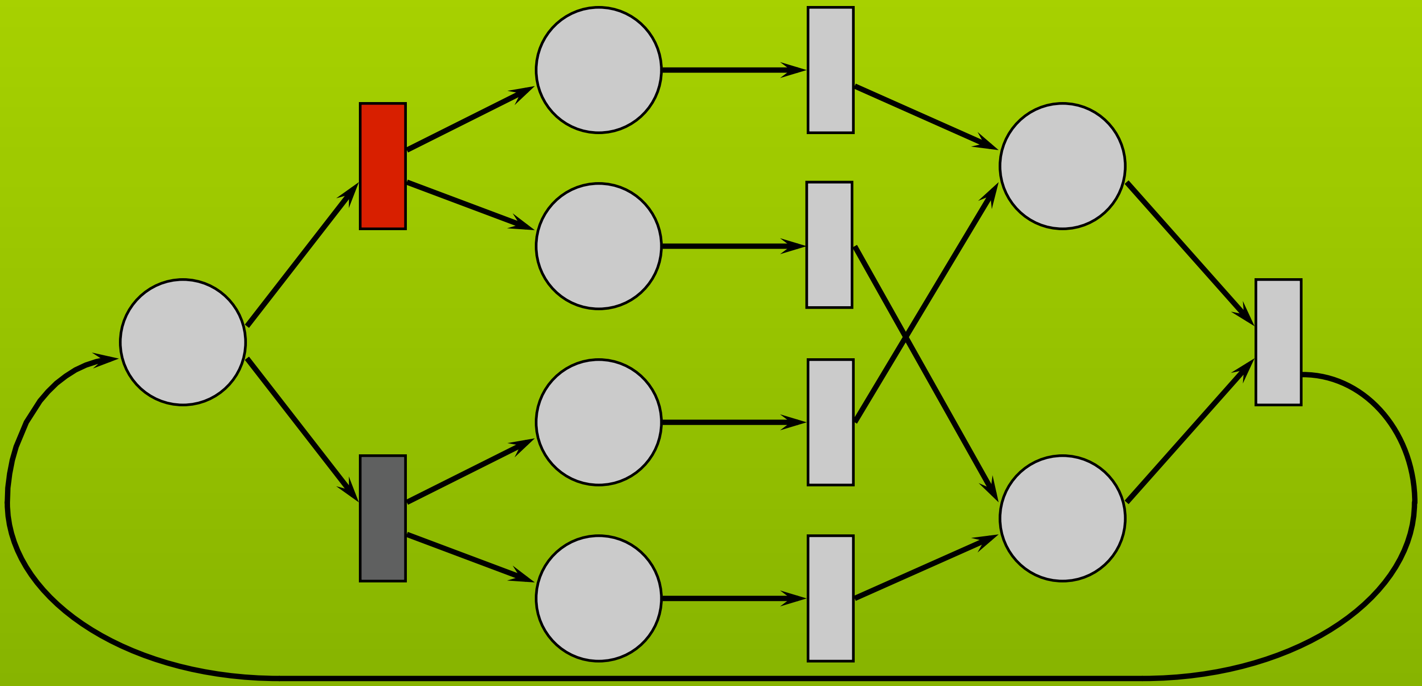
- **Introduced by Hack (‘72)**
- **Extensively studied by Best (‘86) and Desel and Esparza (‘95)**
- **Can express concurrency, causality and choice without confusion**
- **Very strong structural theory**
  - **necessary and sufficient conditions for liveness and safeness, based on decomposition**
  - **exploits duality between MG and SM**

MG (& SM) decomposition

- **An Allocation is a control function that chooses which transition fires among several conflicting ones (A: P T).**
- **Eliminate the subnet that would be inactive if we were to use the allocation...**
- **Reduction Algorithm**
  - **Delete all unallocated transitions**
  - **Delete all places that have all input transitions already deleted**
  - **Delete all transitions that have at least one input place already deleted**
- **Obtain a Reduction (one for each allocation) that is a conflict free subnet**

MG reduction and cover

- **Choose one successor for each conflicting place:**



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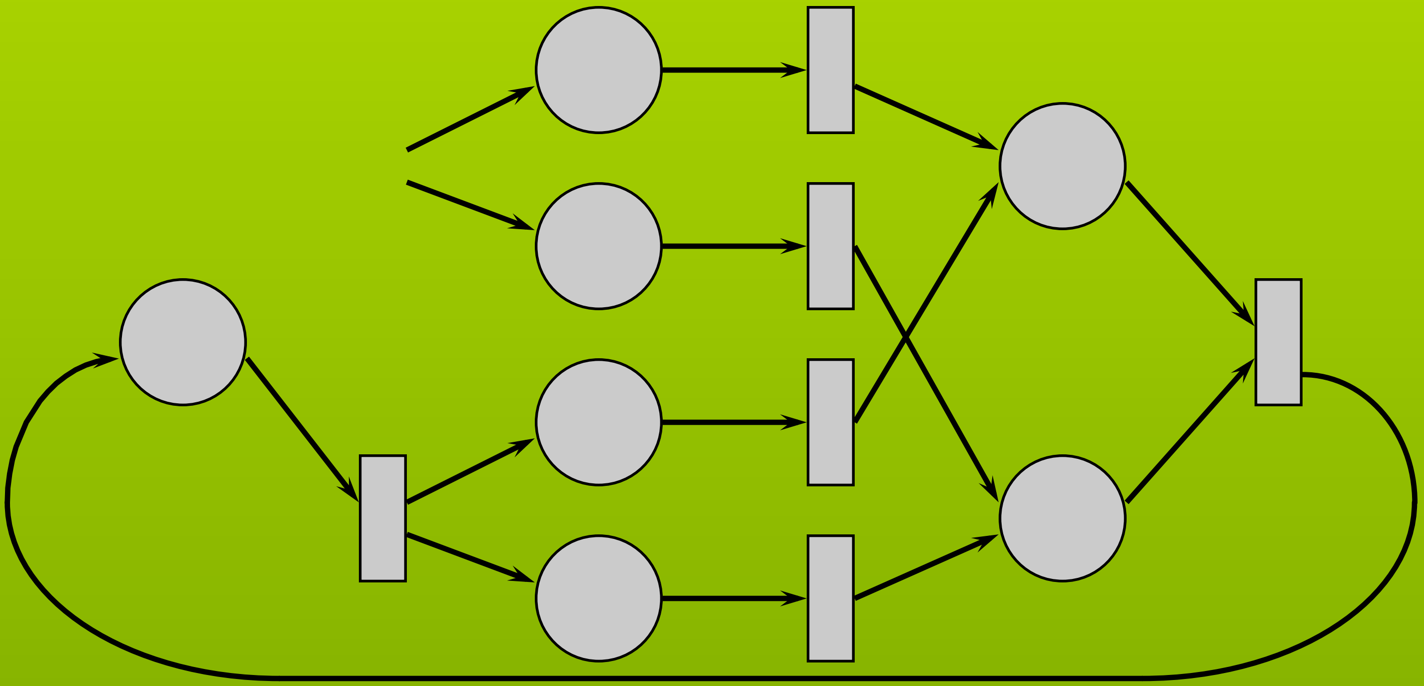

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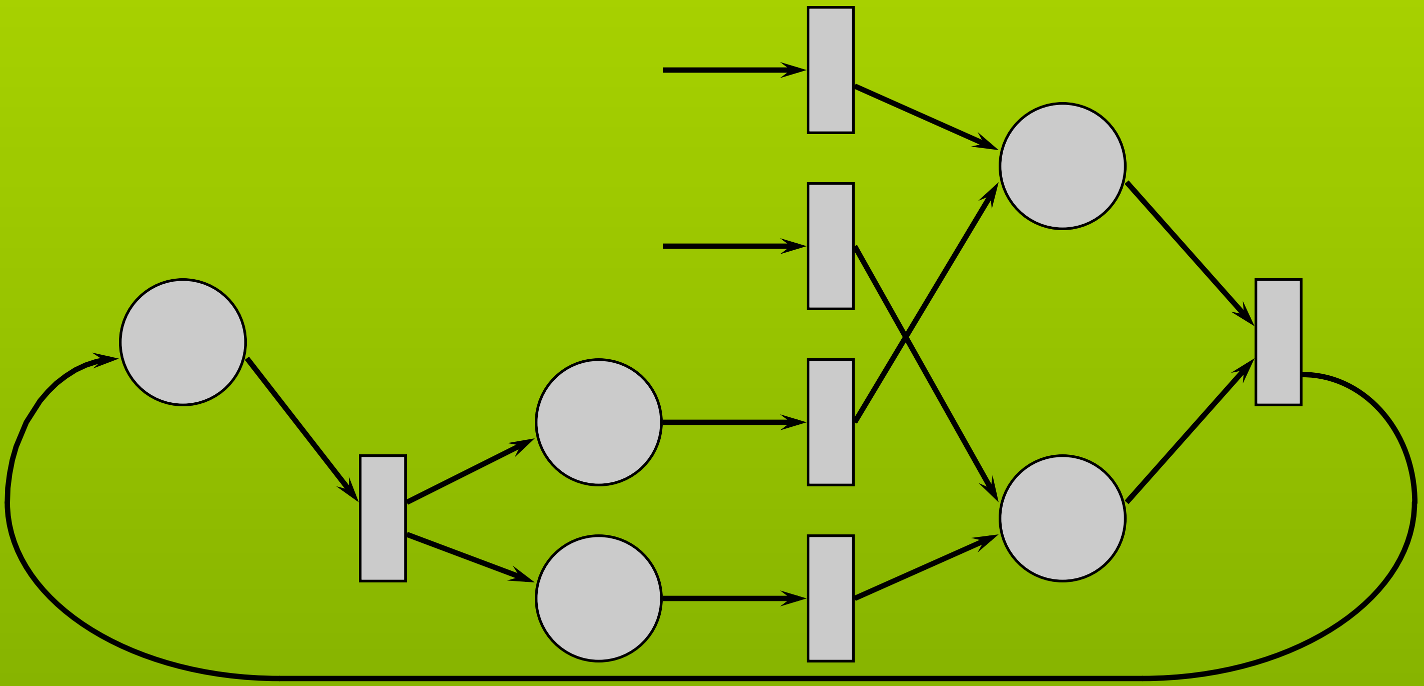

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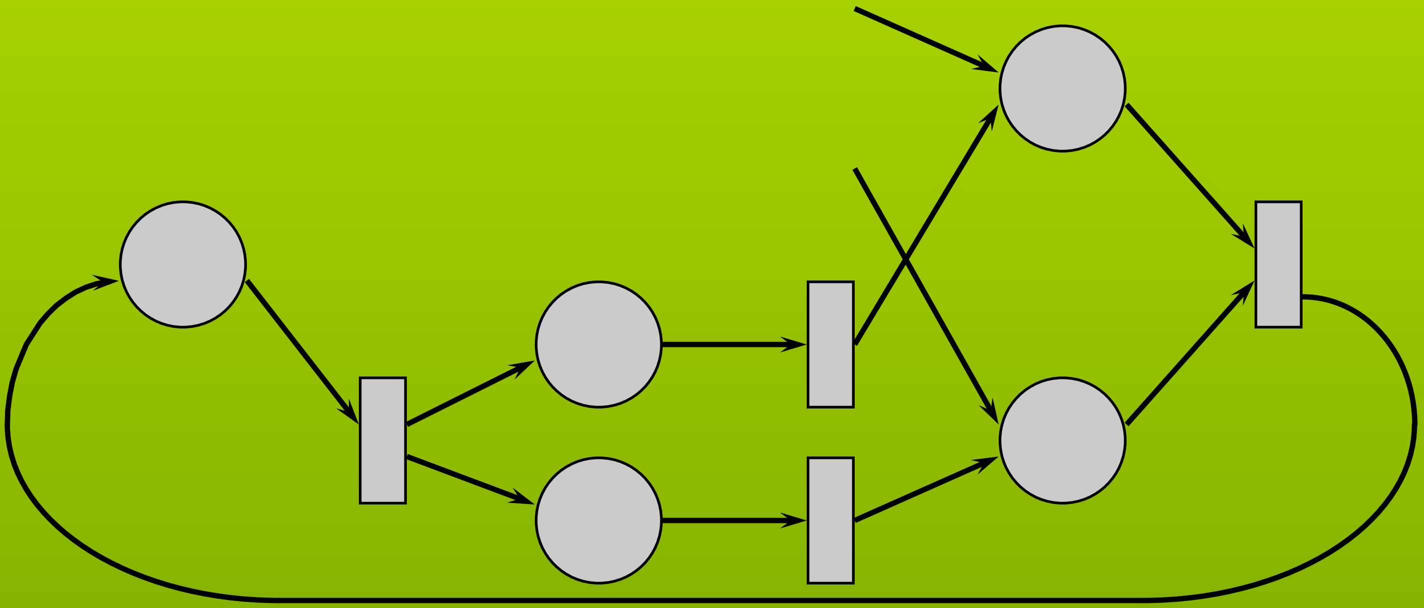

MG reduction and cover

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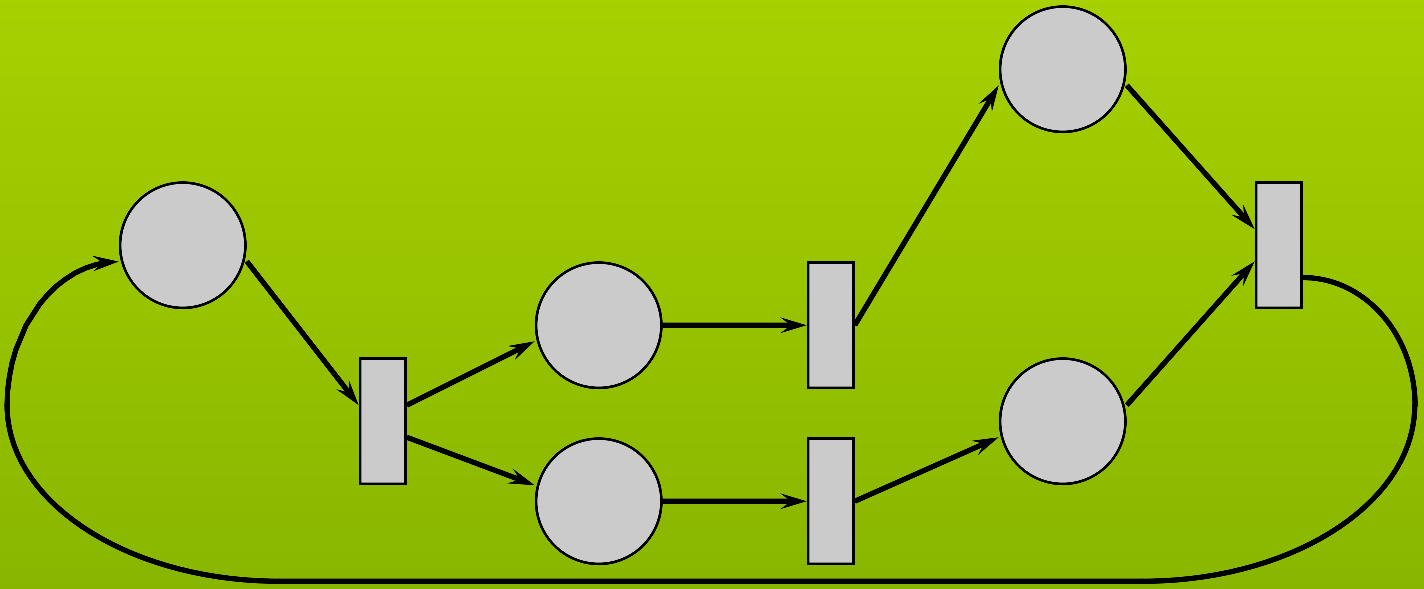
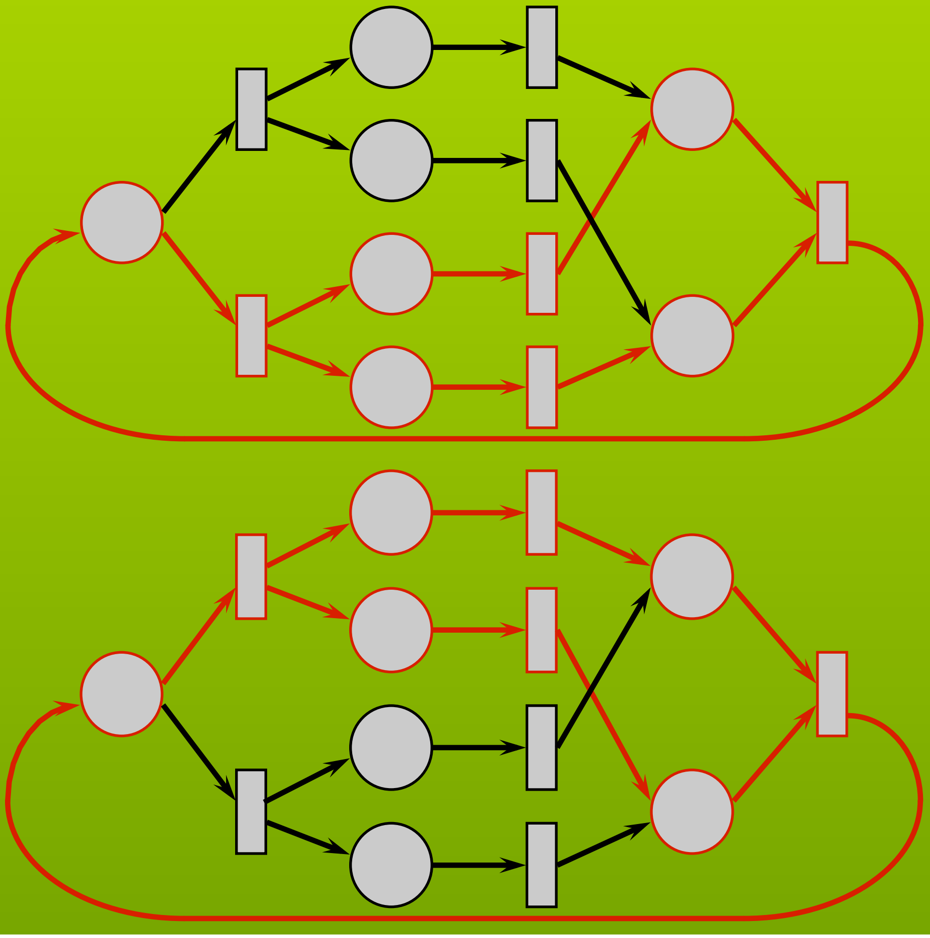


MG reductions

- **The set of all reductions yields a cover of MG components (T-invariants)**



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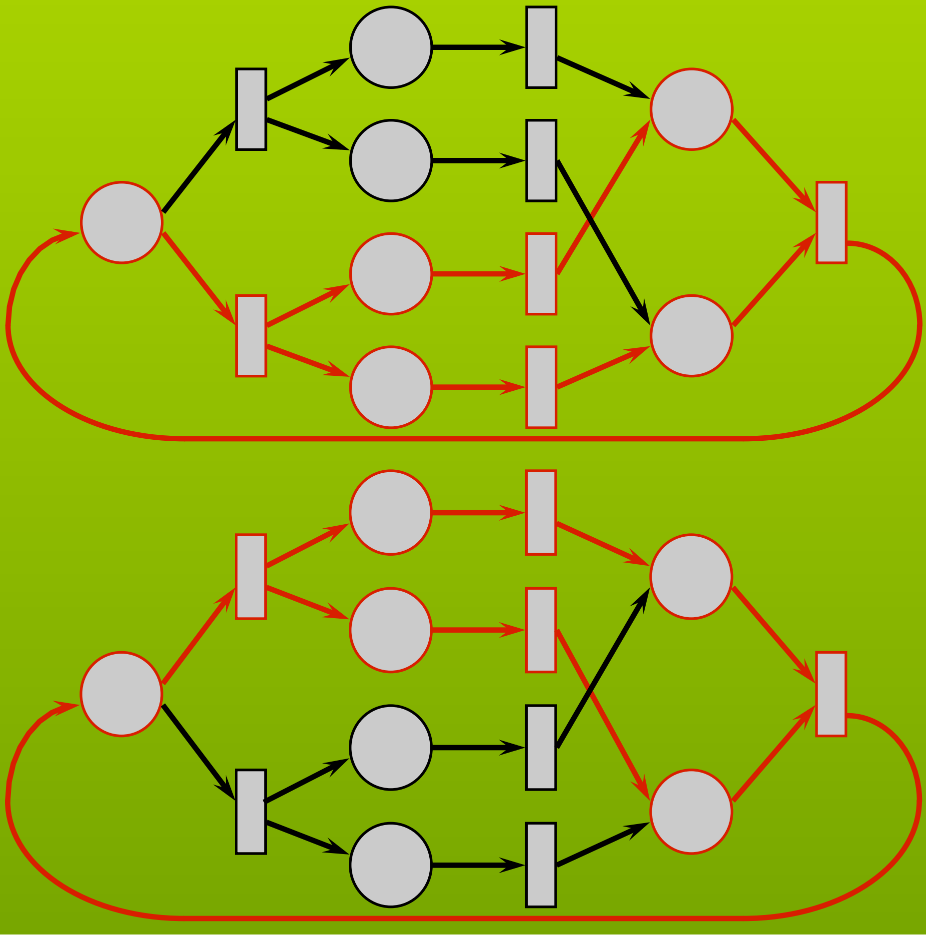

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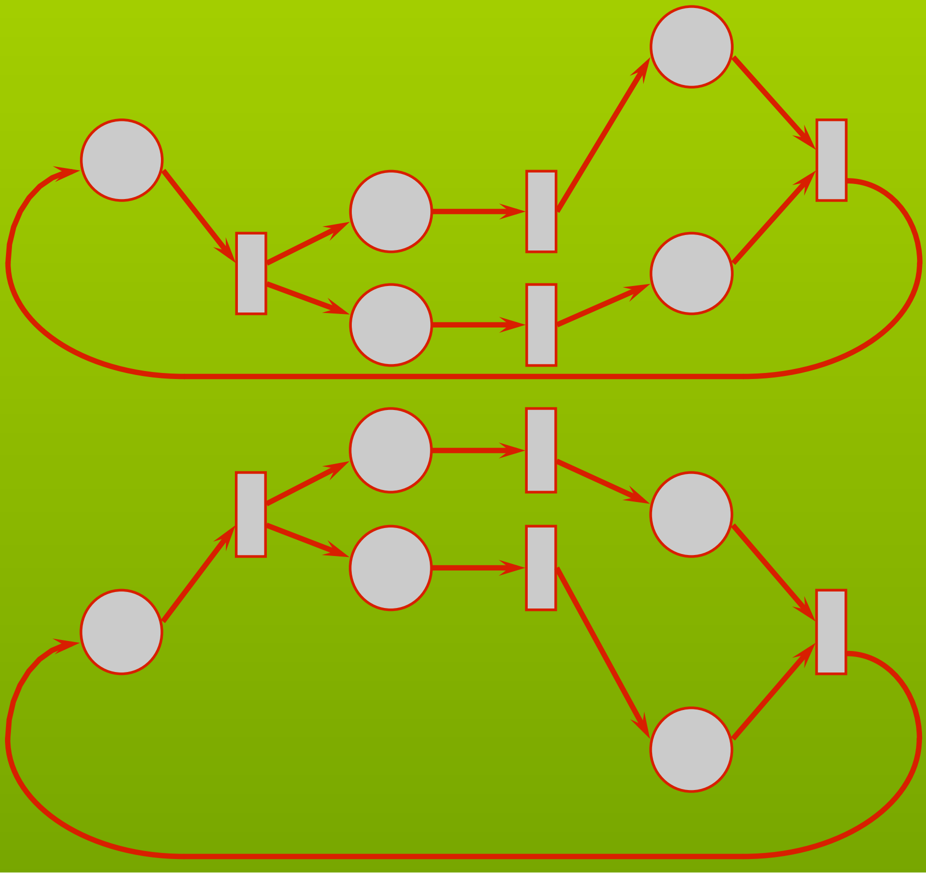


Hack’s theorem (‘72)

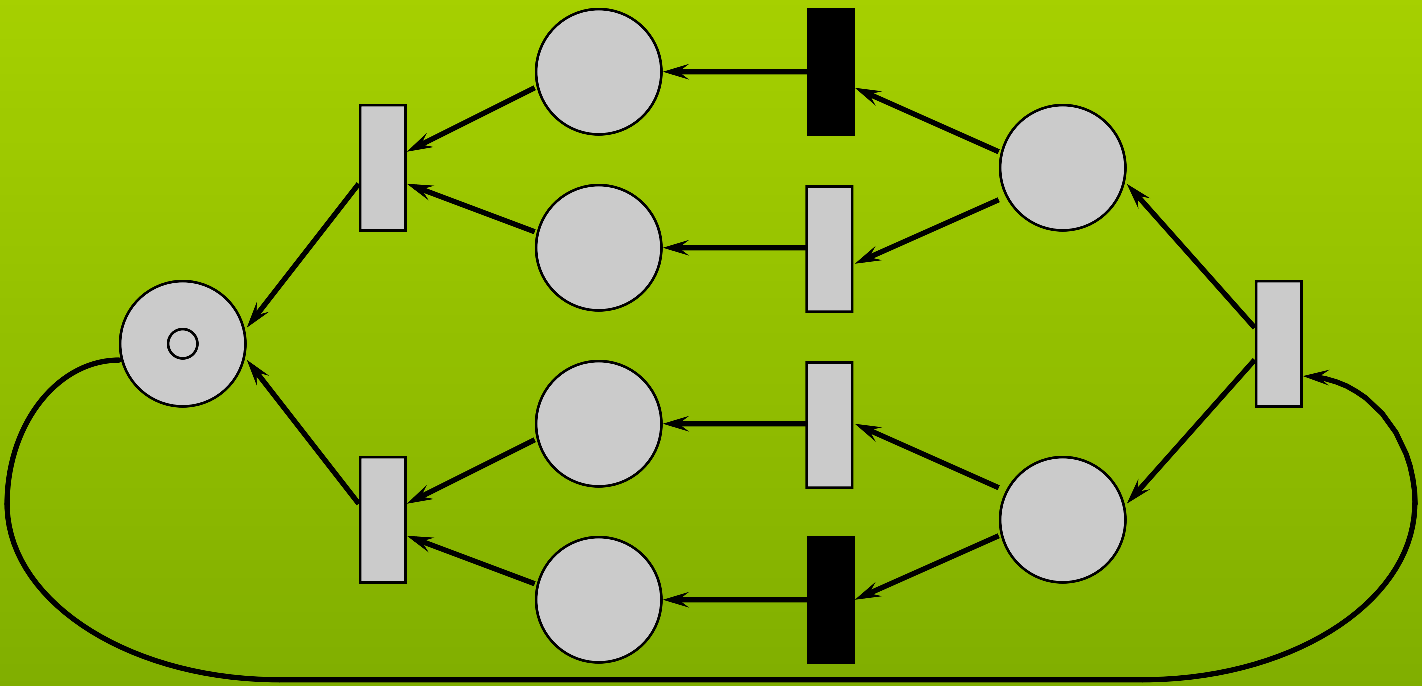
- **Let N be a Free-Choice PN:**
  - **N has a live and safe initial marking (well-formed)**

if and only if

  - **every MG reduction is strongly connected and not empty, and the set of all reductions covers the net**
  - **every SM reduction is strongly connected and not empty, and the set of all reductions covers the net**

Hack’s theorem

- **Example of non-live (but safe) FCN**



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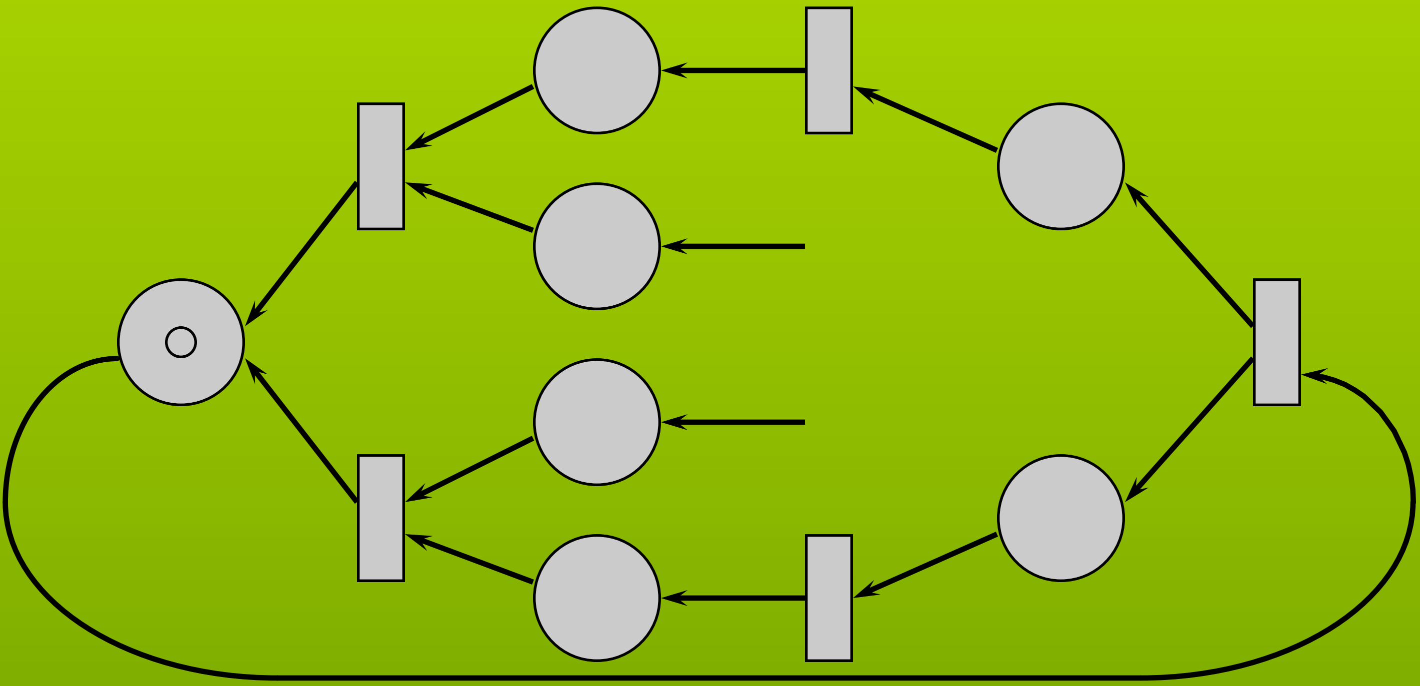

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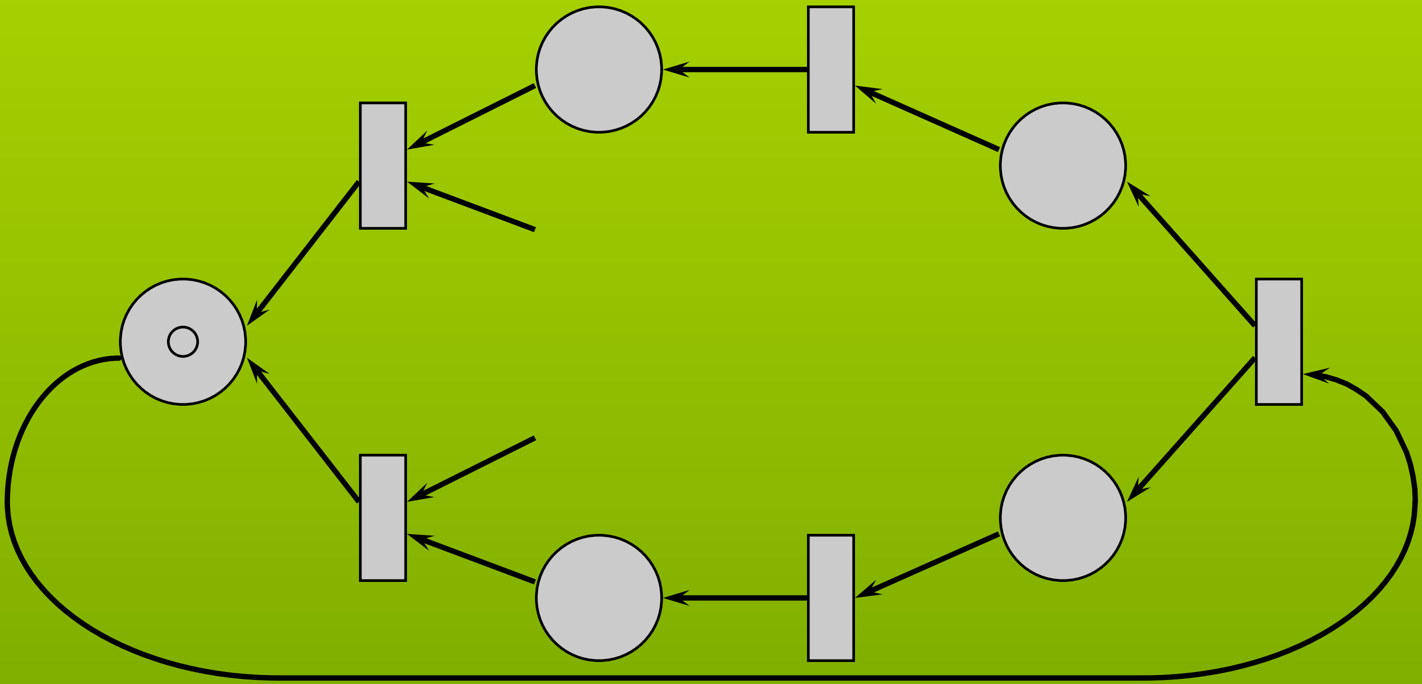

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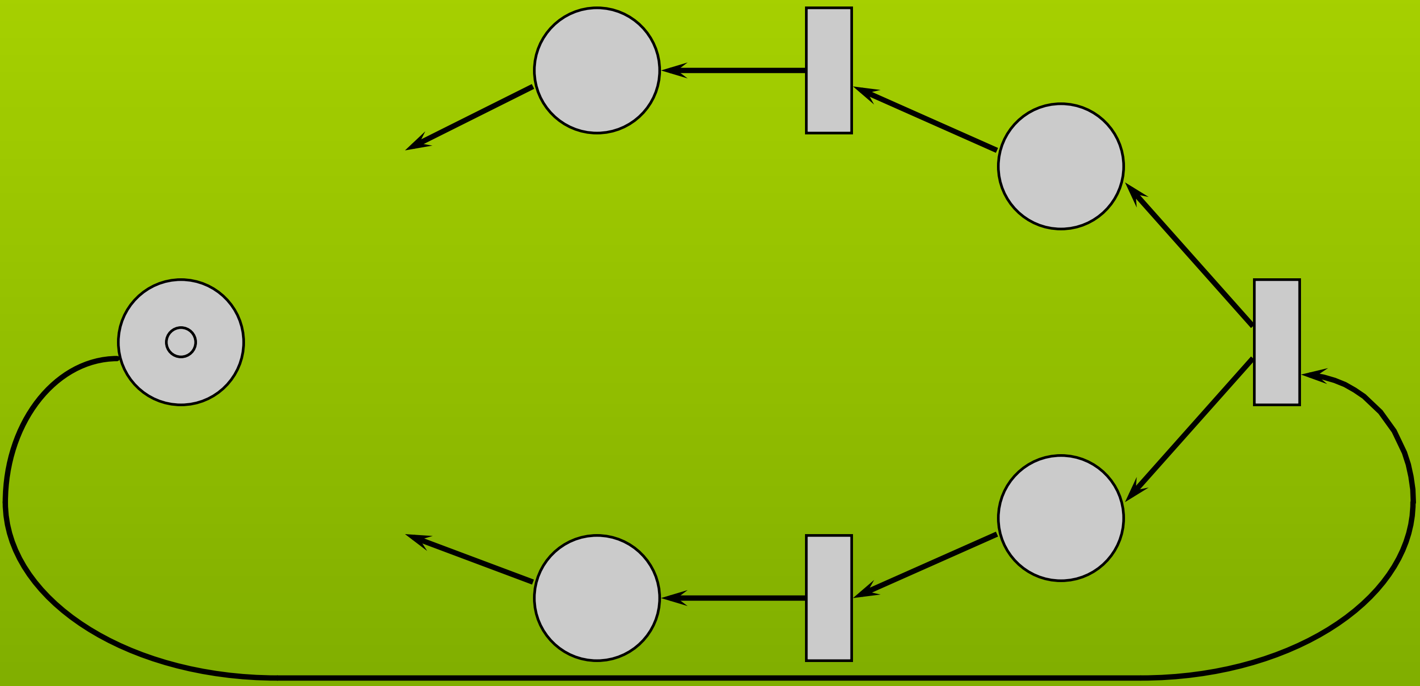

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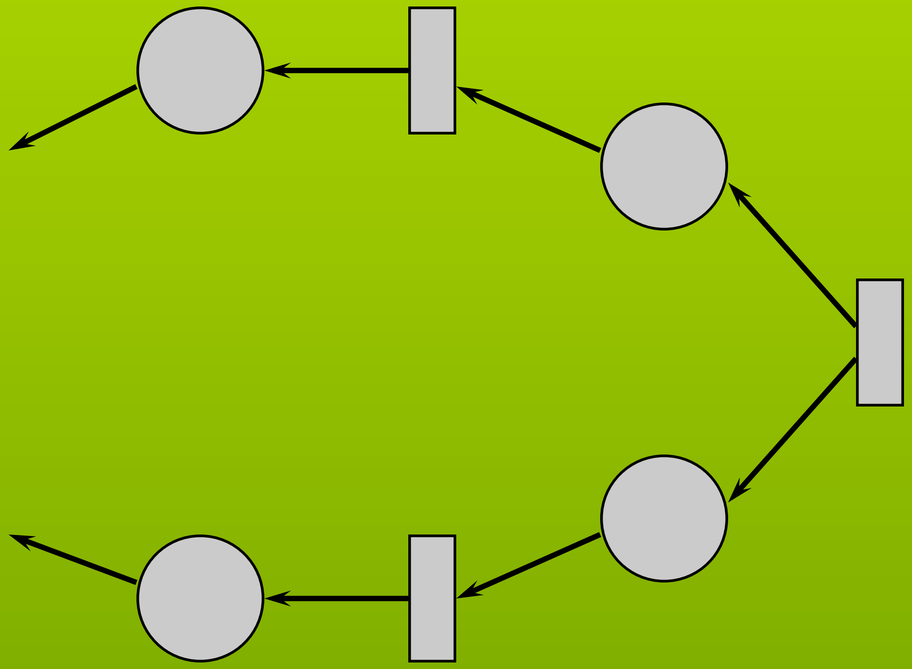

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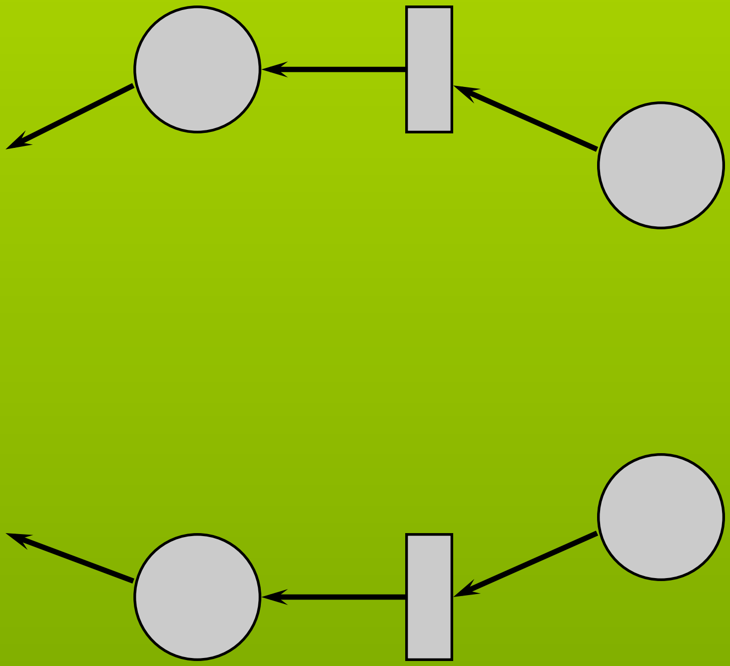

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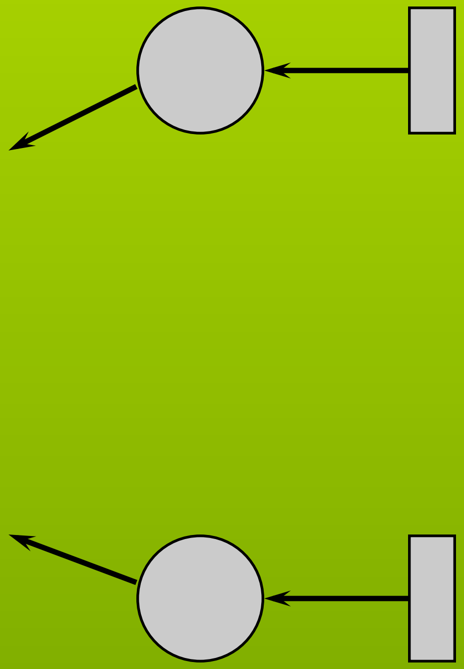

Hack's theorem

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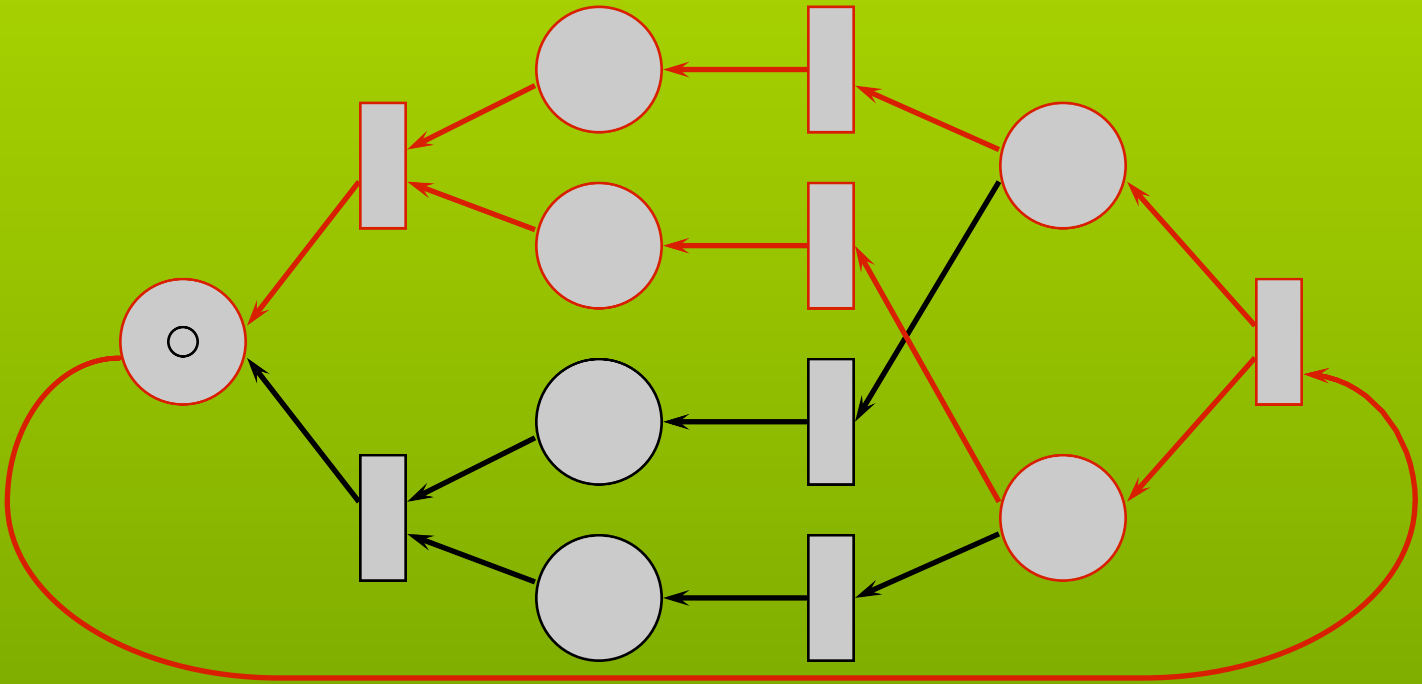

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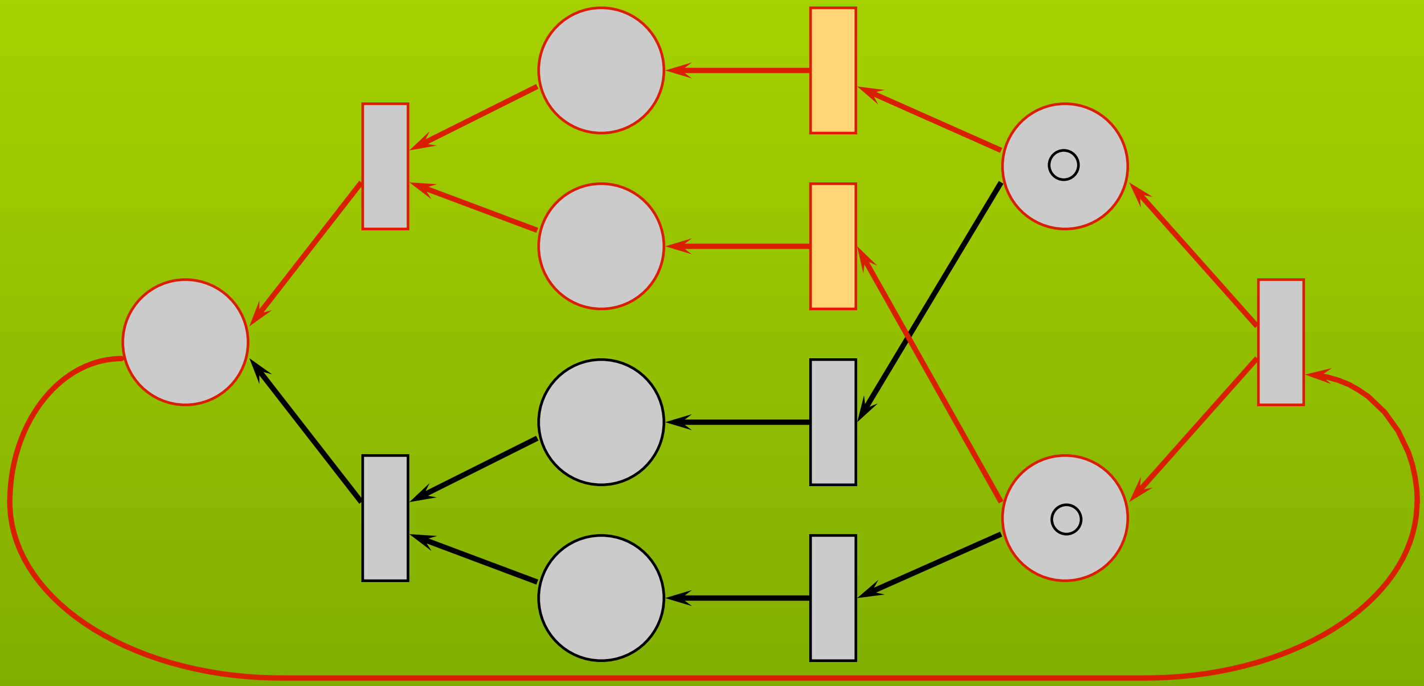

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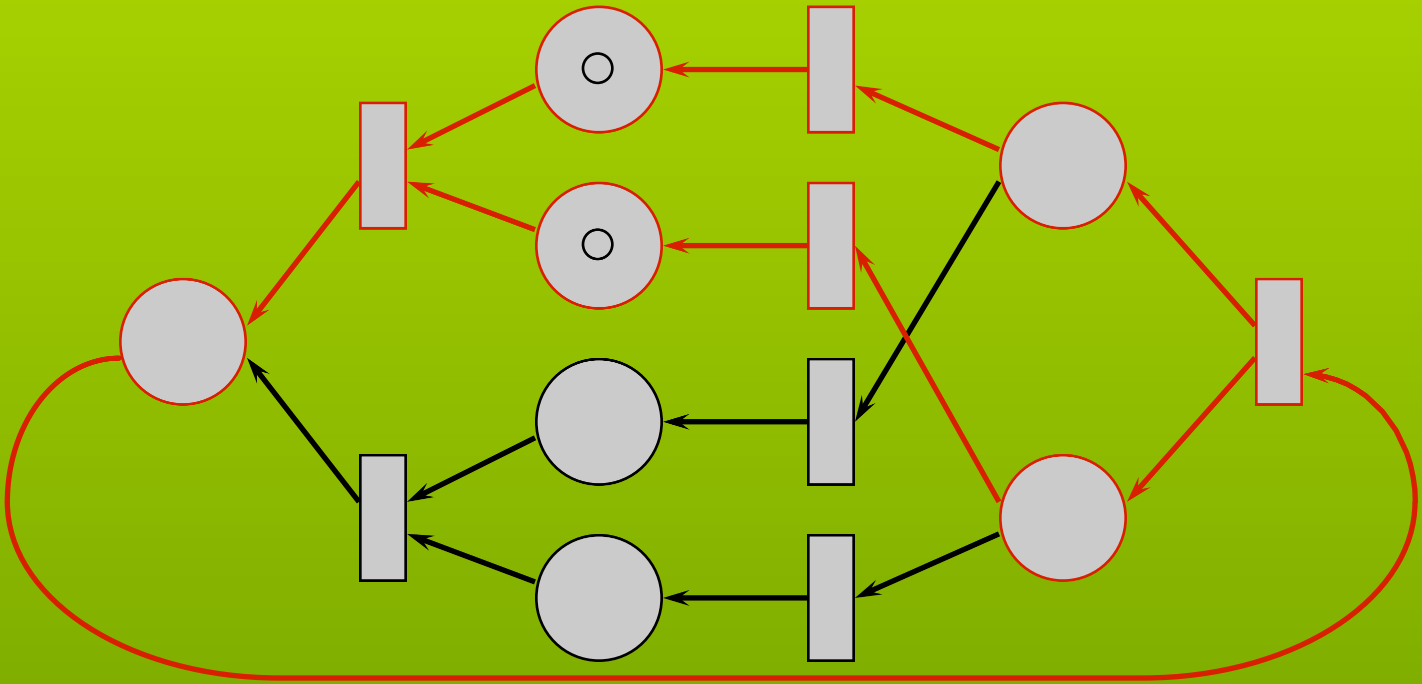

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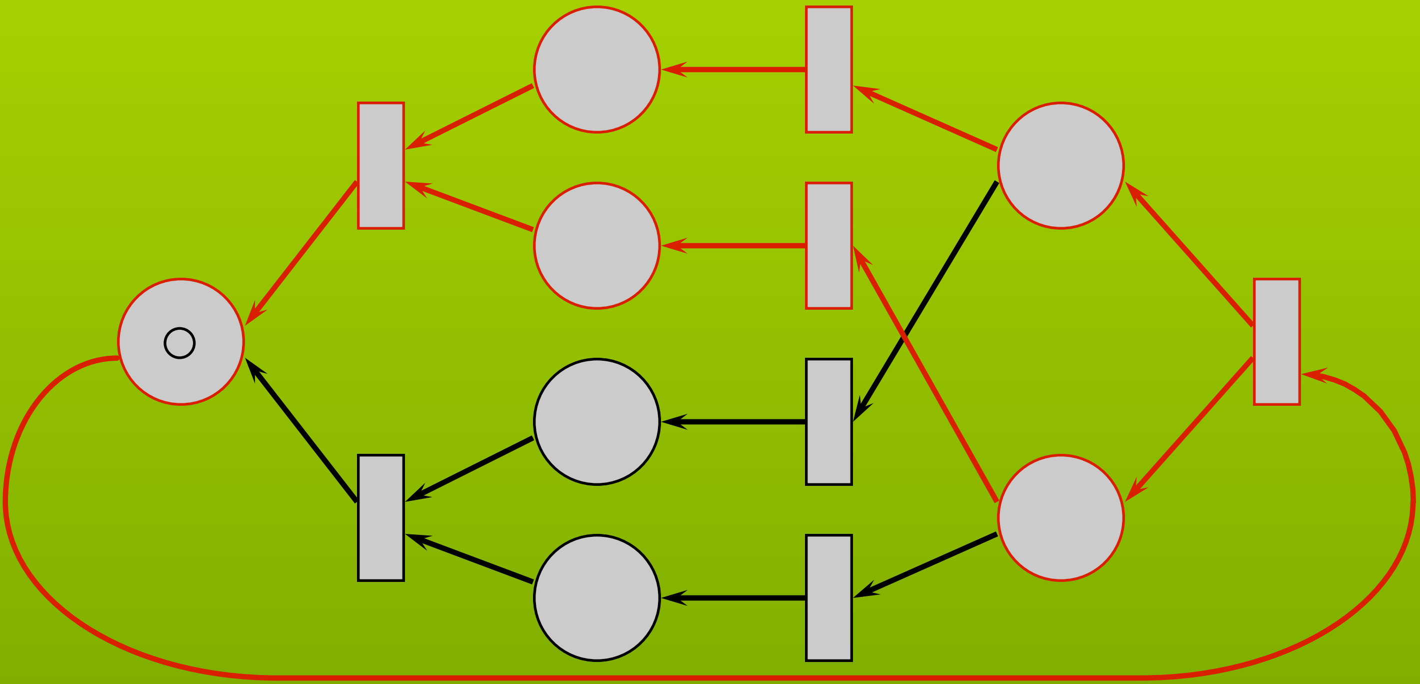

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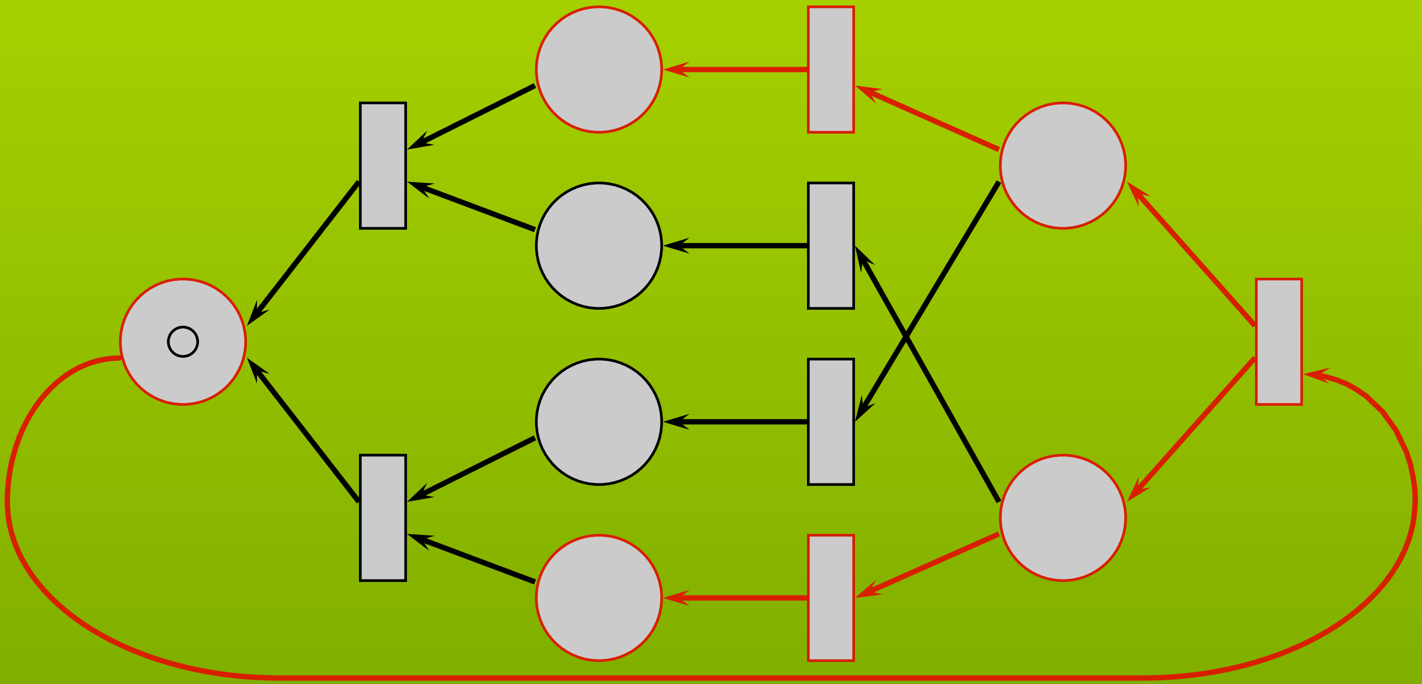

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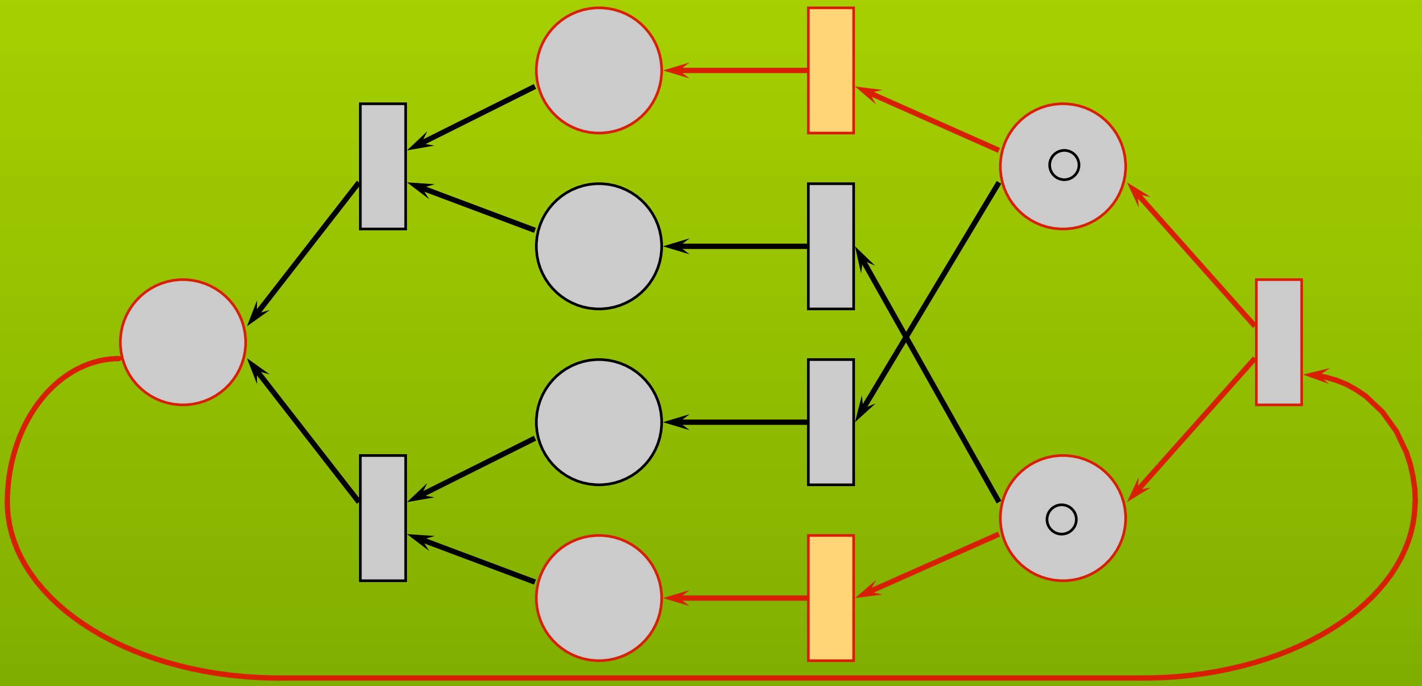

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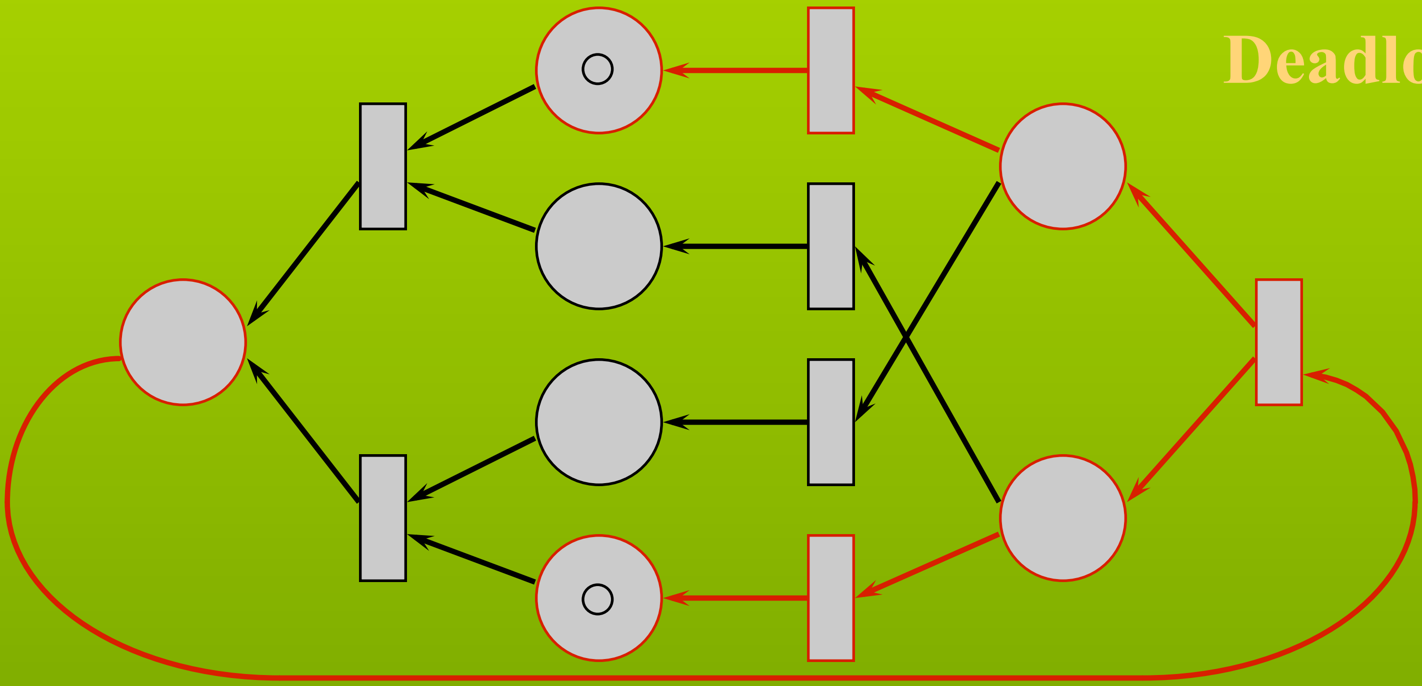


Deadlock

Summary of LSFC nets

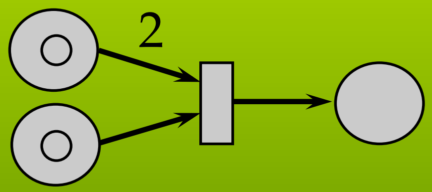
- **Largest class for which structural theory really helps**
- **Structural component analysis may be expensive**

  (exponential number of MG and SM components in the worst case)
- **But…**
  - **number of MG components is generally small**
  - **FC restriction simplifies characterization of behavior**

Petri Net extensions

- Add interpretation to tokens and transitions
  - Colored nets (tokens have value)
- Add time
  - Time/timed Petri Nets (deterministic delay)
    - type (duration, delay)
    - where (place, transition)
  - Stochastic PNs (probabilistic delay)
  - Generalized Stochastic PNs (timed and immediate transitions)
- Add hierarchy
  - Place Charts Nets

PNs Summary



- PN Graph: places (buffers), transitions (actions), tokens (data)
- Firing rule: transition enabled if there are enough tokens in each input place
- Properties
  - Structural (consistency, structural boundedness…)
  - Behavioral (reachability, boundedness, liveness…)
- Analysis techniques
  - Structural (only CN or CS): State equations, Invariants
  - Behavioral: coverability tree
- Reachability
- Subclasses: Marked Graphs, State Machines, Free-Choice PNs

References

- T. Murata **Petri Nets: Properties, Analysis and Applications**
- http://www.daimi.au.dk/PetriNets/