



Outline

- **Petri nets**
 - **Introduction**
 - **Examples**
 - **Properties**
 - **Analysis techniques**

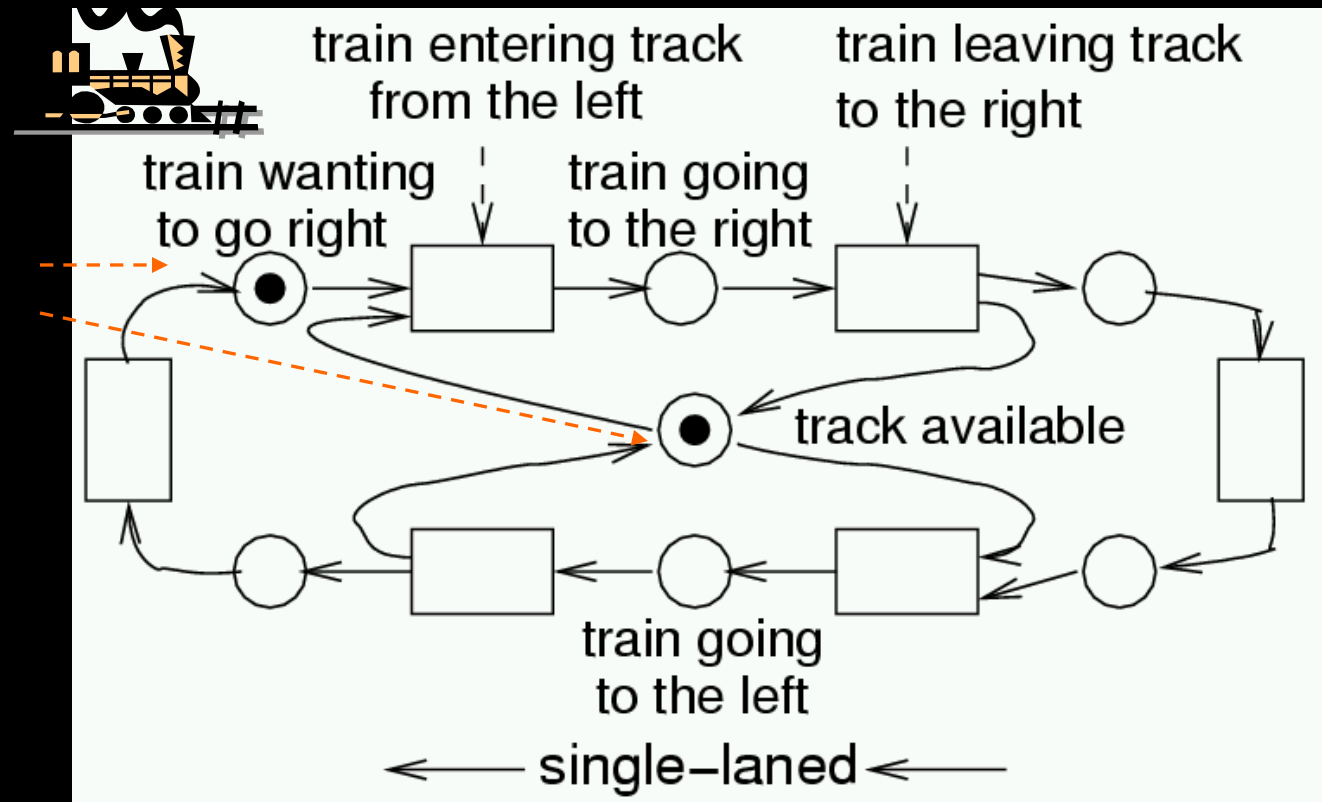


Petri Nets (PNs)

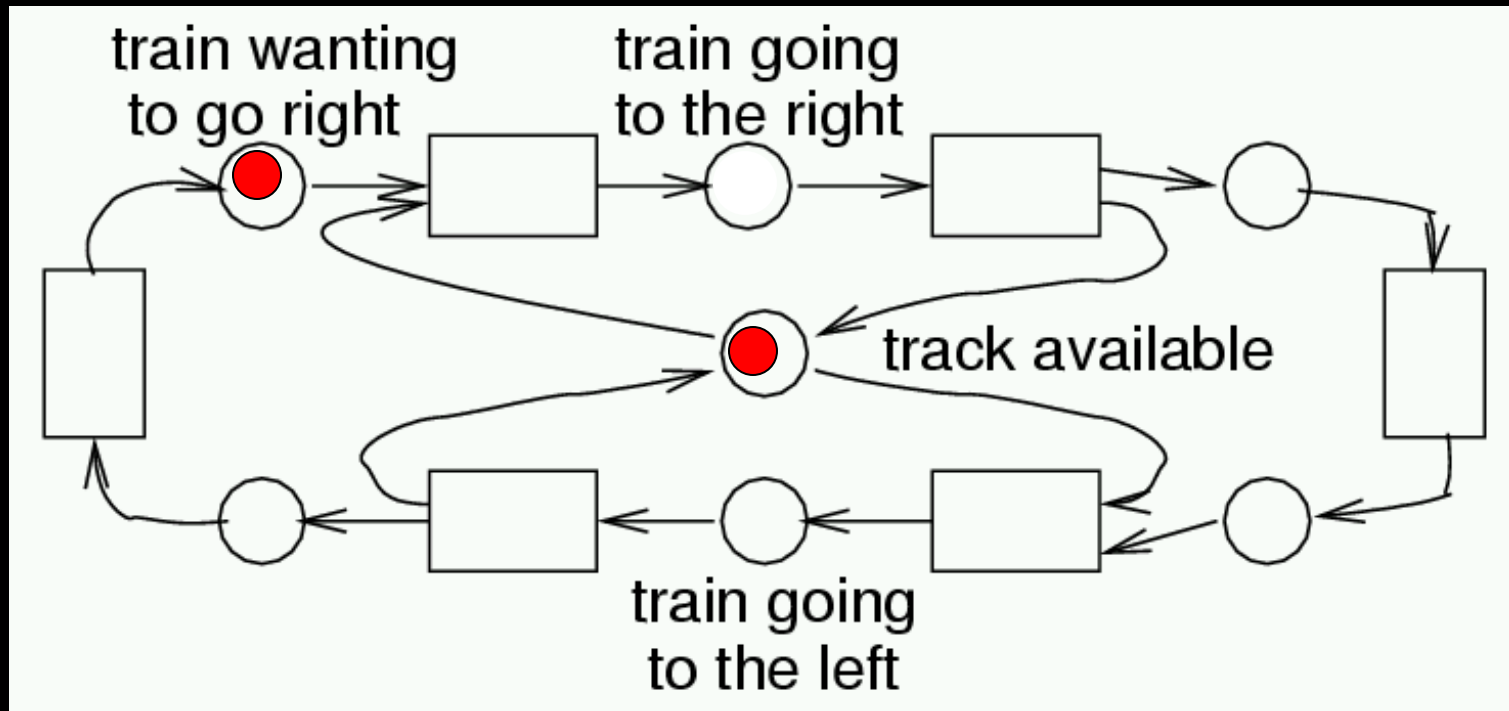
- **Model introduced by C.A. Petri in 1962**
 - Ph.D. Thesis: “Communication with Automata”
- **Applications: distributed computing, manufacturing, control, communication networks, transportation...**
- **PNs describe explicitly and graphically:**
 - sequencing/causality
 - conflict/non-deterministic choice
 - concurrency
- **Basic PN model**
 - Asynchronous model (partial ordering)
 - Main drawback: **no hierarchy**

Example: Synchronization at single track rail segment

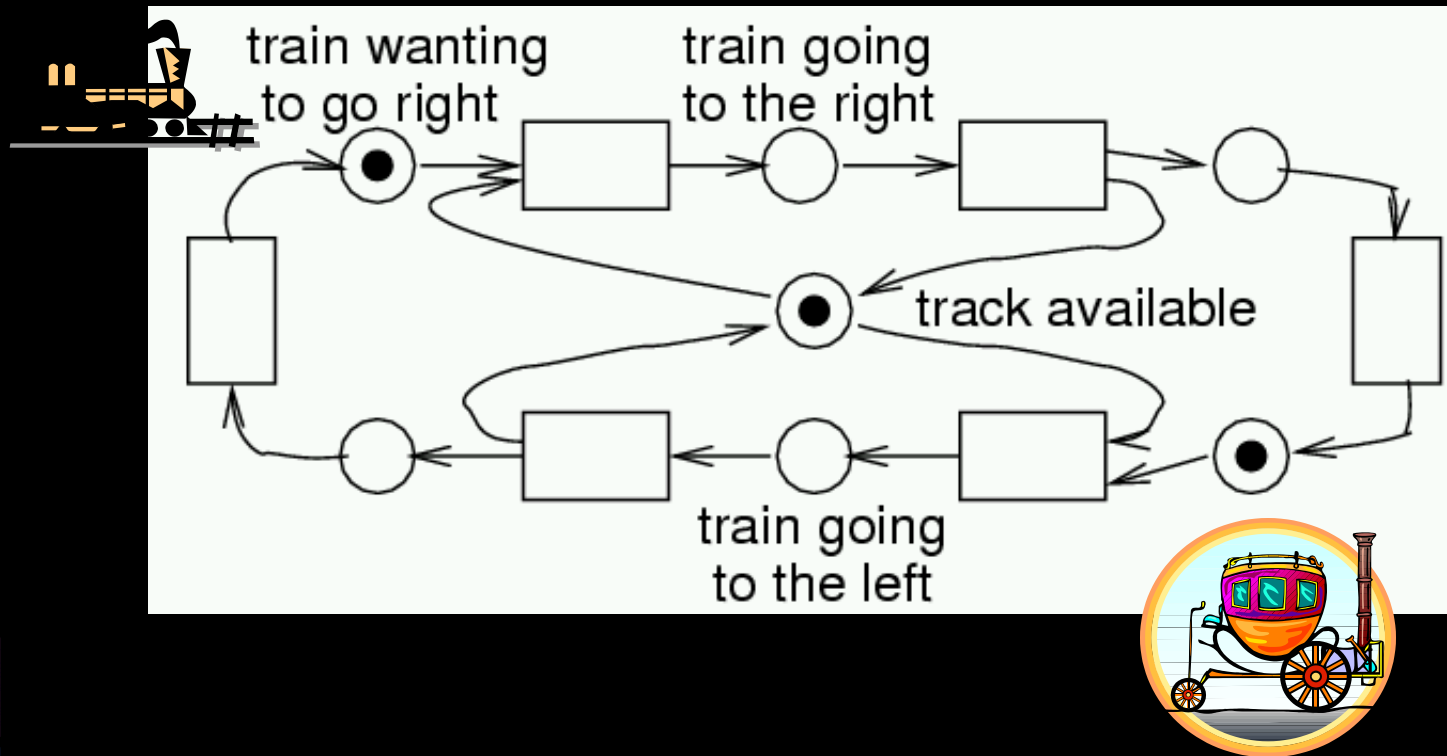
- „Preconditions“



Playing the „token game“

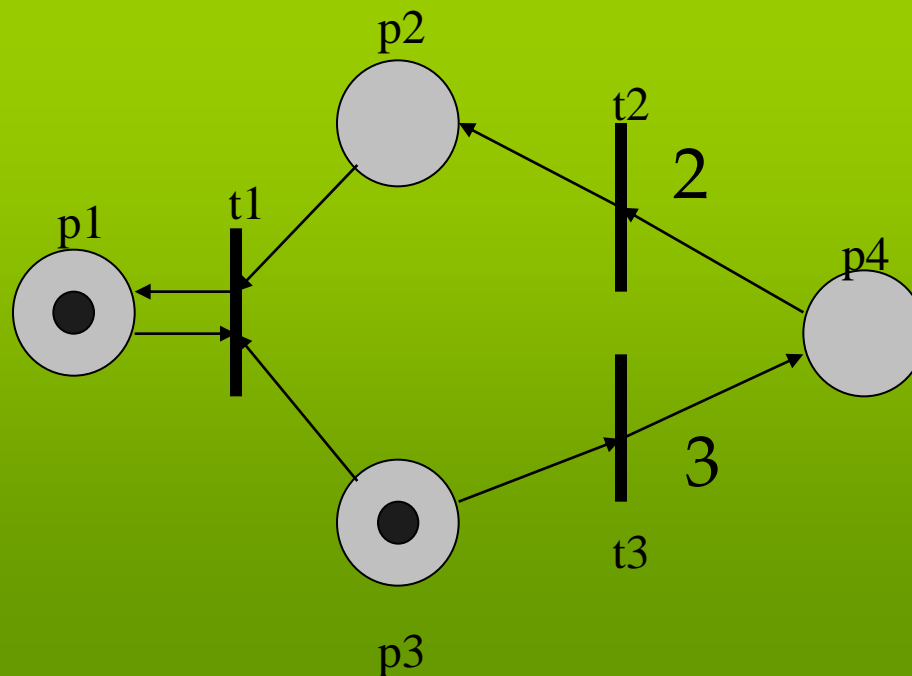


Conflict for resource „track“



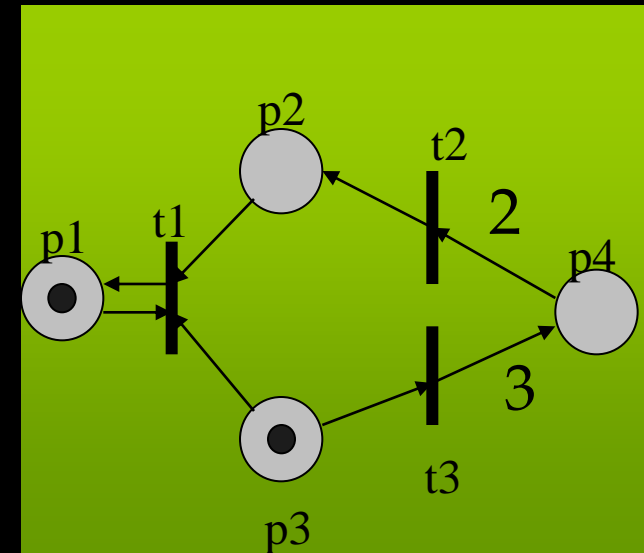
Petri Net Graph

- **Bipartite weighted directed graph:**
 - **Places:** circles
 - **Transitions:** bars or boxes
 - **Arcs:** arrows labeled with weights
- **Tokens:** black dots



Petri Net

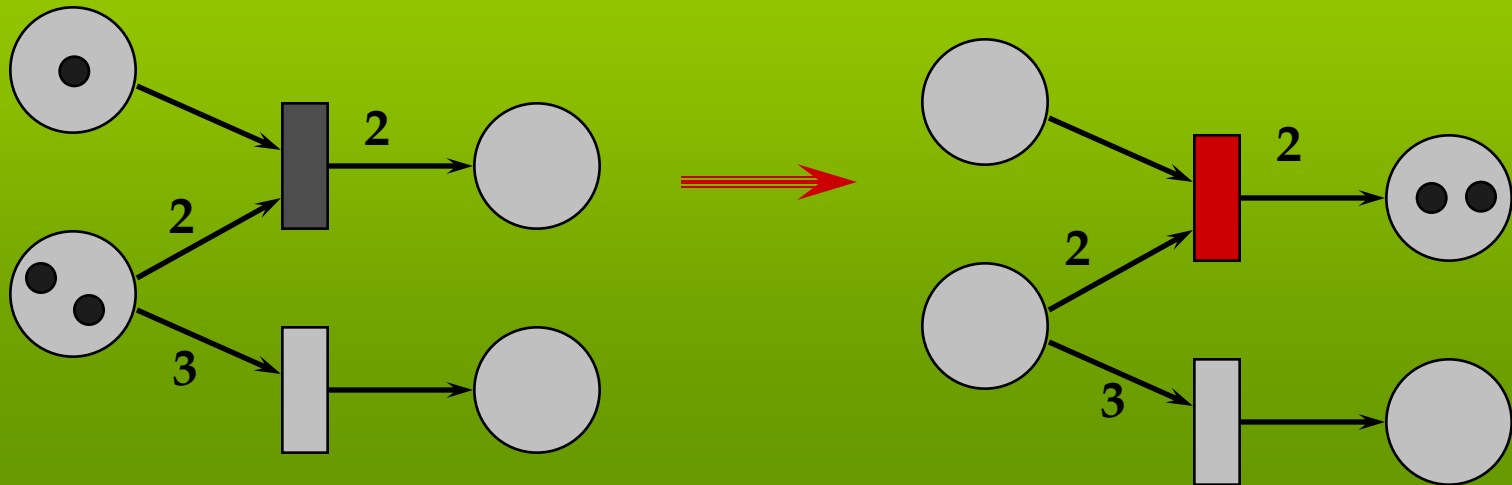
- A PN (N, M_0) is a Petri Net Graph N
 - **places**: represent distributed state by holding tokens
 - marking (state) M is an n -vector (m_1, m_2, m_3, \dots) , where m_i is the non-negative number of tokens in place p_i .
 - initial marking (M_0) is initial state
 - **transitions**: represent actions/events
 - enabled transition: enough tokens in predecessors
 - firing transition: modifies marking
- ...and an initial marking M_0 .



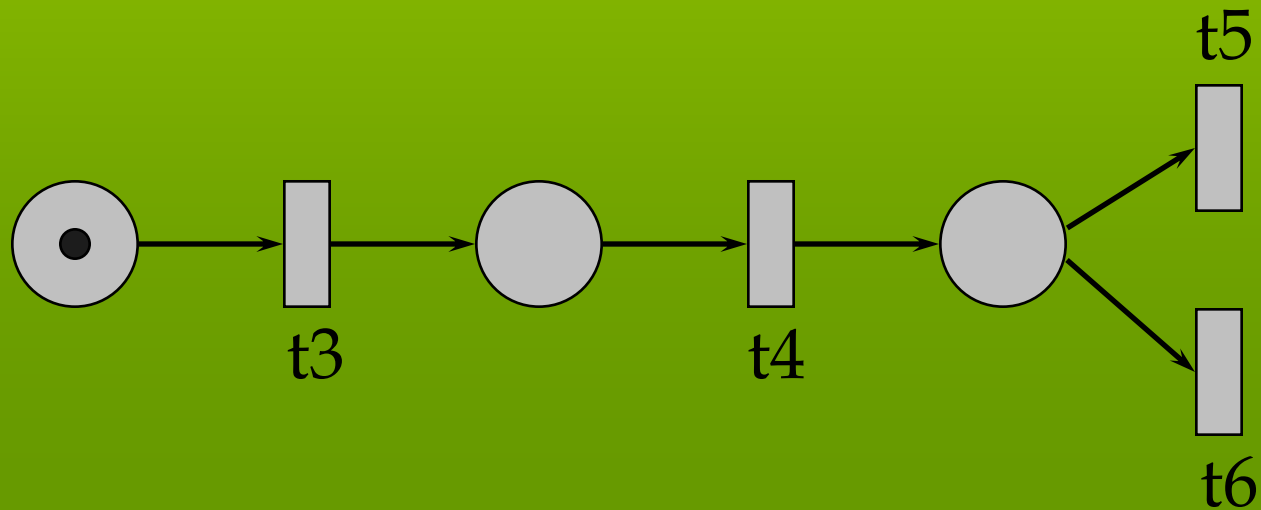
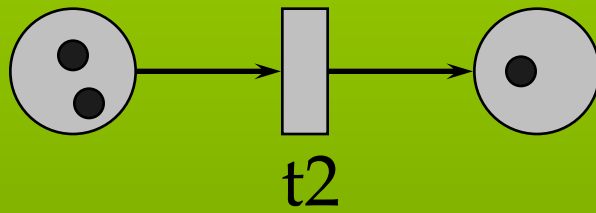
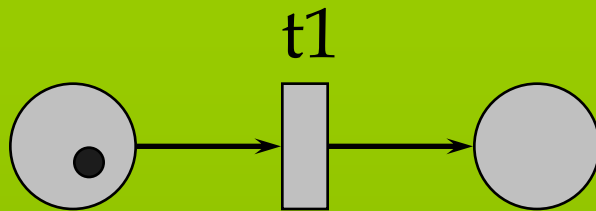
Places/Transitions: conditions/events

Transition firing rule

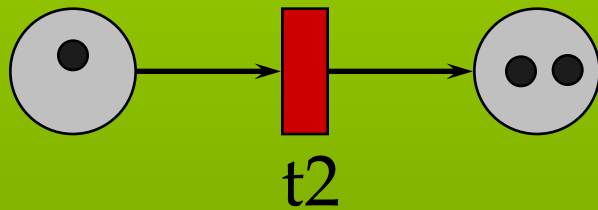
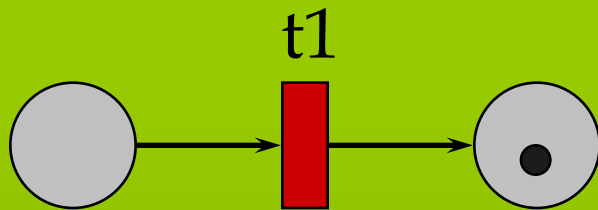
- A marking is changed according to the following rules:
 - A transition is **enabled** if there are enough tokens in each input place
 - An enabled transition **may or may not** fire
 - The **firing** of a transition modifies marking by **consuming** tokens from the input places and **producing** tokens in the output places



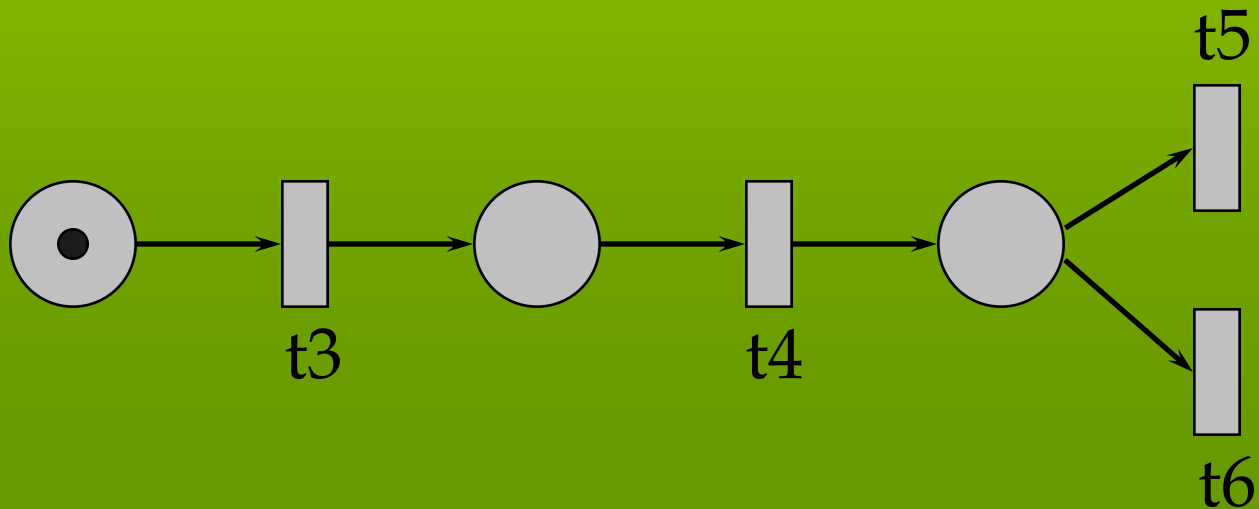
Concurrency, causality, choice



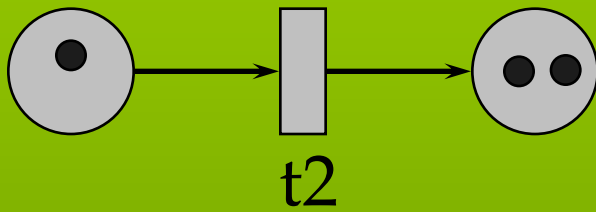
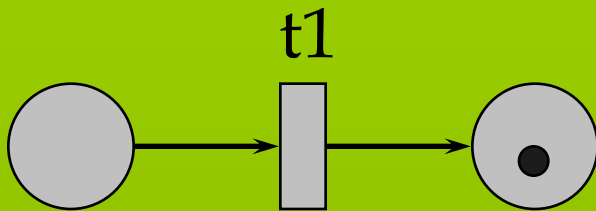
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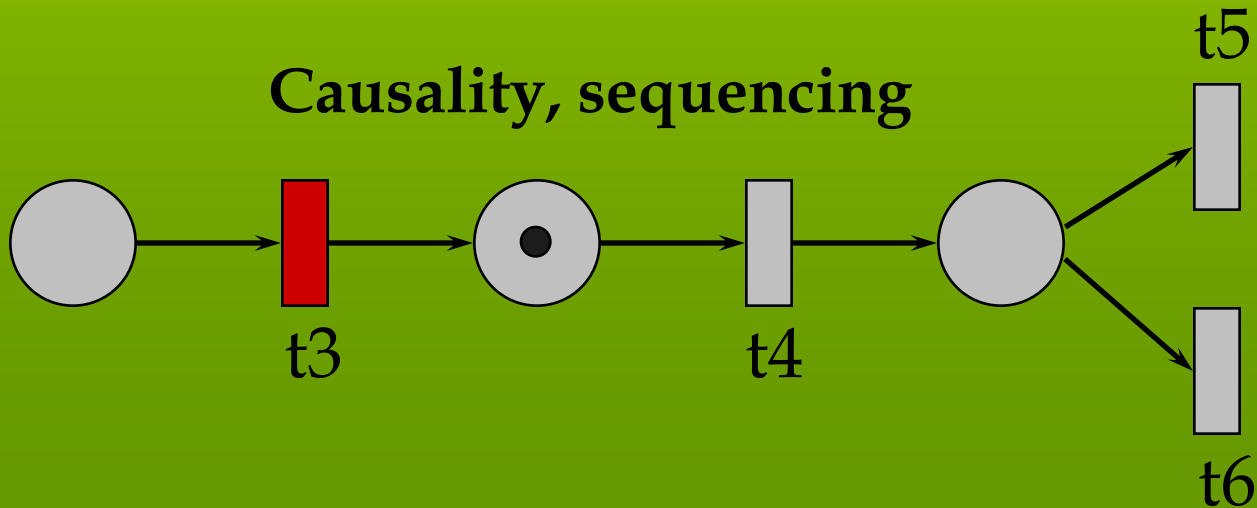
Concurrency



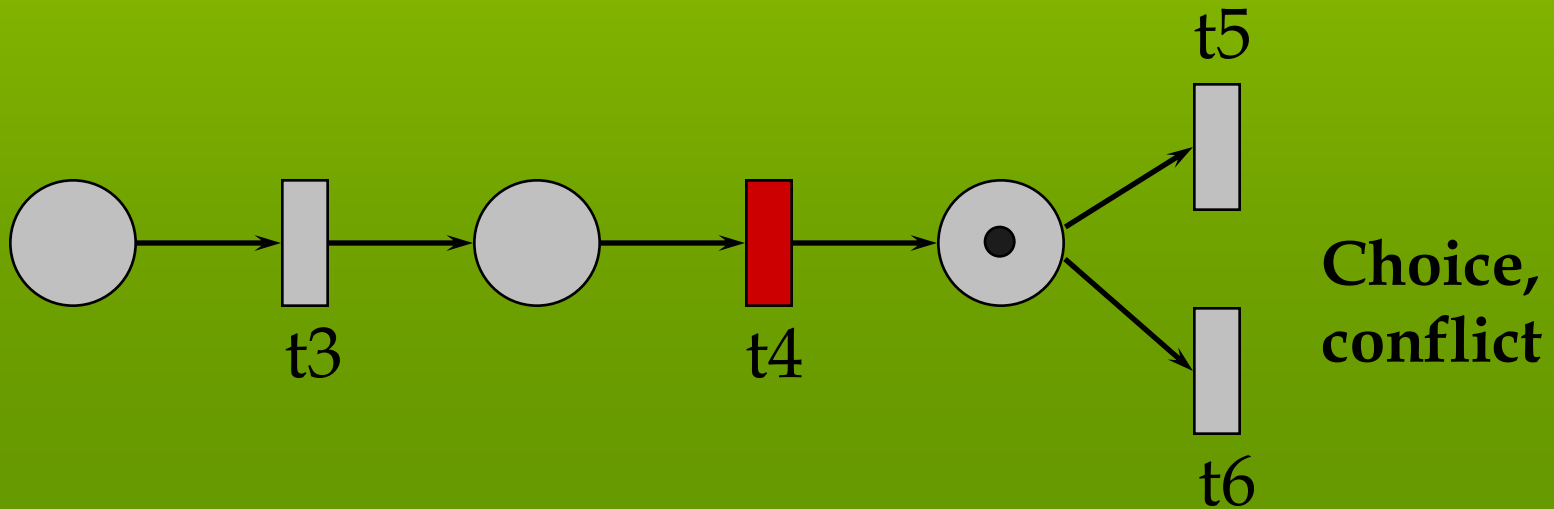
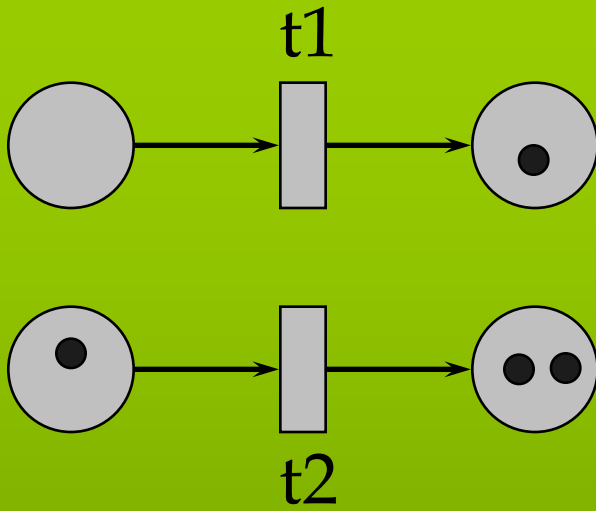
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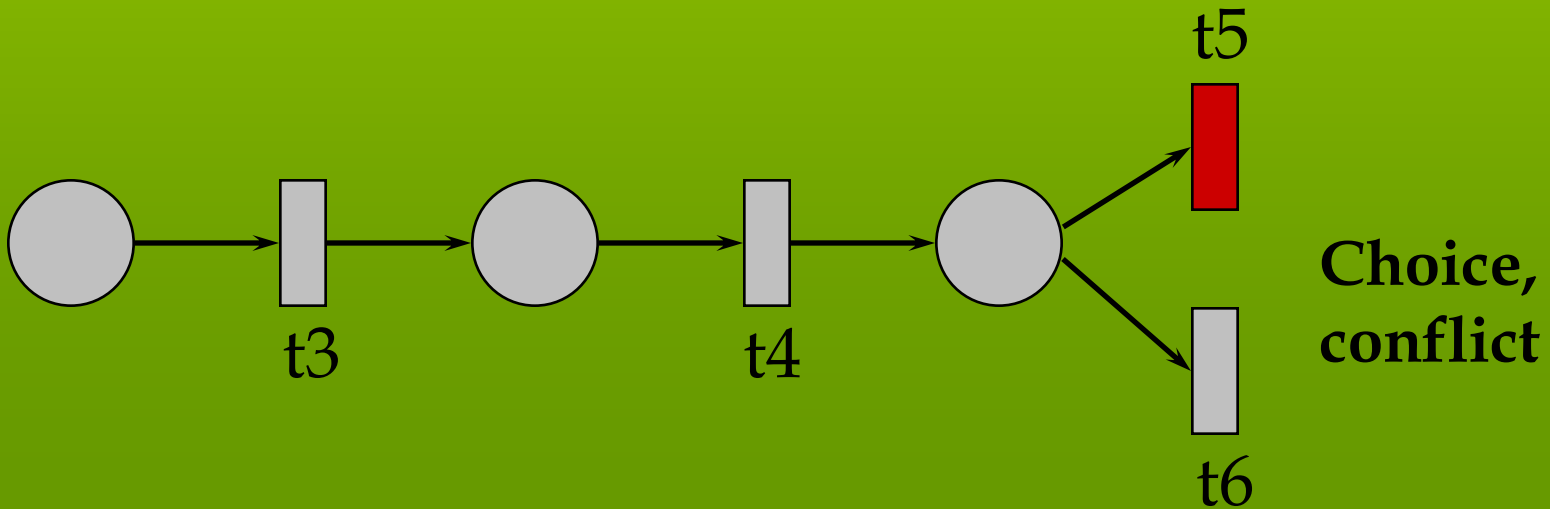
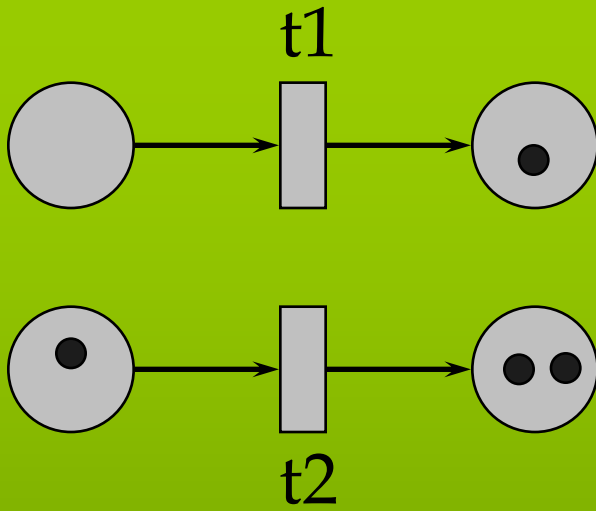
Causality, sequencing



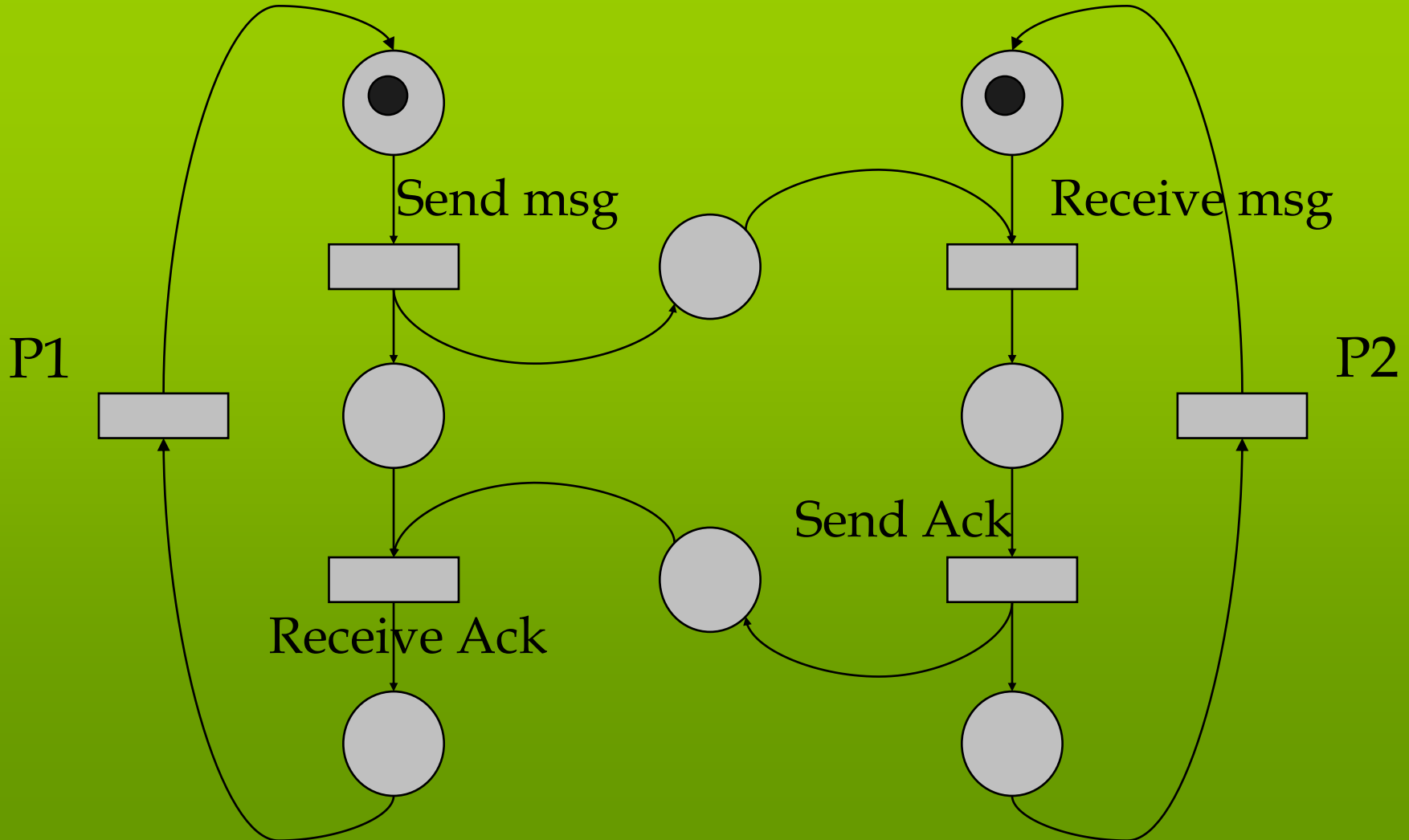
Concurrency, causality, choice



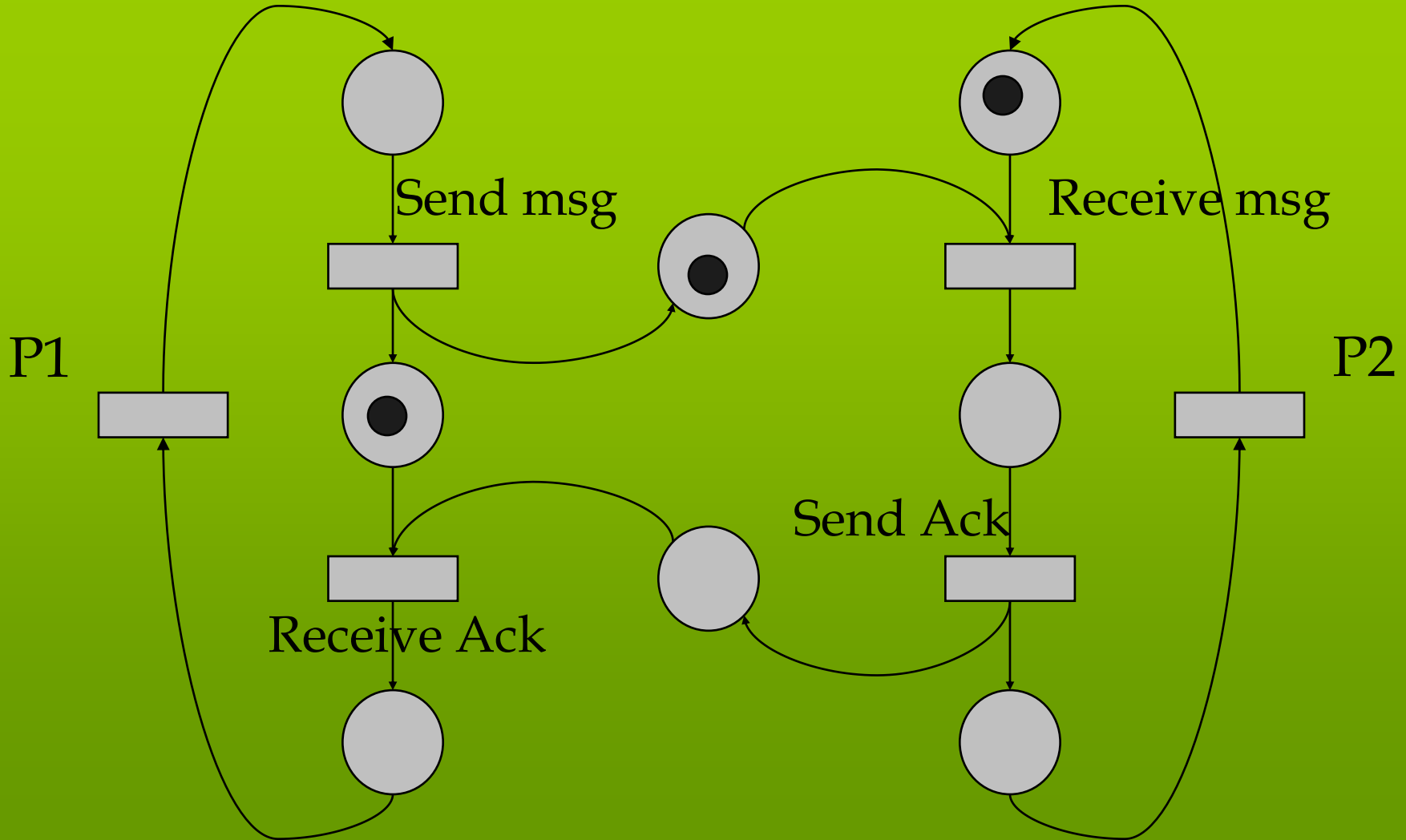
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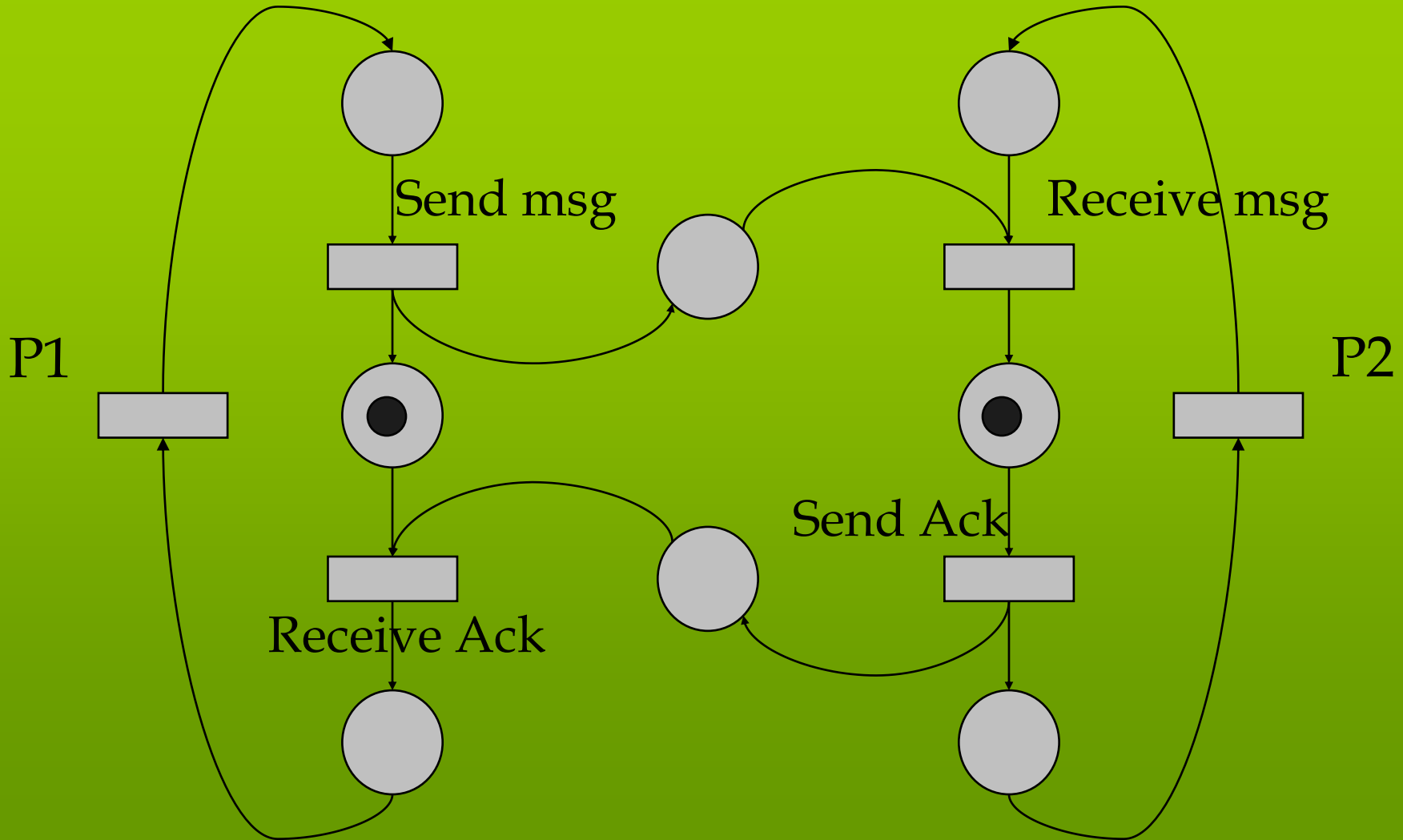
Communication Protocol



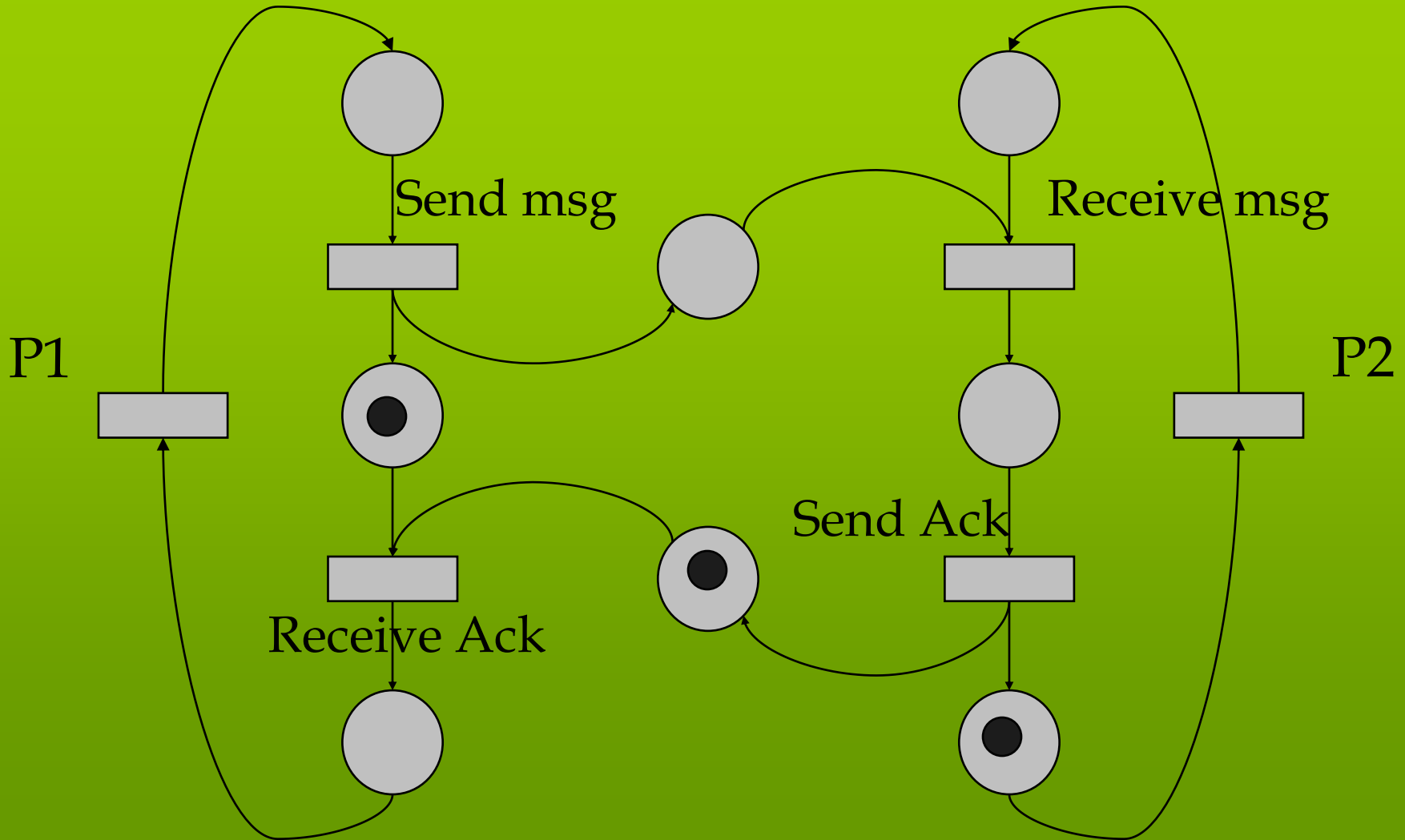
Communication Protocol



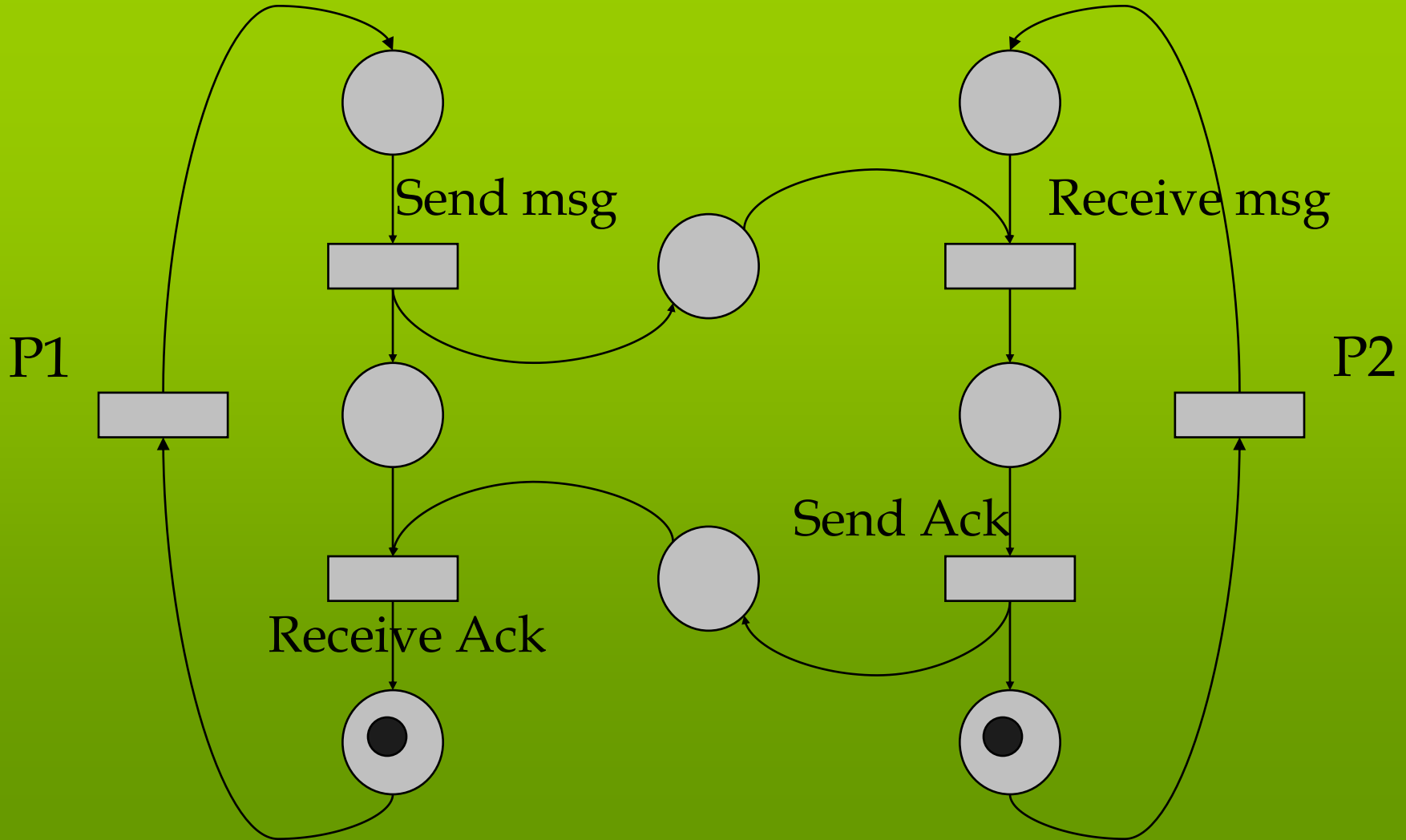
Communication Protocol



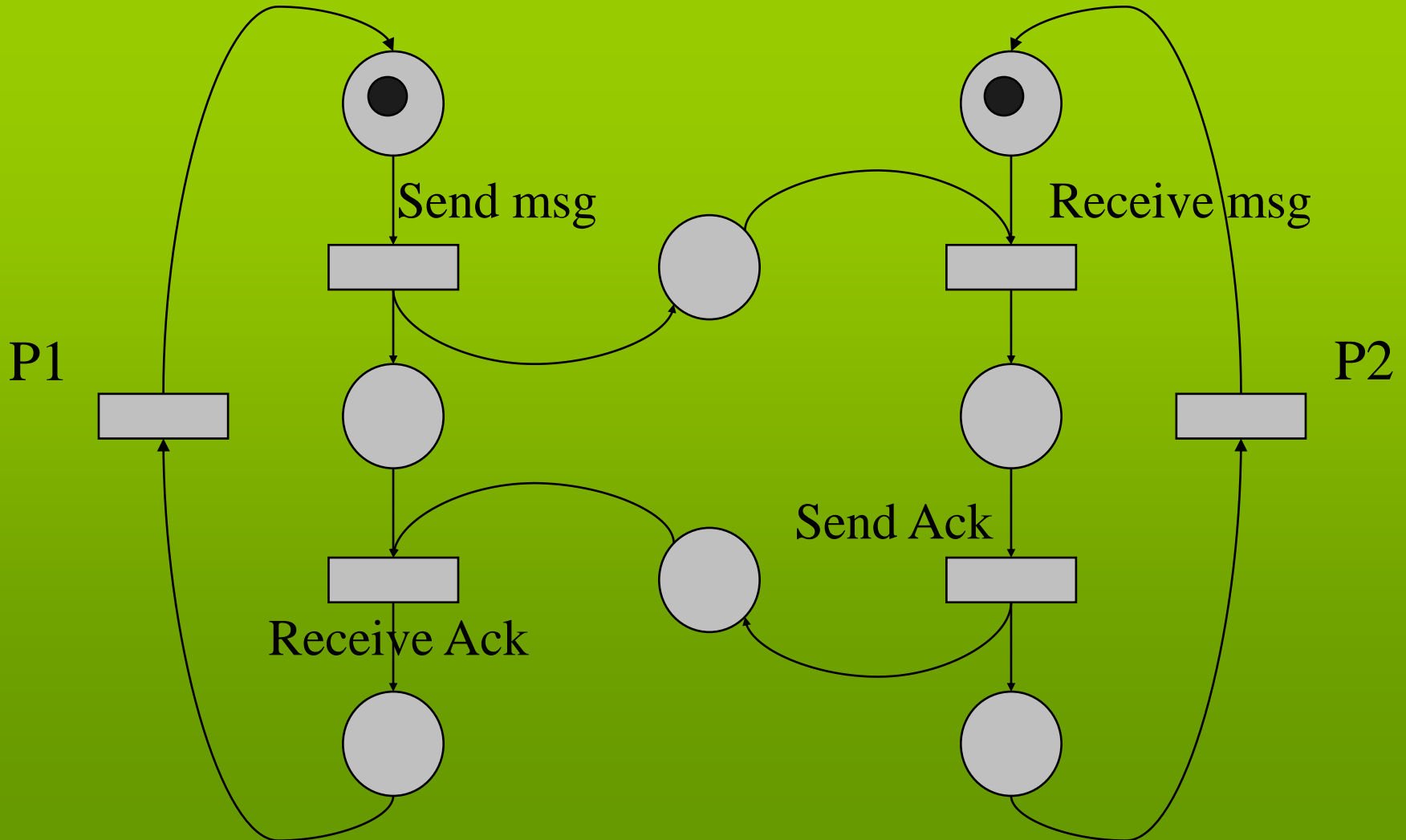
Communication Protocol



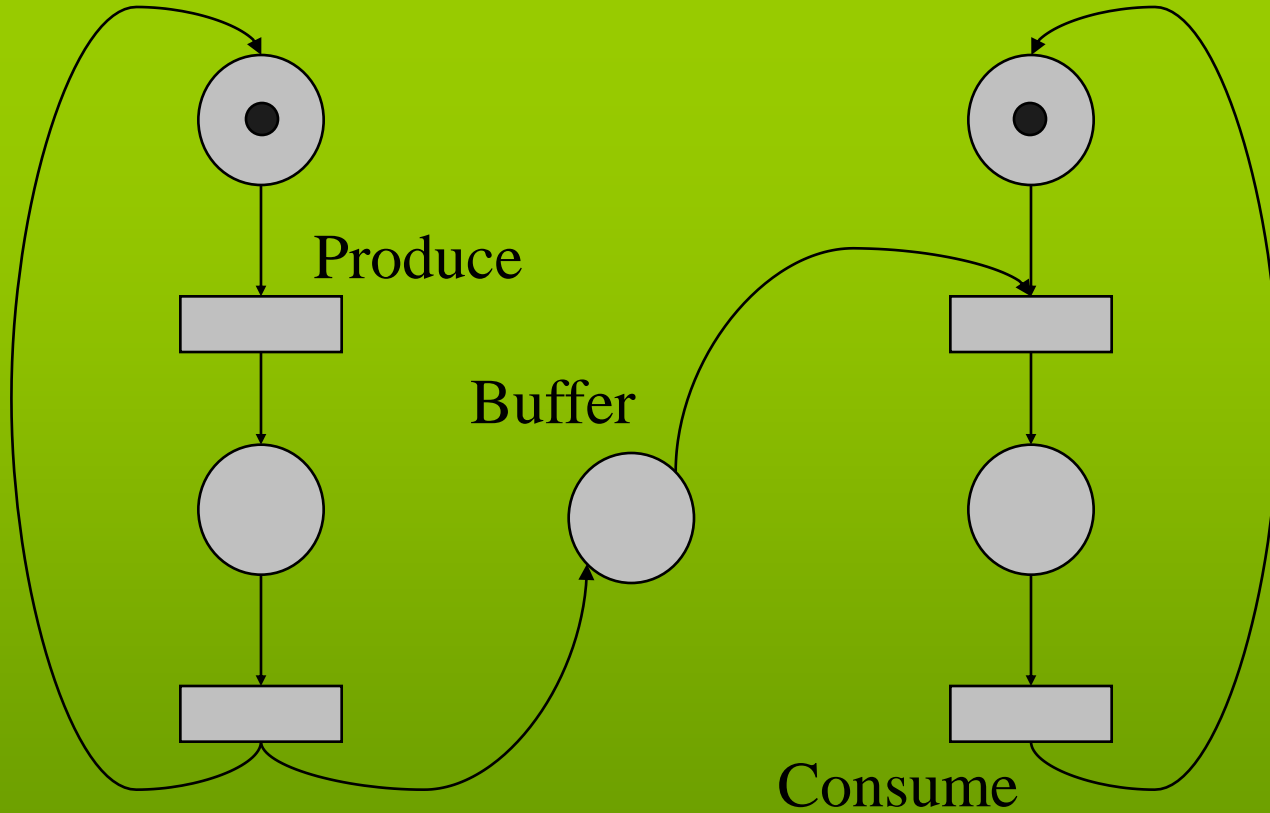
Communication Protocol



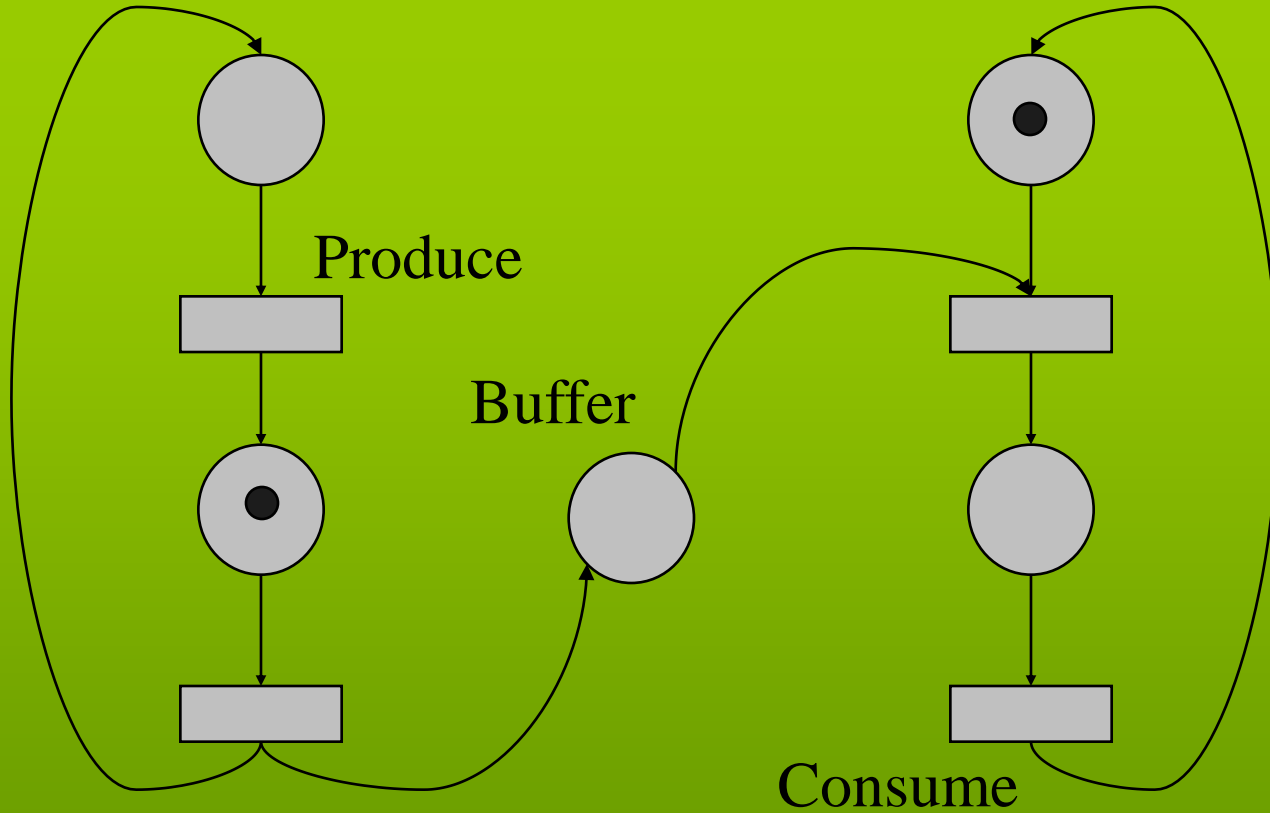
Communication Protocol



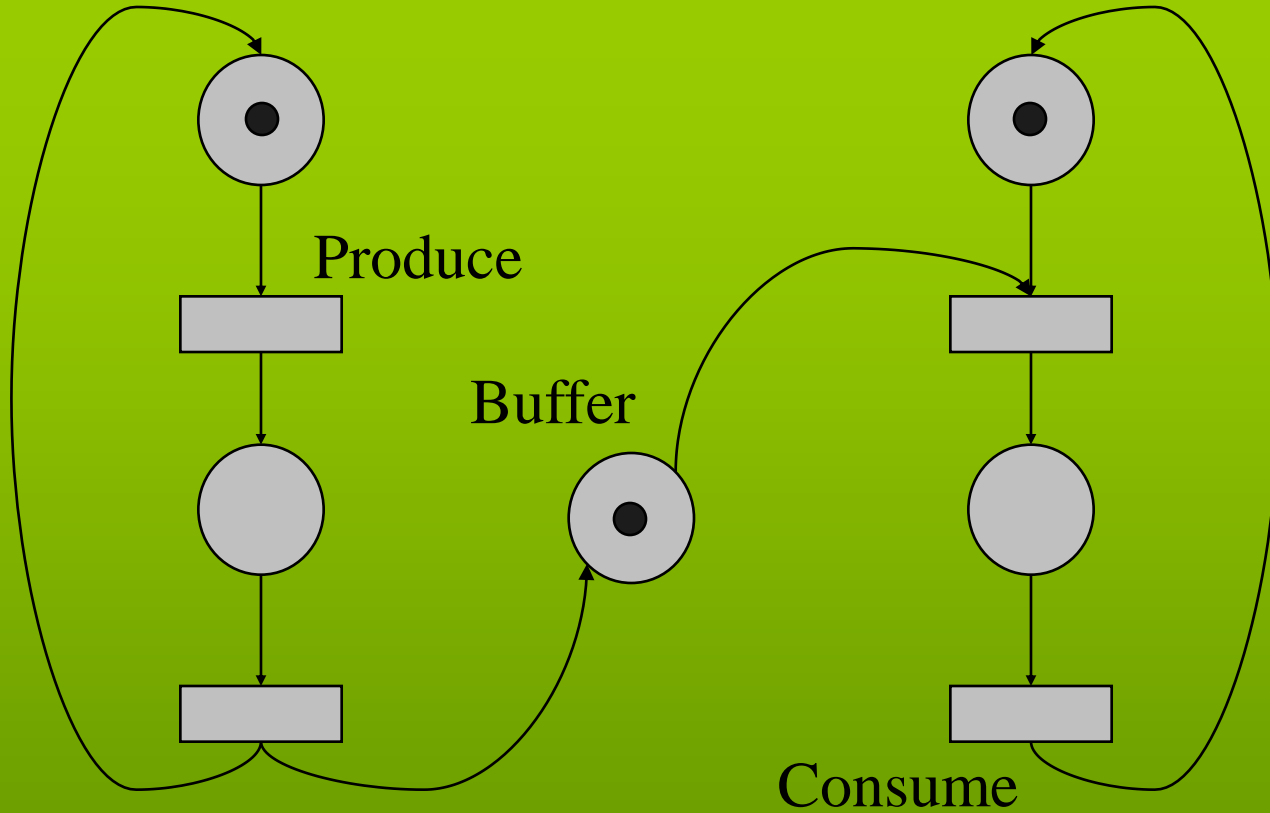
Producer-Consumer Problem



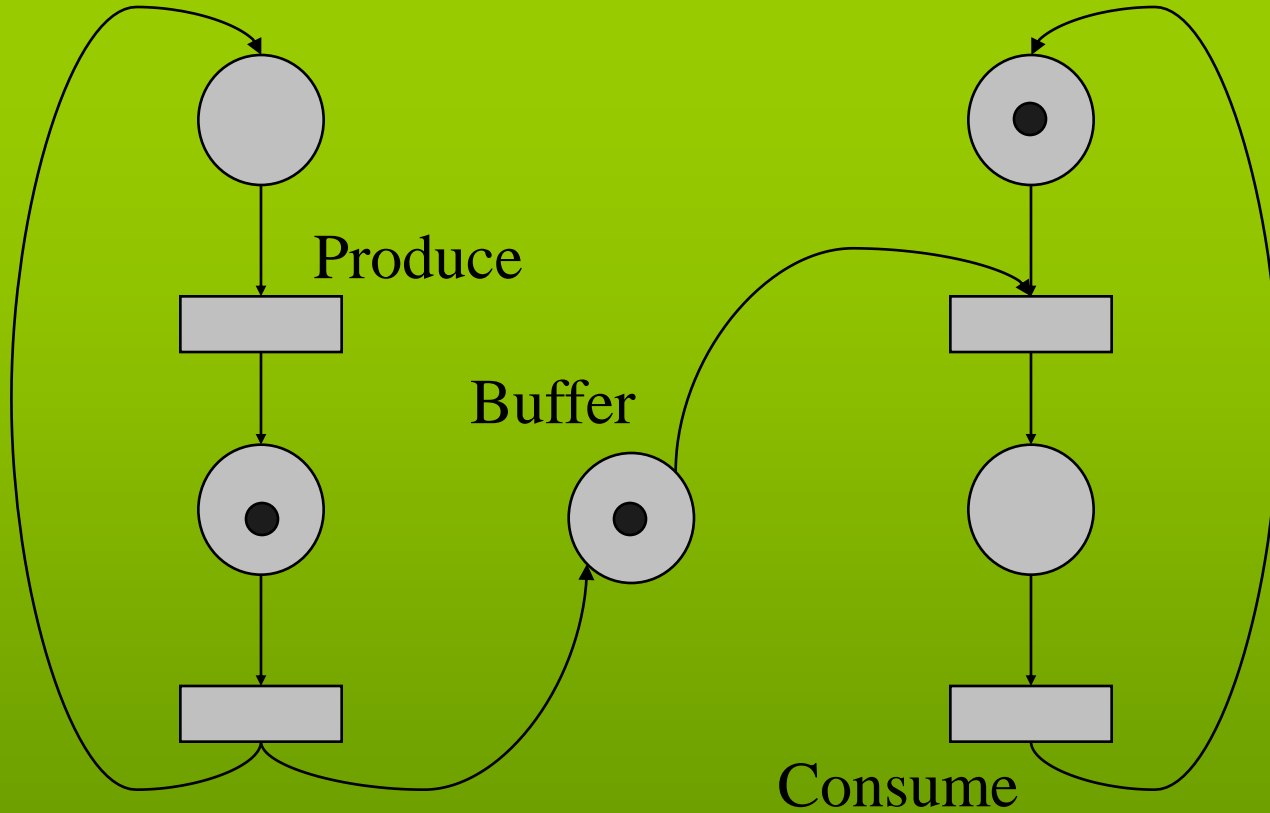
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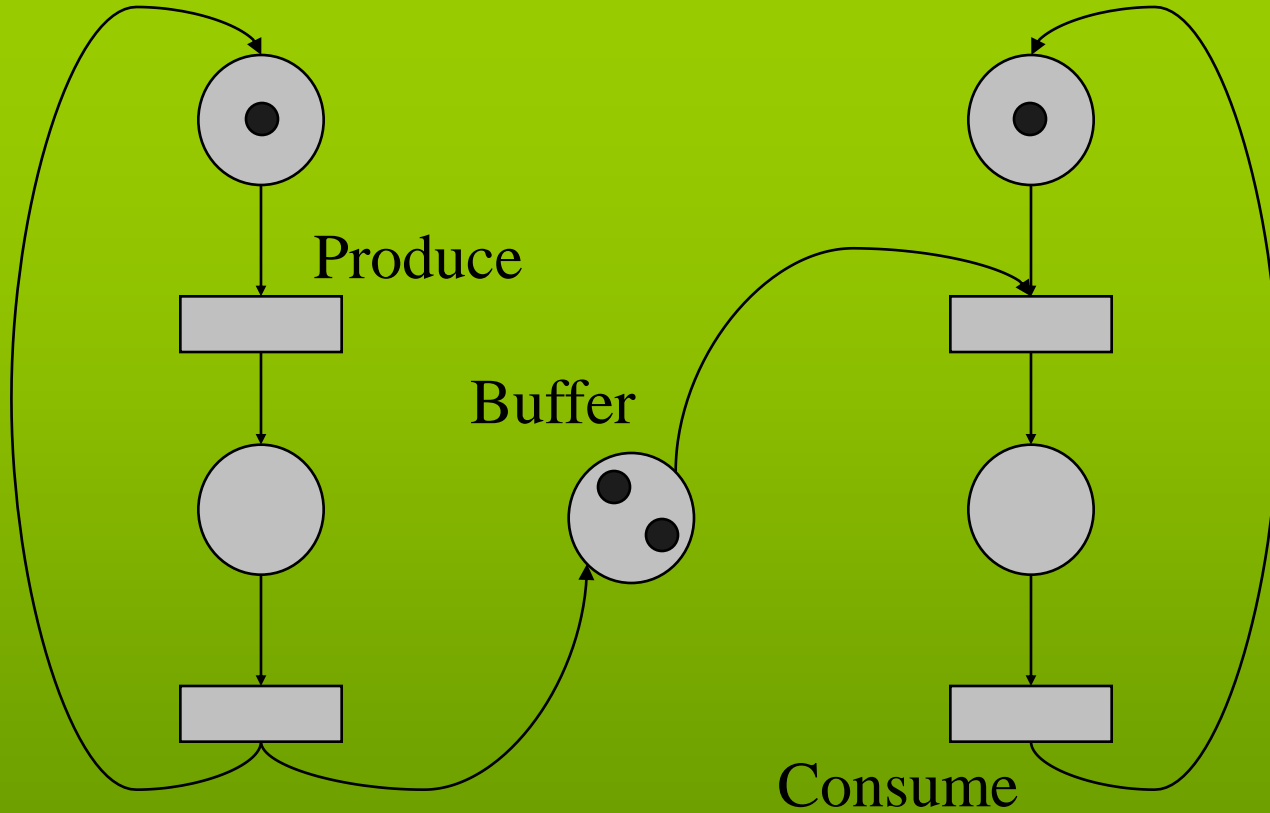
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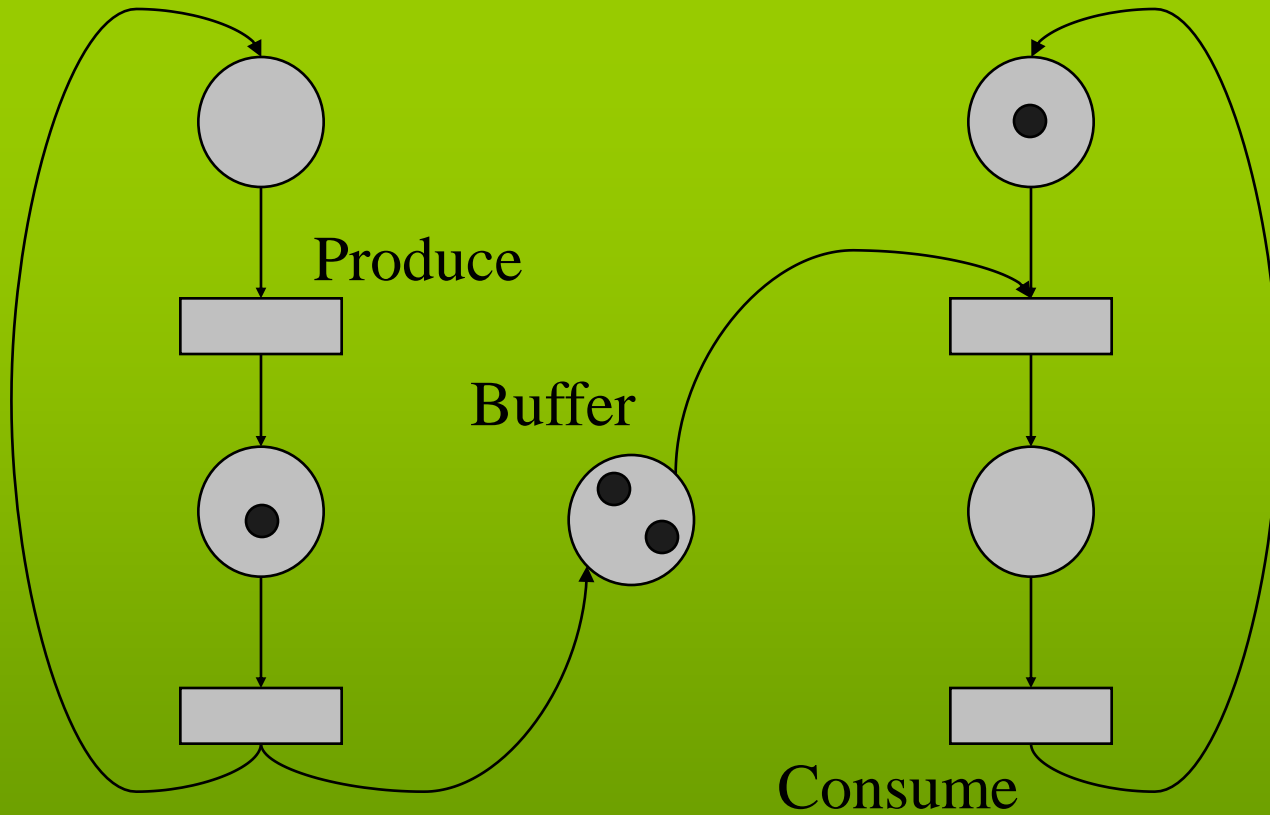
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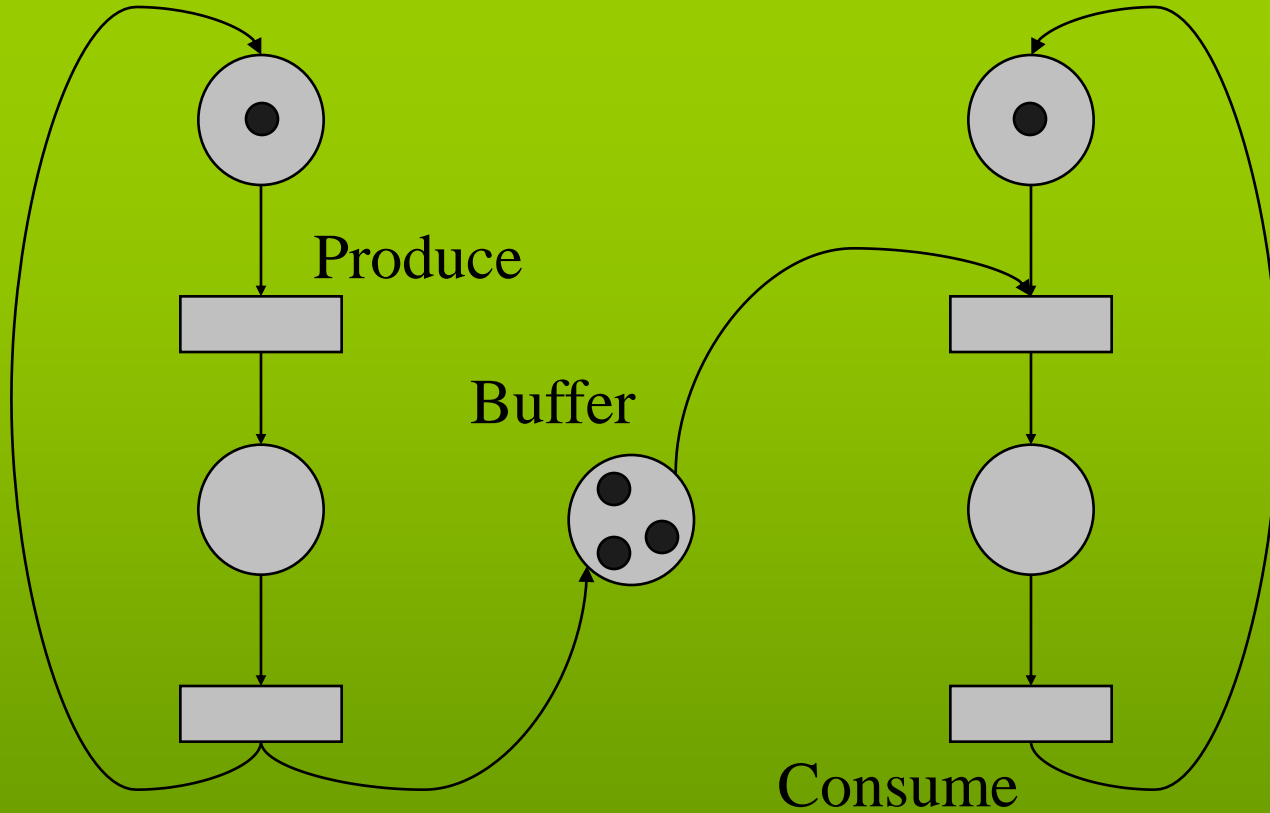
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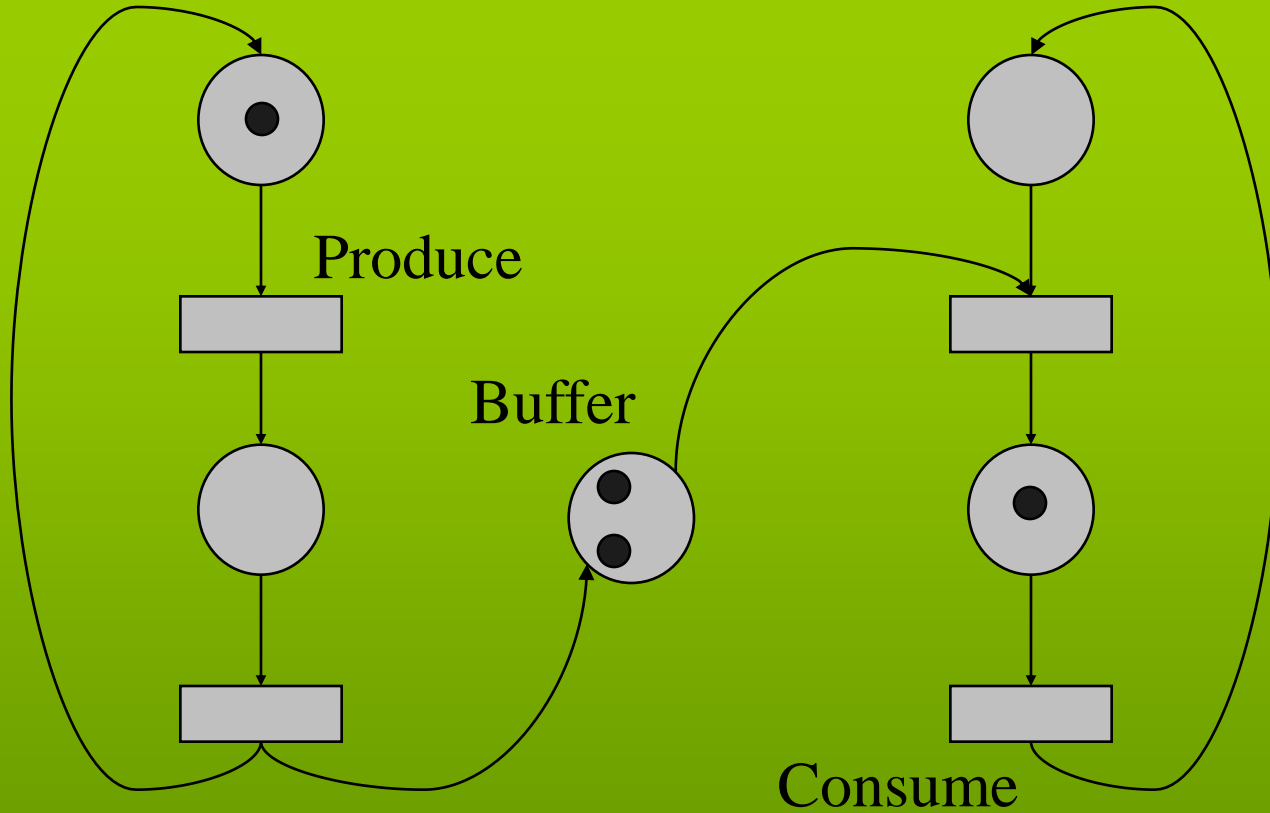
Producer-Consumer Problem



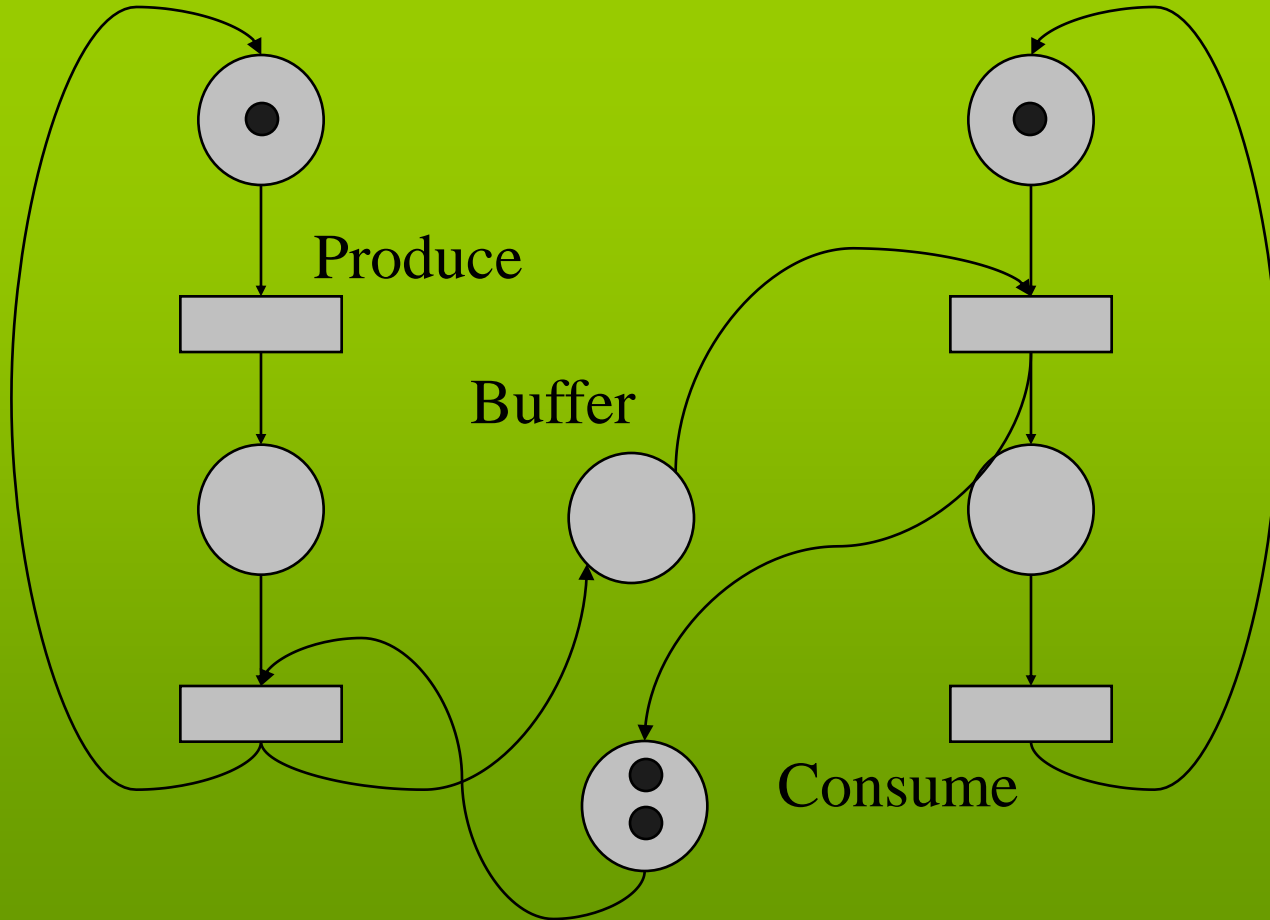
Producer-Consumer Problem



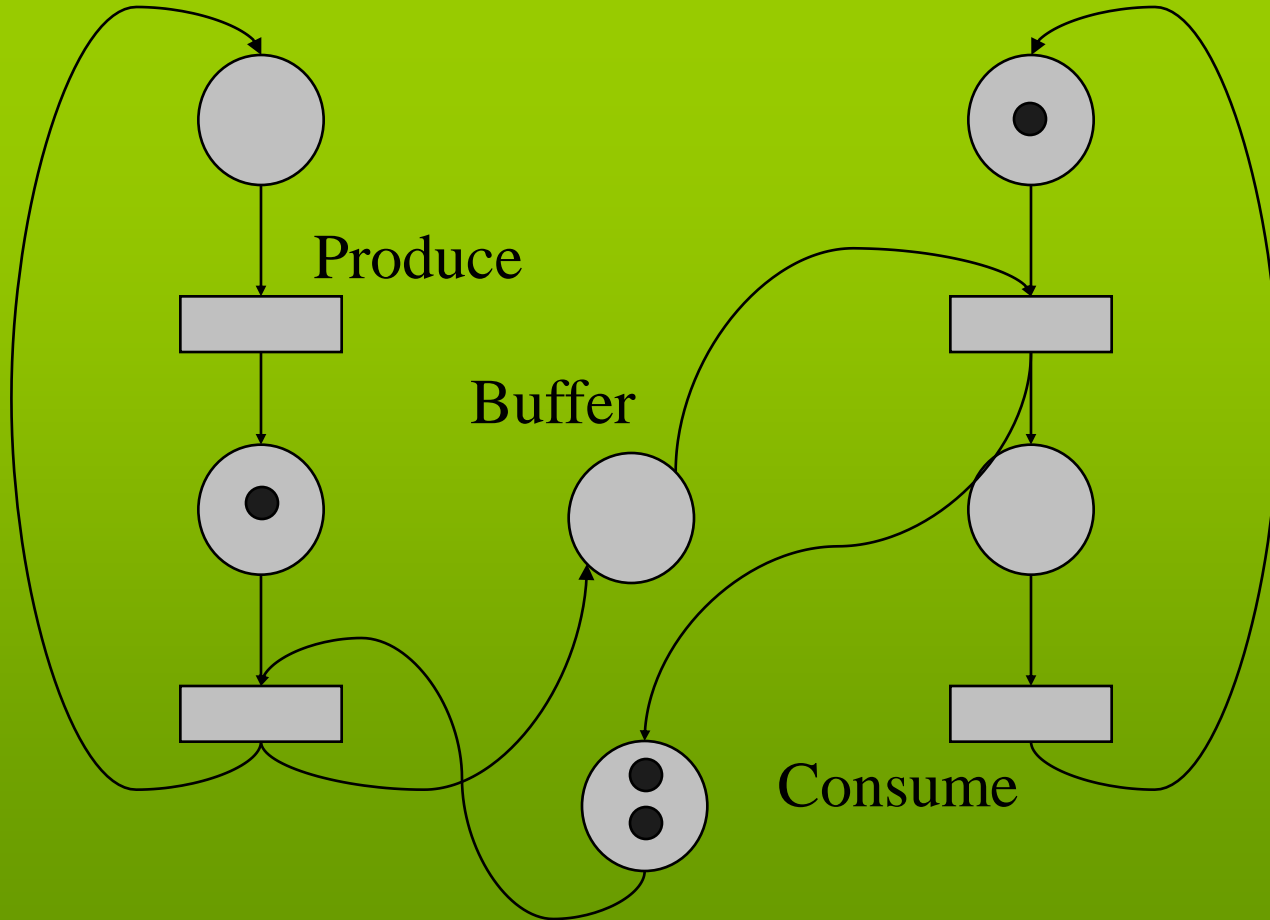
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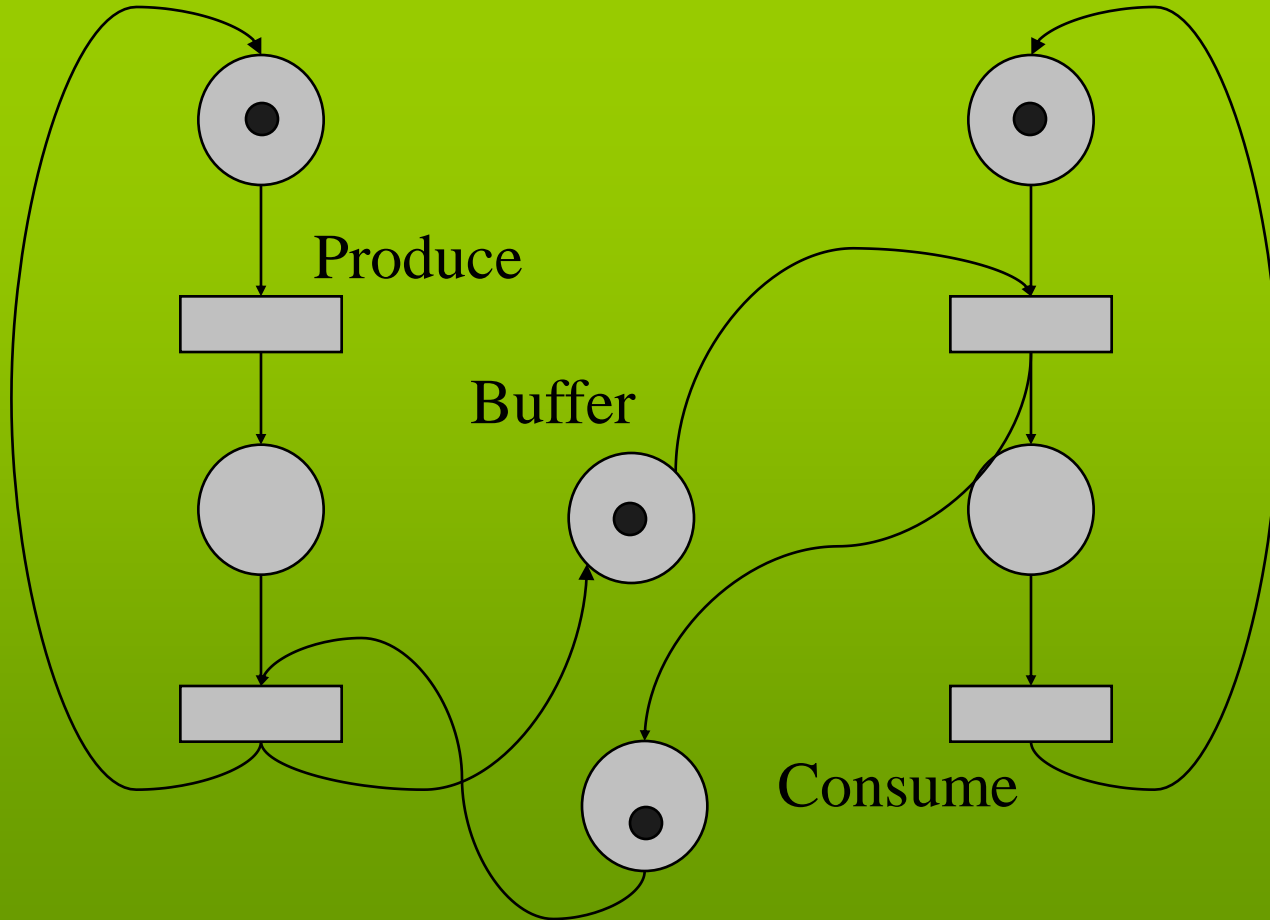
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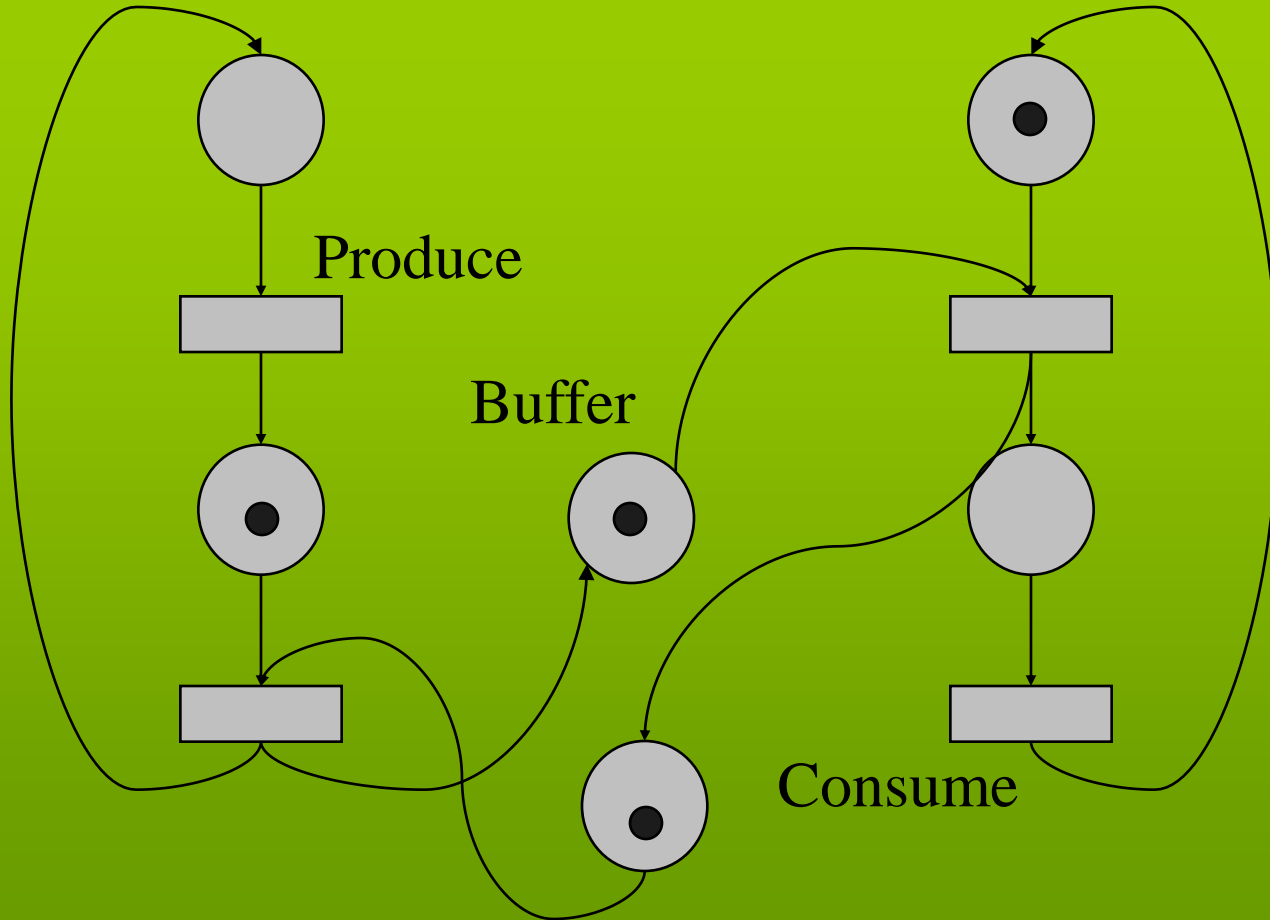
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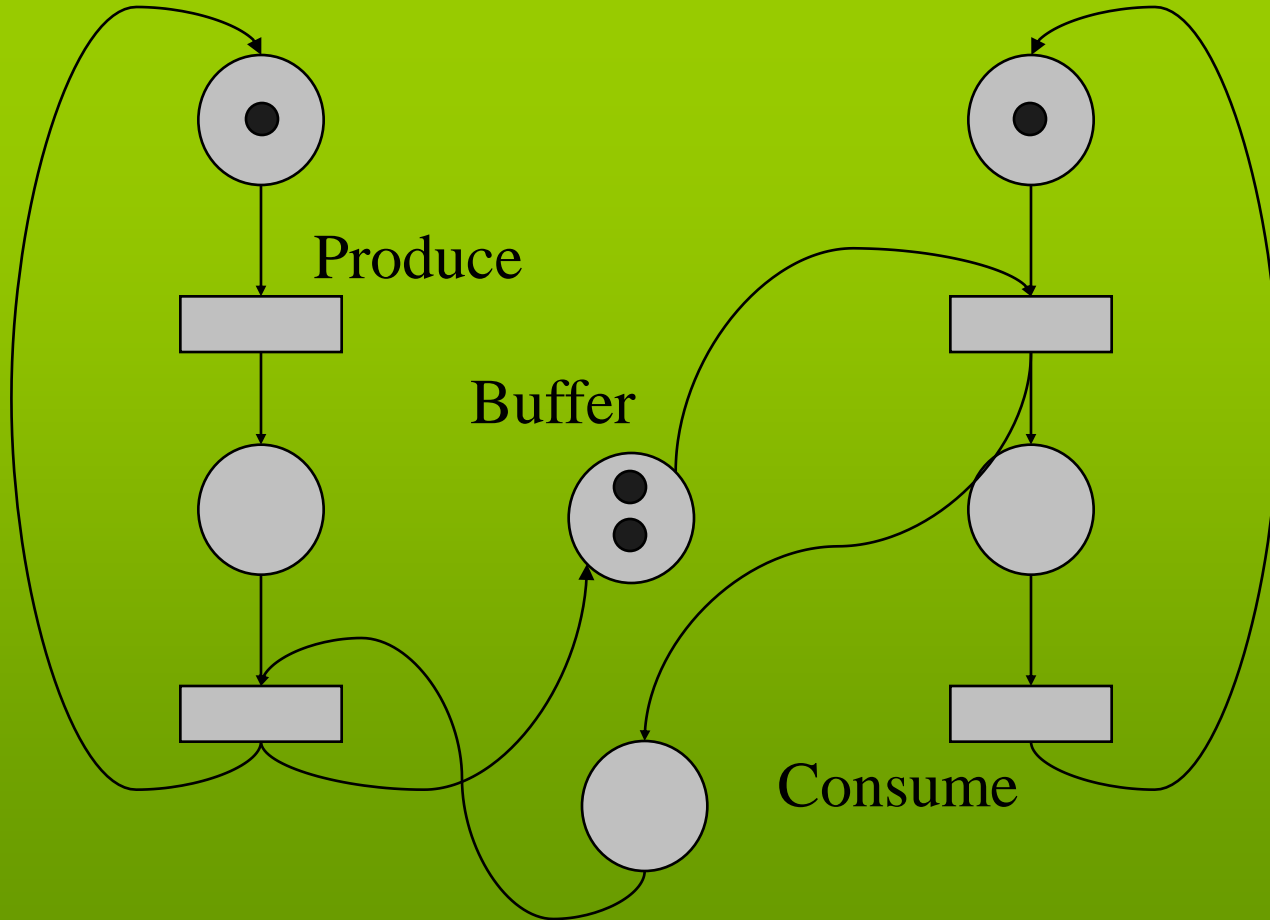
Producer-Consumer Problem



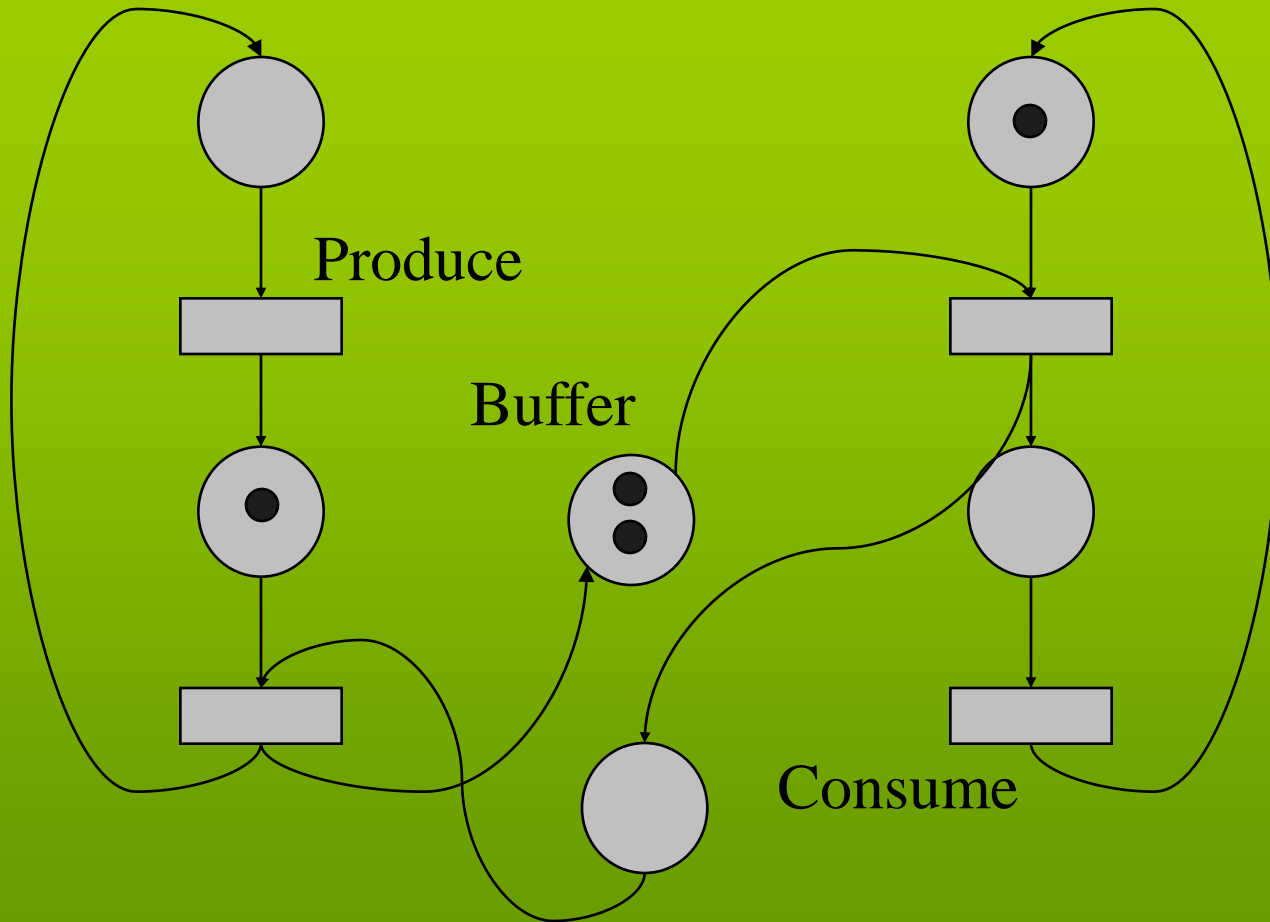
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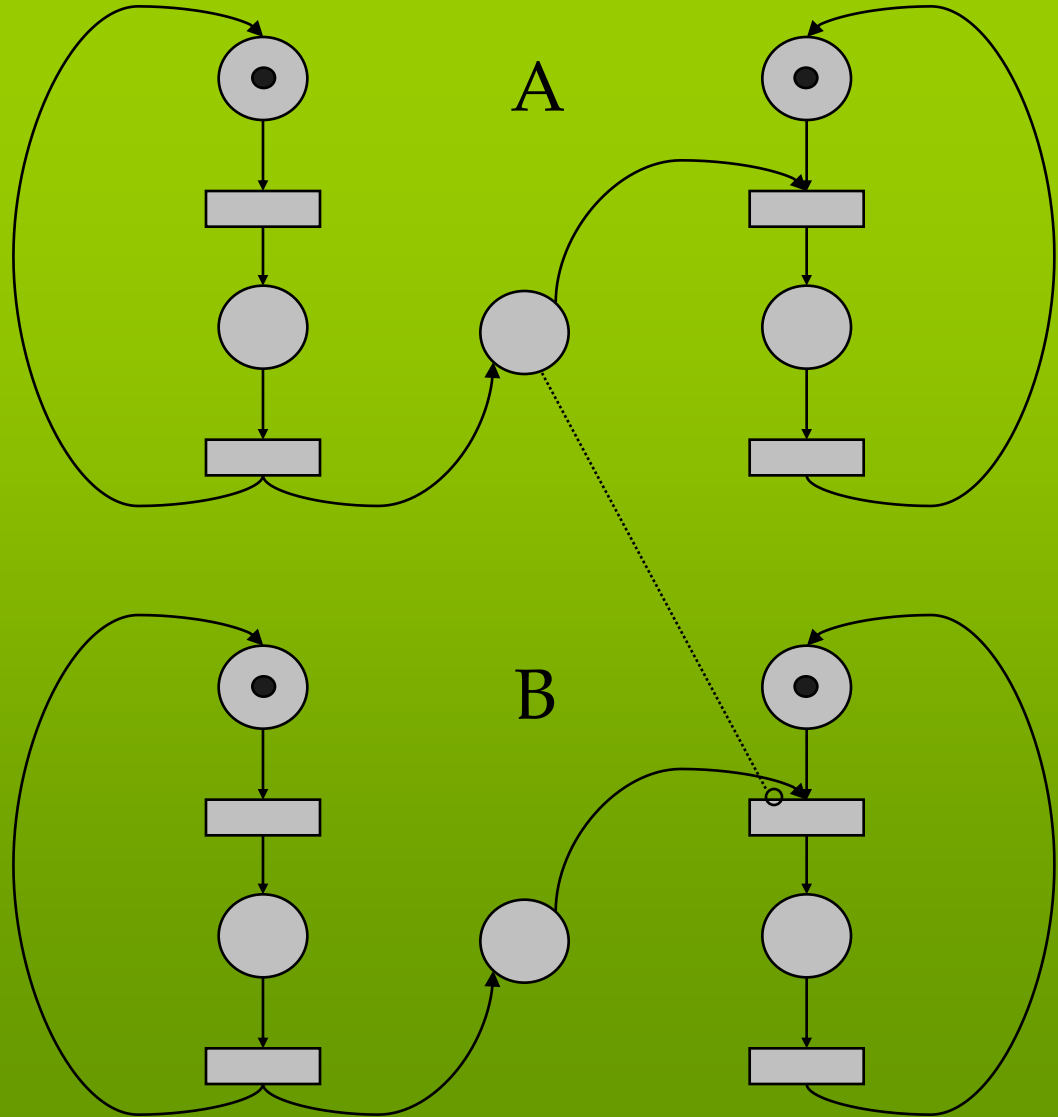
Producer-Consumer Problem



Producer-Consumer with priority

**Consumer B can
consume only if
buffer A is empty**

Inhibitor arcs



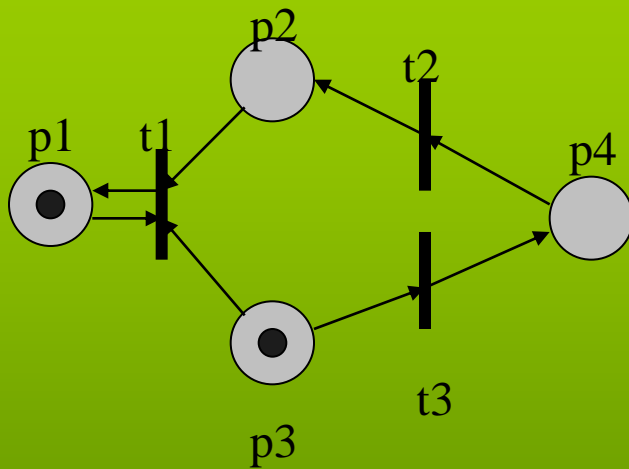


PN properties

- **Behavioral: depend on the initial marking (most interesting)**
 - Reachability
 - Boundedness
 - Schedulability
 - Liveness
 - Conservation
- **Structural: do not depend on the initial marking (often too restrictive)**
 - Consistency
 - Structural boundedness

Reachability

- Marking M is **reachable** from marking M_0 if there exists a **sequence of firings** $\sigma = M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \dots M$ that transforms M_0 to M .
- The reachability problem is decidable.



$$M_0 = (1, 0, 1, 0)$$

$$M = (1, 1, 0, 0)$$

$$M_0 = (1, 0, 1, 0)$$

$$\downarrow t_3$$

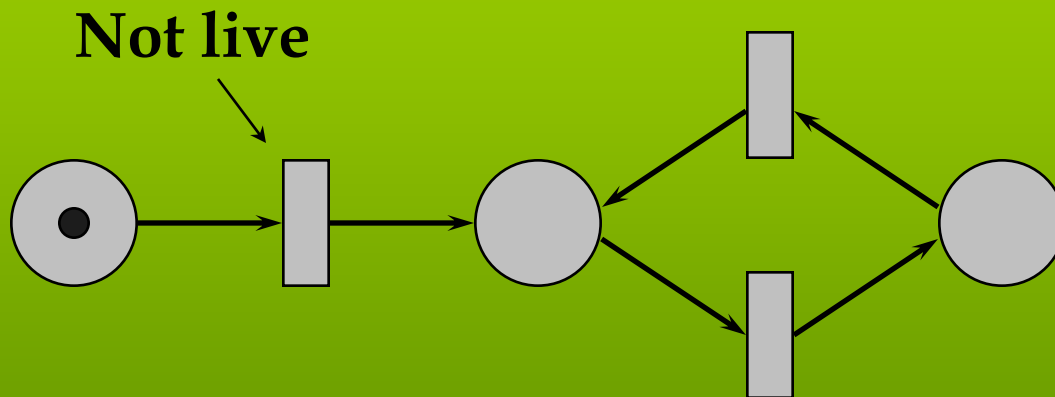
$$M_1 = (1, 0, 0, 1)$$

$$\downarrow t_2$$

$$M = (1, 1, 0, 0)$$

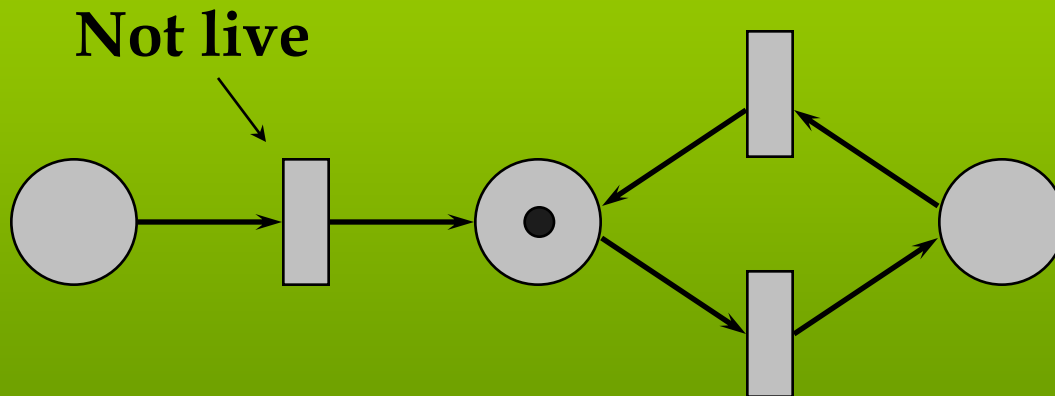
Liveness

- **Liveness**: from any marking any transition can become fireable
 - Liveness implies deadlock freedom, not viceversa



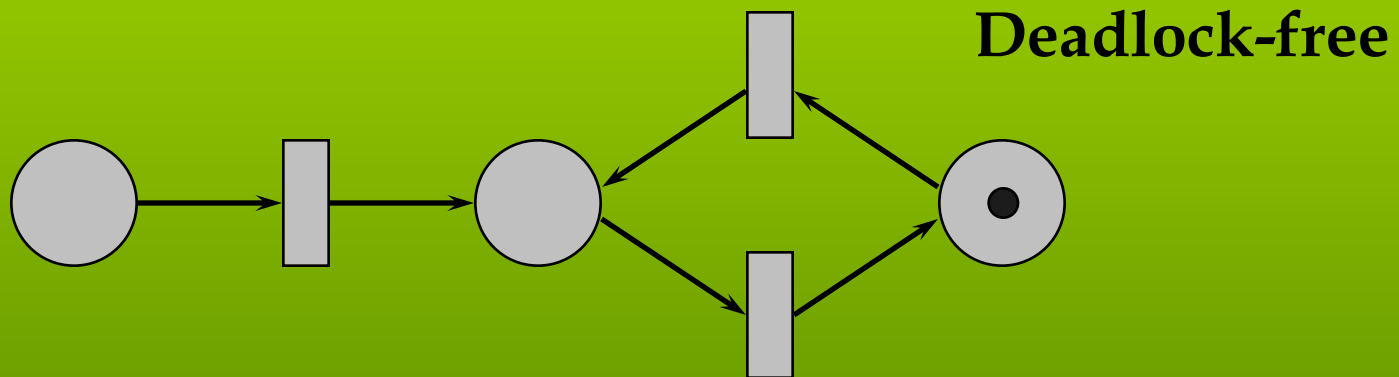
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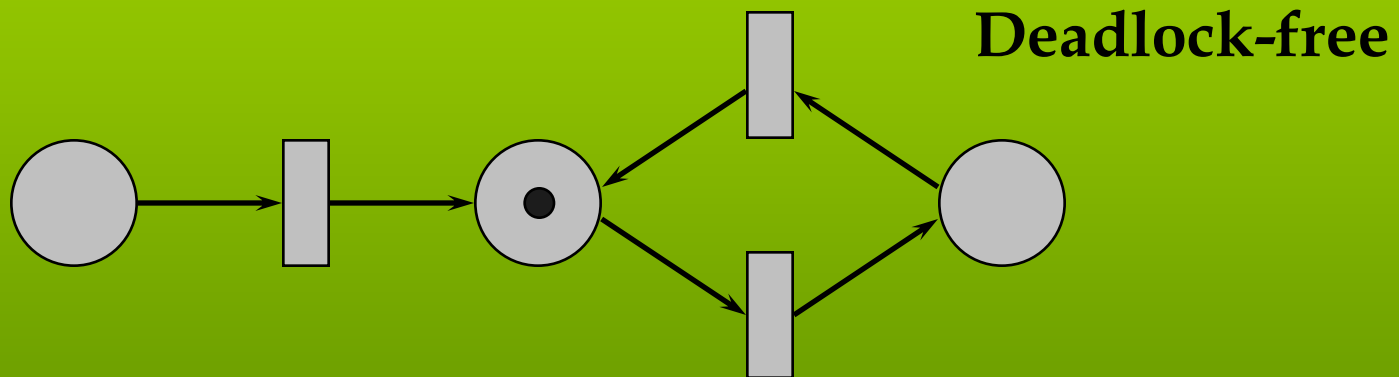
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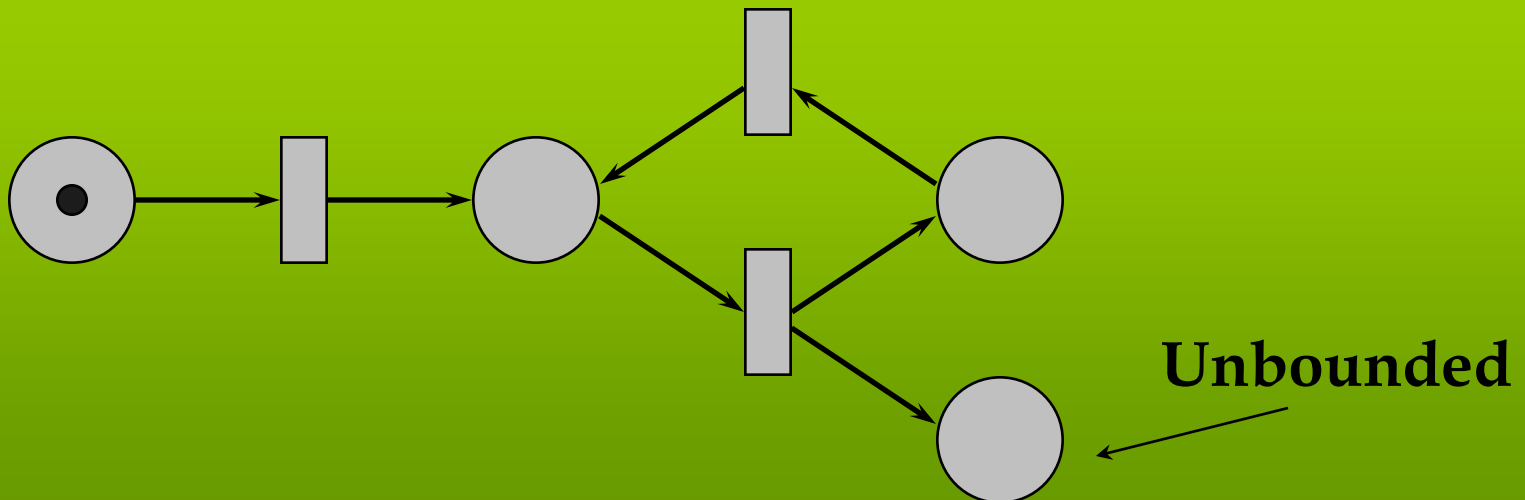
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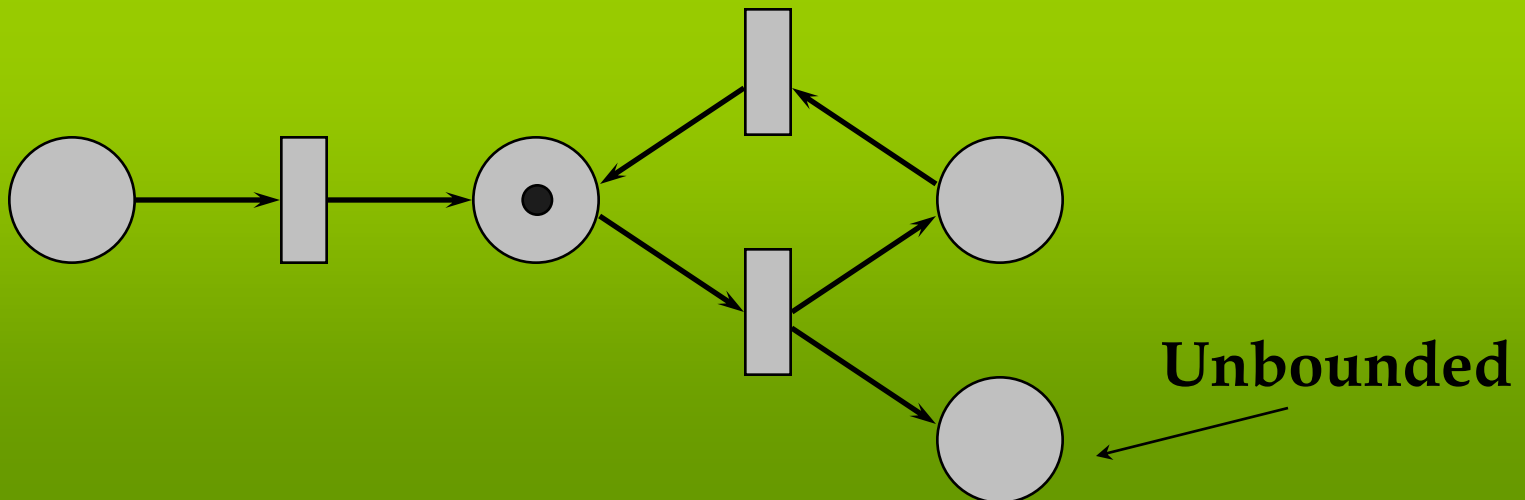
Boundedness

- **Boundedness**: the number of tokens in any place cannot grow indefinitely
 - (1-bounded also called *safe*)
 - Application: places represent buffers and registers (check there is no overflow)



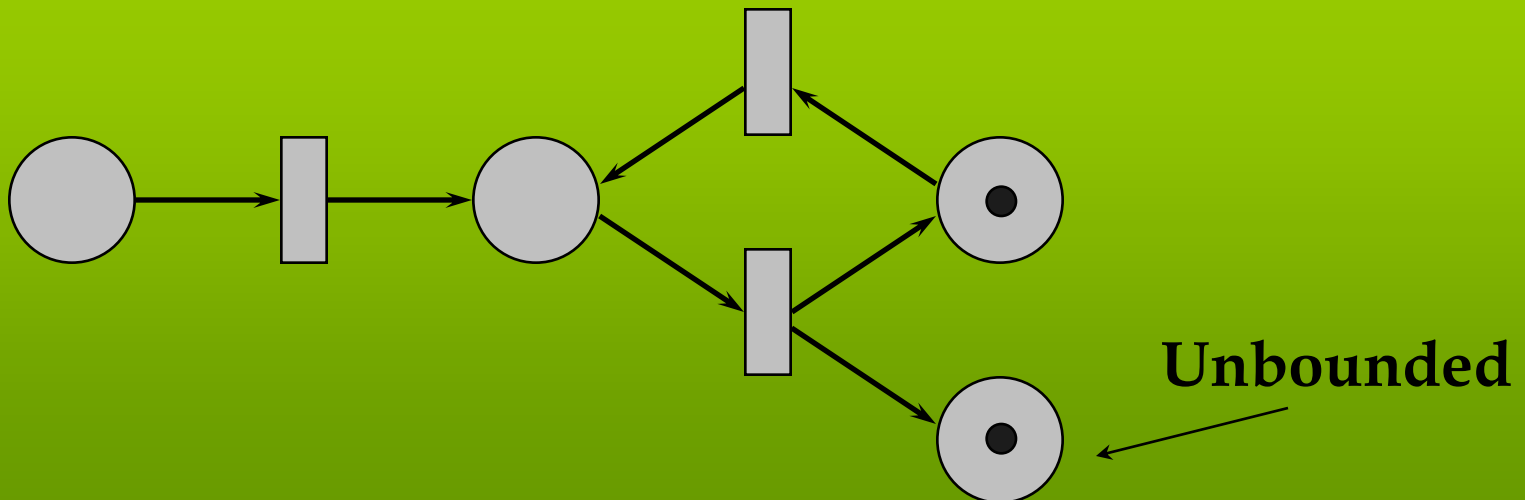
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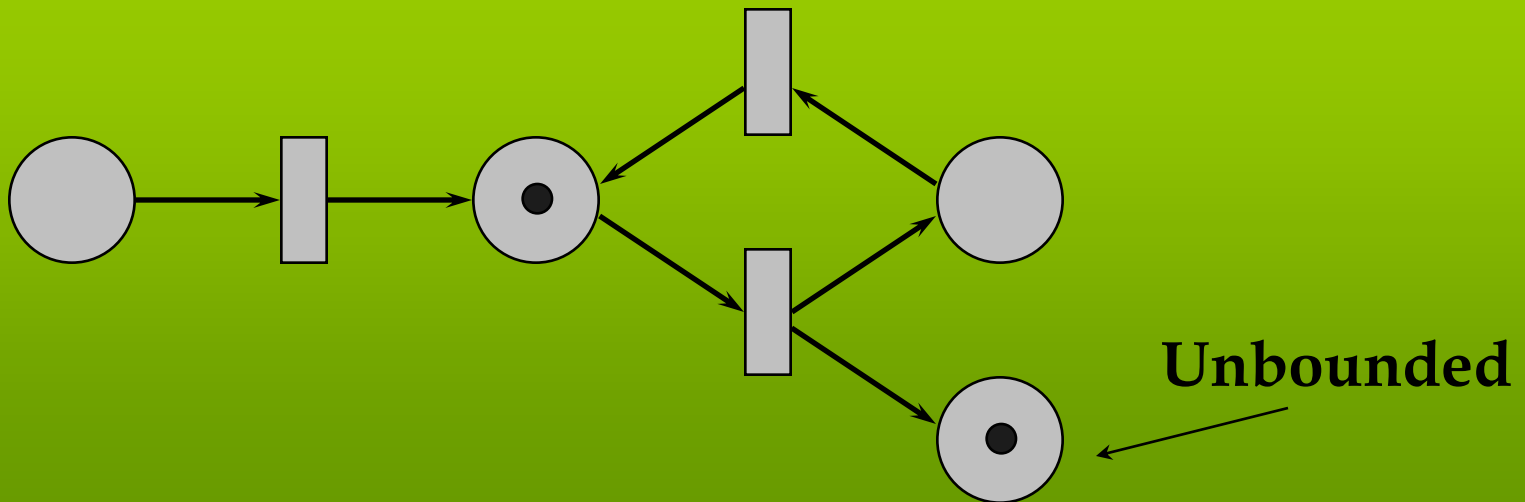
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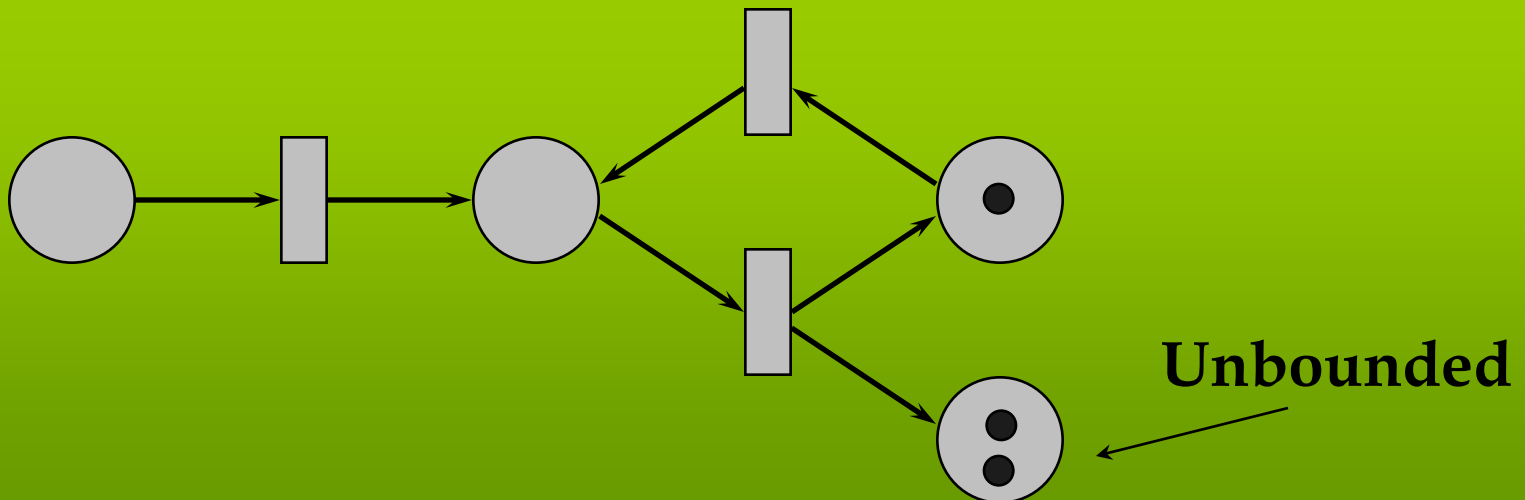
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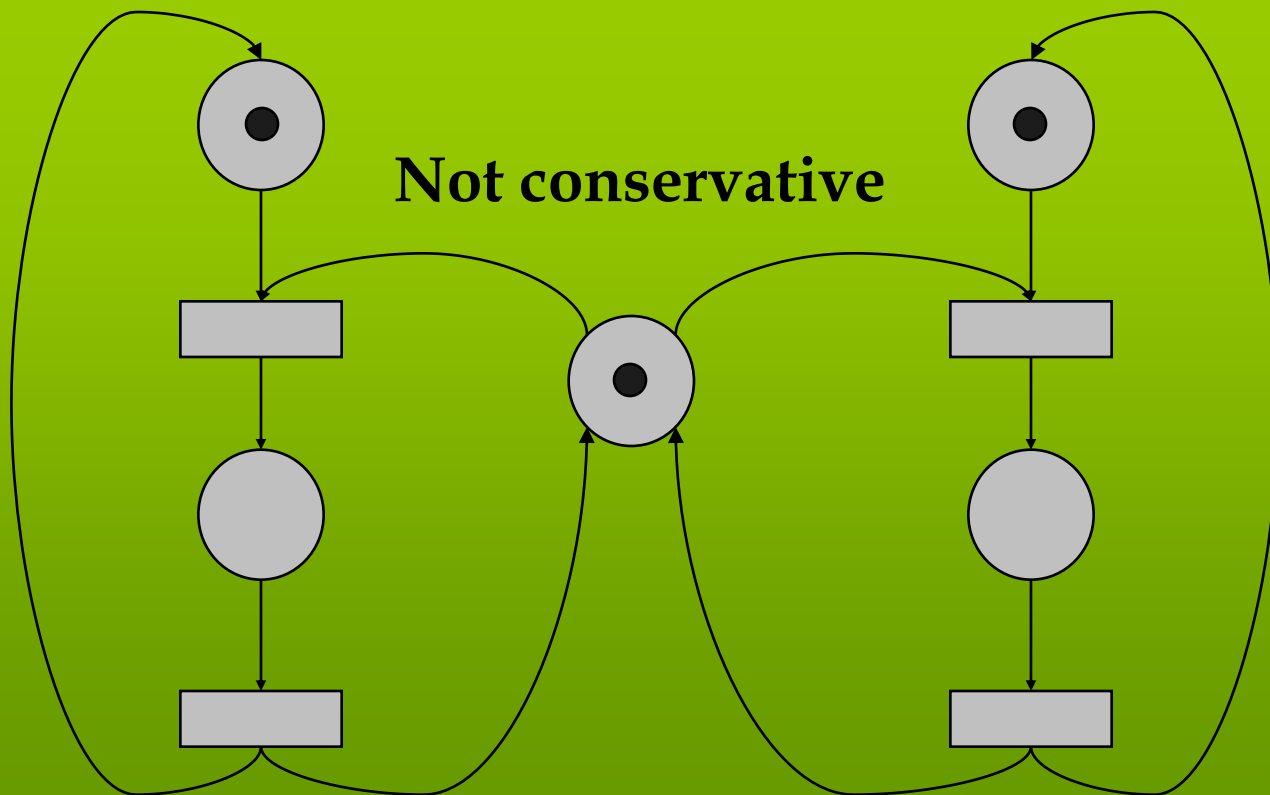
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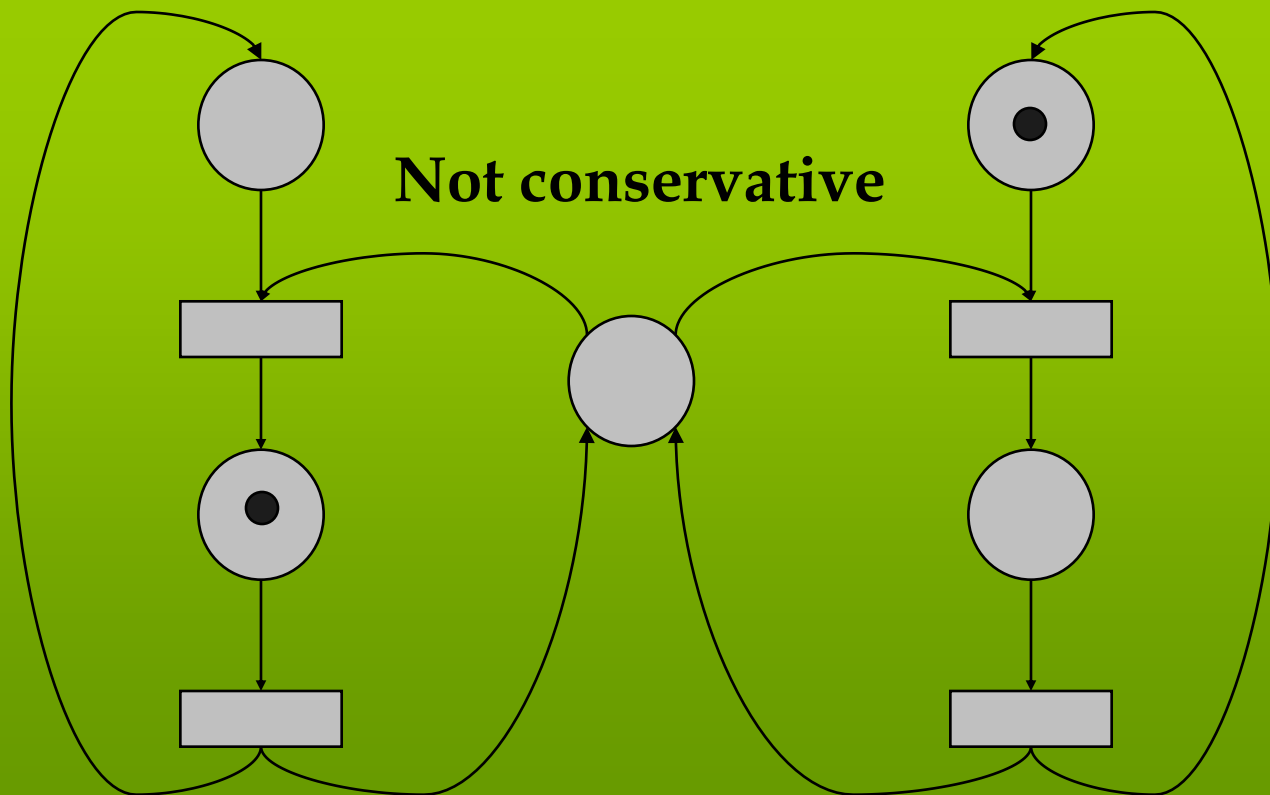
Conservation

- **Conservation**: the total number of tokens in the net is constant



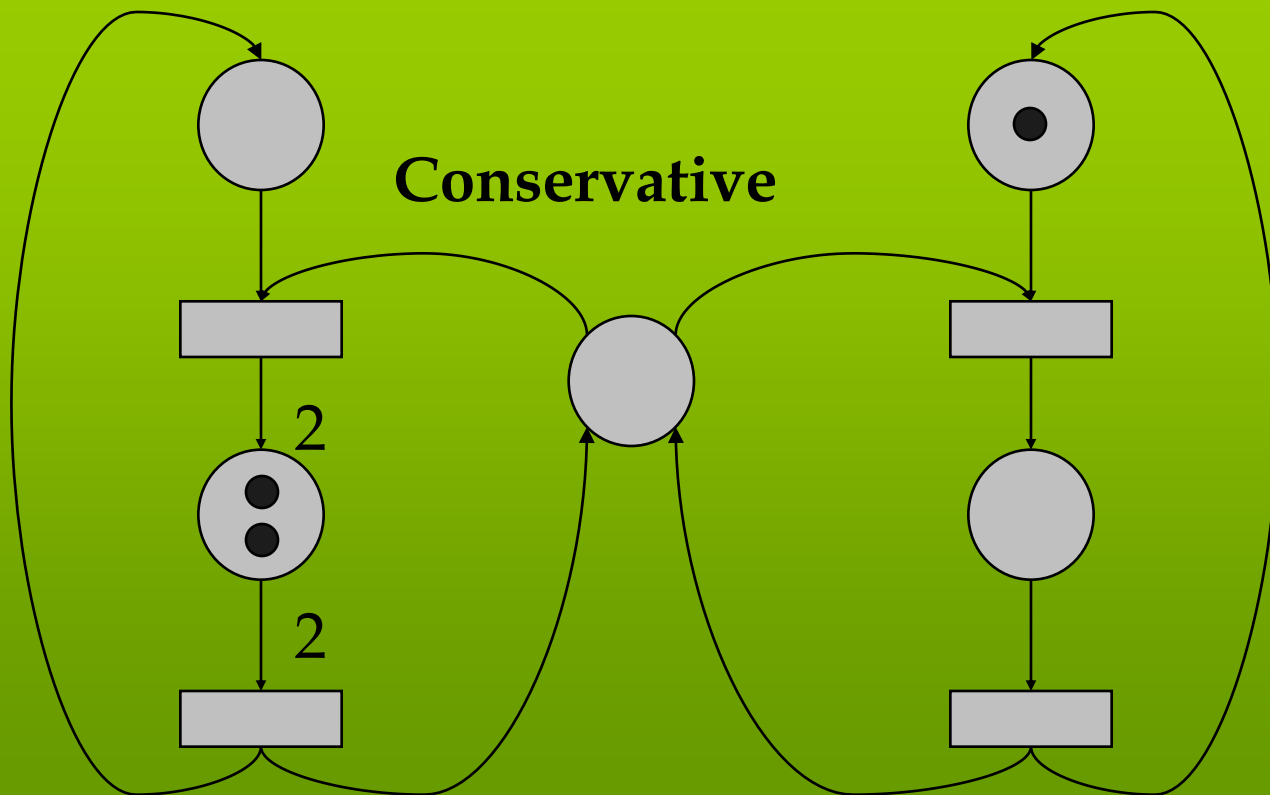
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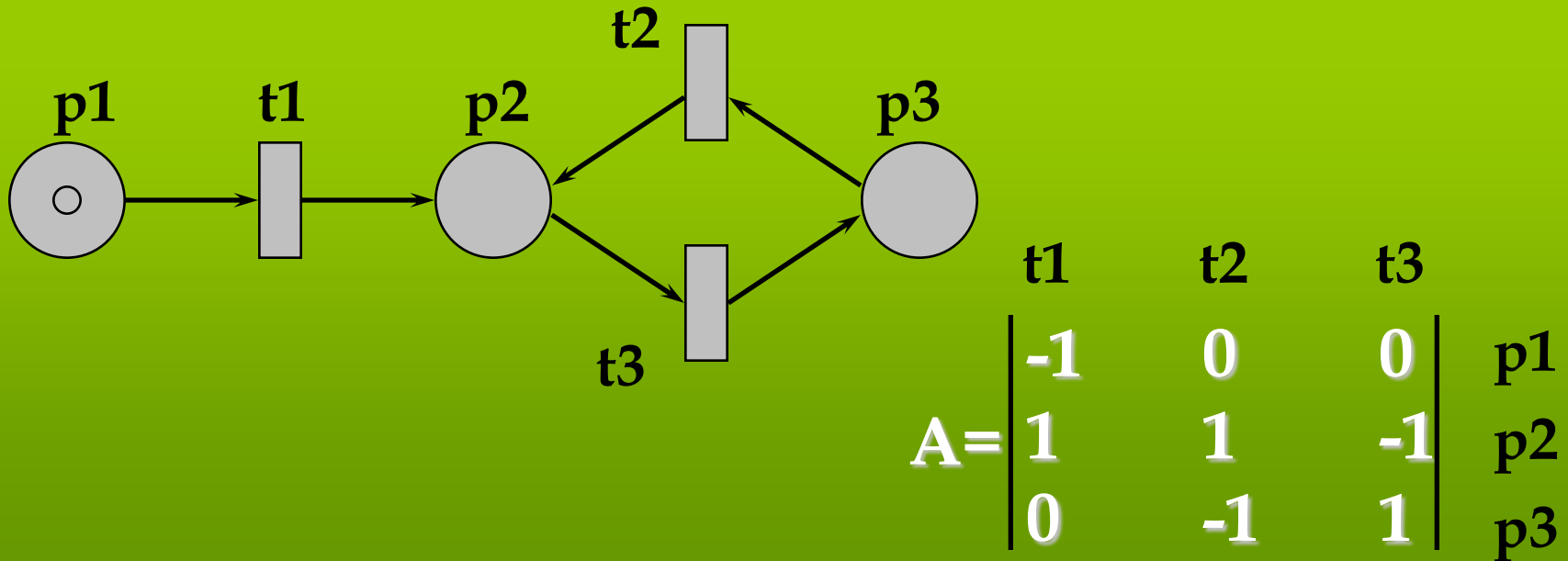




Analysis techniques

- **Structural analysis techniques**
 - Incidence matrix
 - T- and S- Invariants
- **State Space Analysis techniques**
 - Coverability Tree
 - Reachability Graph

Incidence Matrix



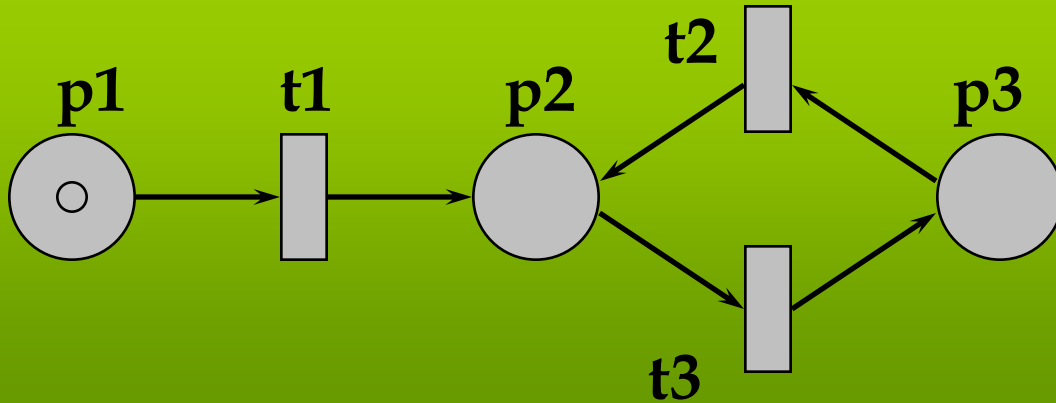
- Necessary condition for marking M to be reachable from initial marking M_0 :

there exists **firing vector** v s.t.:

$$M = M_0 + A v$$

State equations

- E.g. reachability of $M = |0\ 0\ 1|^T$ from $M_0 = |1\ 0\ 0|^T$

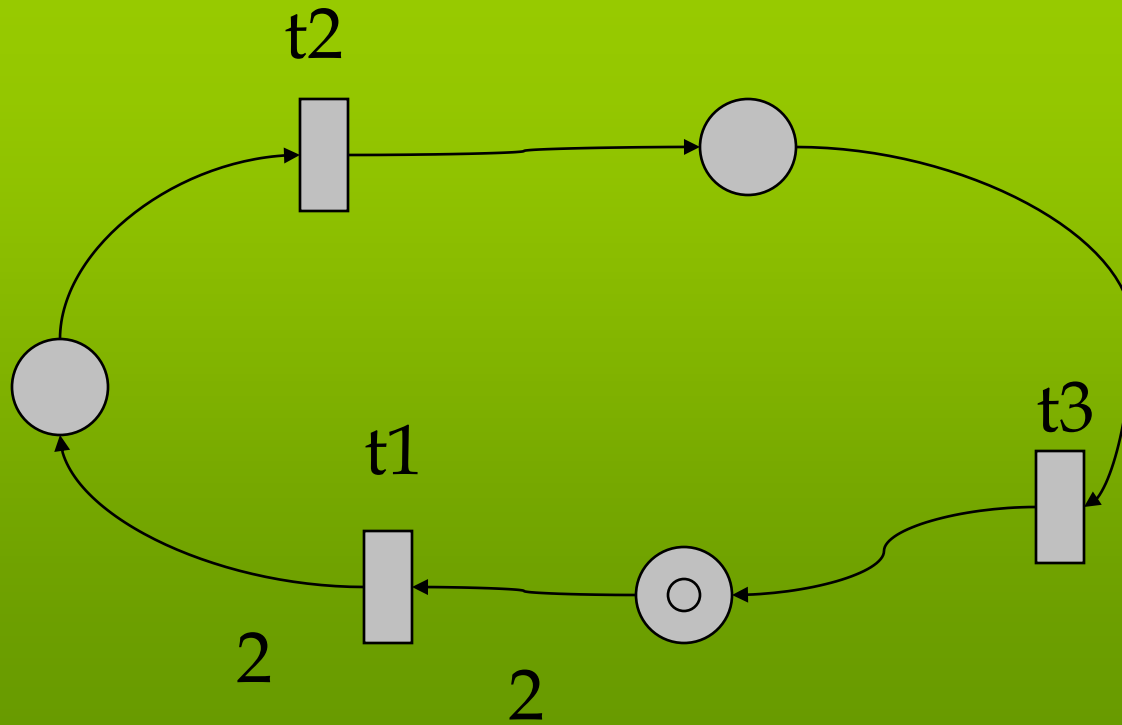


$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

but also $v_2 = |1\ 1\ 2|^T$ or any $v_k = |1\ (k)\ (k+1)|^T$

Necessary Condition only



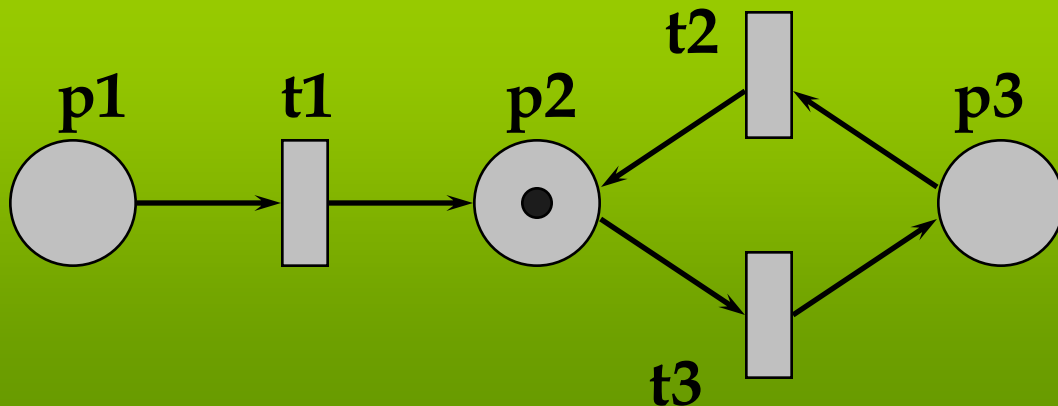
Deadlock!!

State equations and invariants

- Solutions of $Ax = 0$ (in $M = M_0 + Ax$, $M = M_0$)

T-invariants

- sequences of transitions that (if fireable) bring back to original marking
- periodic schedule in SDF
- e.g. $x = | 0 \ 1 \ 1 |^T$



$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$



Application of T-invariants

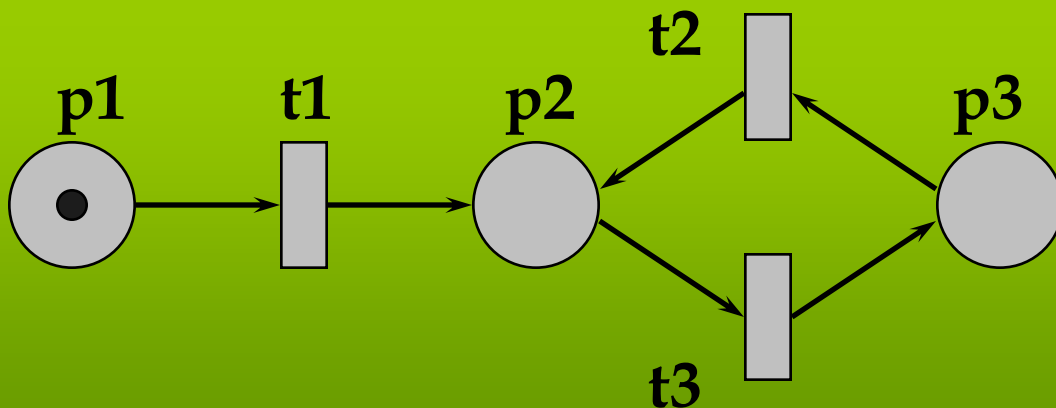
- Scheduling
 - **Cyclic schedules**: need to return to the initial state

State equations and invariants

- Solutions of $yA = 0$

S-invariants

- sets of places whose weighted total token count does not change after the firing of any transition ($y M = y M'$)
- e.g. $y = | 1 1 1 |^T$



$$A^T = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$



Application of S-invariants

- **Structural Boundedness: bounded for any finite initial marking M_0**
- **Existence of a positive S-invariant is sufficient condition for structural boundedness**
 - initial marking is finite
 - weighted token count does not change



Summary of algebraic methods

- **Extremely efficient**
(polynomial in the size of the net)
- **Generally provide only **necessary** or **sufficient** information**
- **Excellent for ruling out **some** deadlocks or otherwise dangerous conditions**
- **Can be used to infer structural boundedness**



Coverability Tree

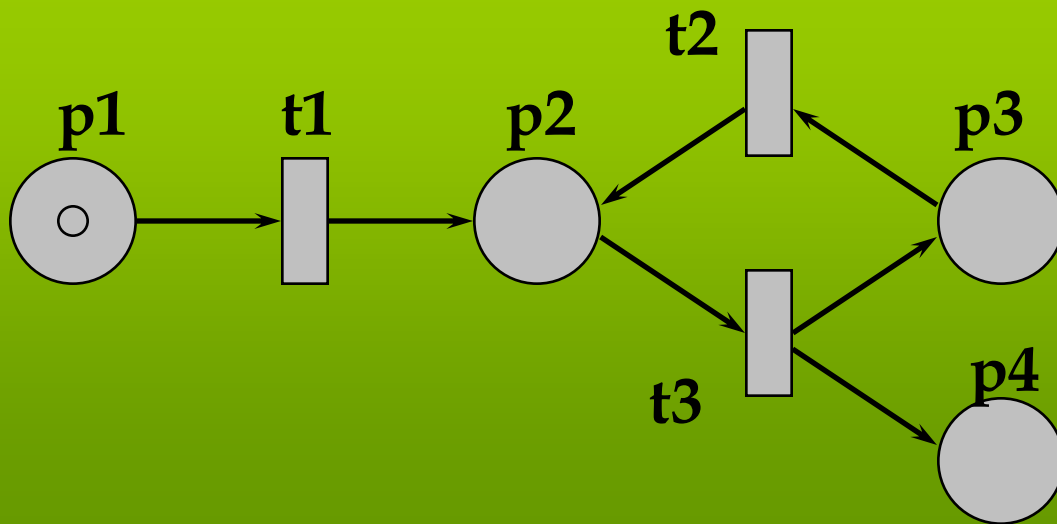
- **Build a (finite) tree representation of the markings**

Karp-Miller algorithm

- **Label initial marking M_0 as the root of the tree and tag it as *new***
- **While new markings exist do:**
 - **select a new marking M**
 - **if M is identical to a marking on the path from the root to M , then tag M as *old* and go to another new marking**
 - **if no transitions are enabled at M , tag M *dead-end***
 - **while there exist enabled transitions at M do:**
 - **obtain the marking M' that results from firing t at M**
 - **on the path from the root to M if there exists a marking M'' such that $M'(p) \geq M''(p)$ for each place p and M' is different from M'' , then replace $M'(p)$ by ω for each p such that $M'(p) > M''(p)$**
 - **introduce M' as a node, draw an arc with label t from M to M' and tag M' as *new*.**

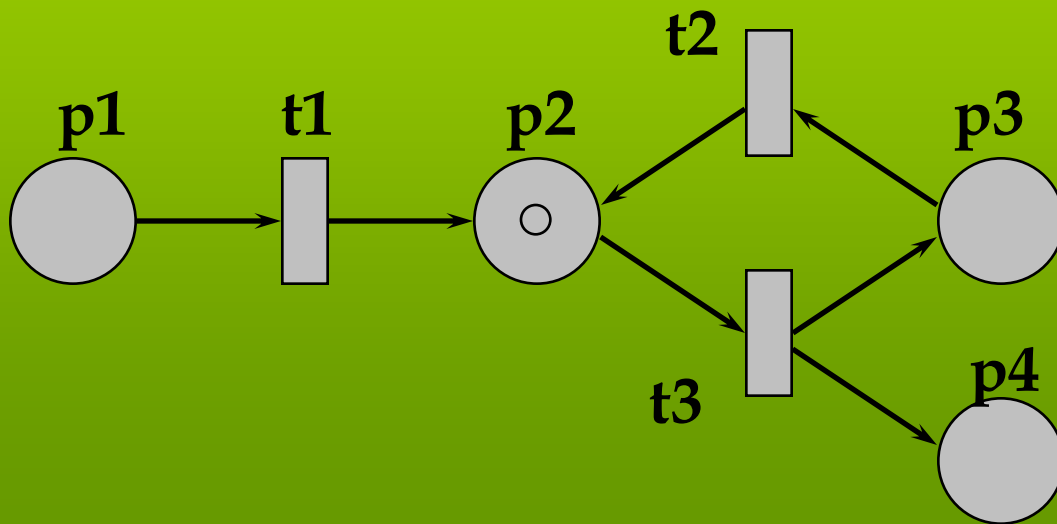
Coverability Tree

- Boundedness is decidable with *coverability tree*



Coverability Tree

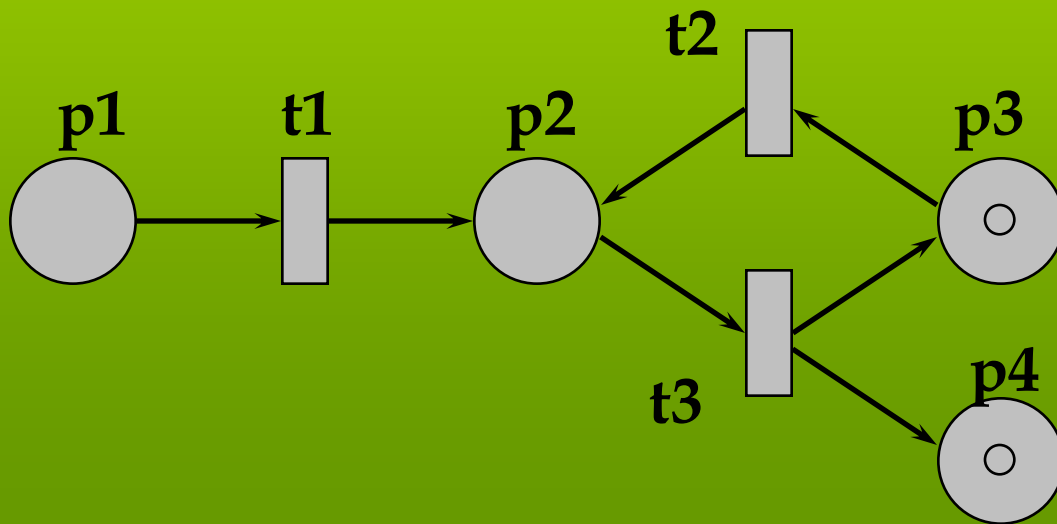
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1000
↓ t1
0100

Coverability Tree

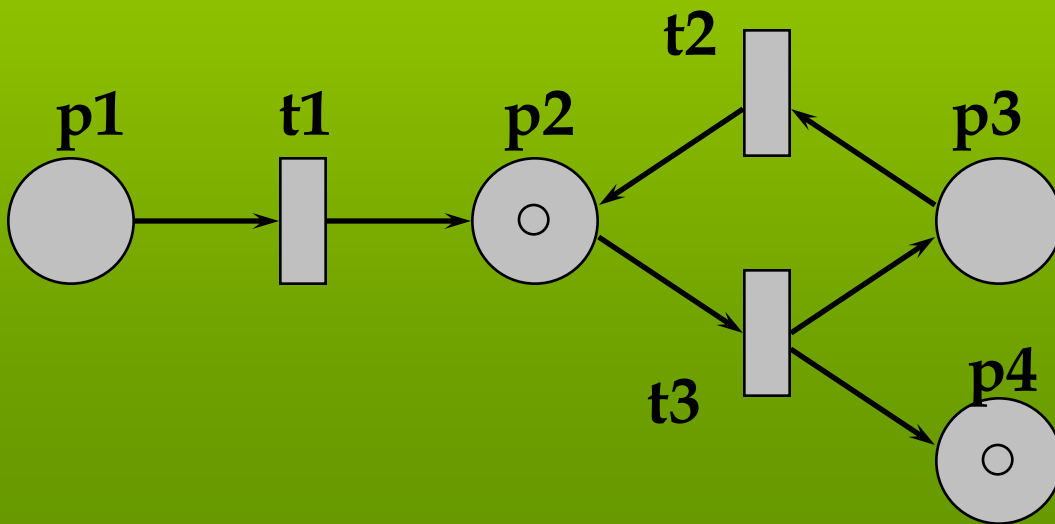
- Boundedness is decidable
with *coverability tree*



1000
↓ t1
0100
↓ t3
0011

Coverability Tree

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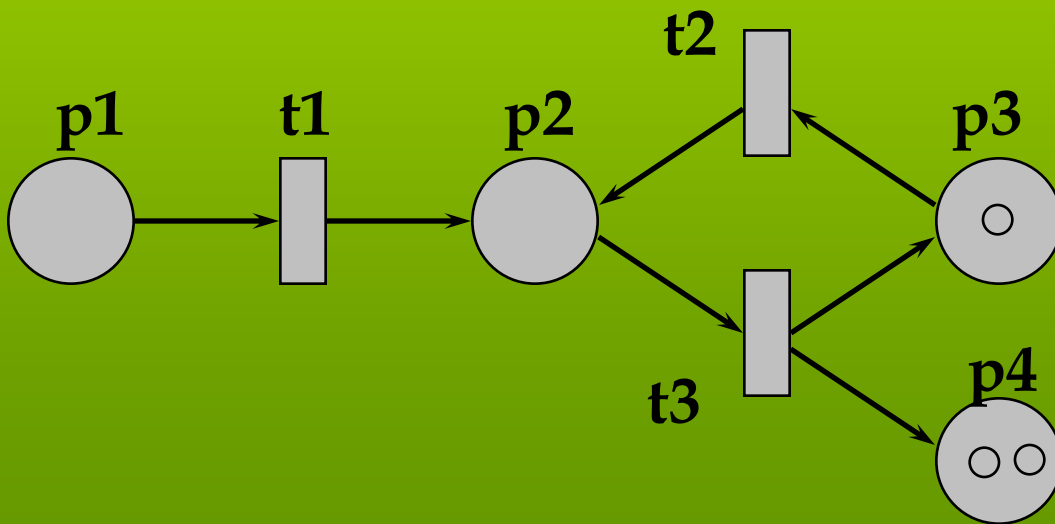


```

1000
  ↓ t1
0100
  ↓ t3
0011
  ↓ t2
0101
  
```

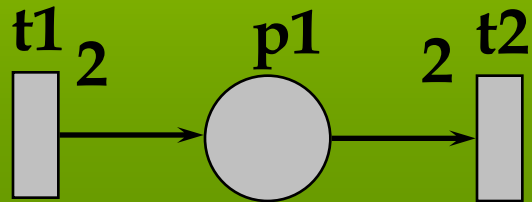
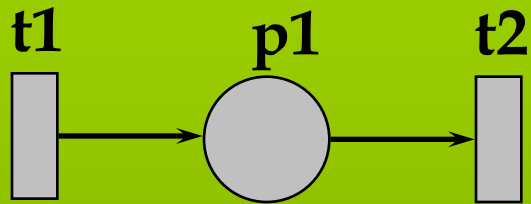
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with *coverability tree*



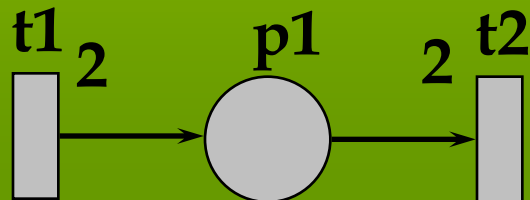
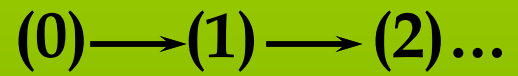
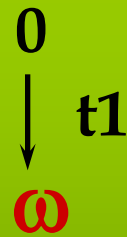
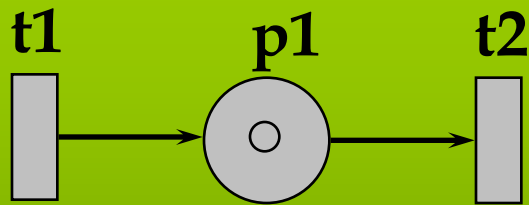
Coverability Tree

- Is (1) reachable from (0)?



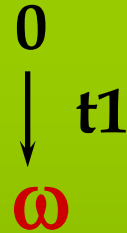
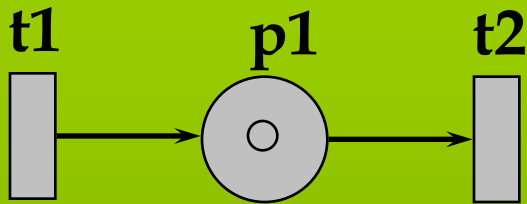
Coverability Tree

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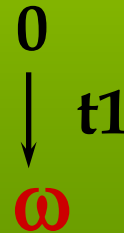
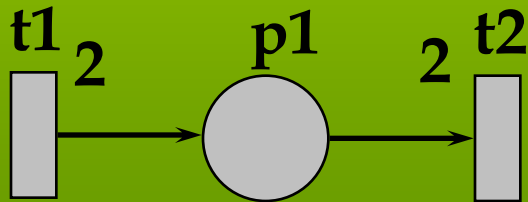


Coverability Tree

- Is (1) reachable from (0)?



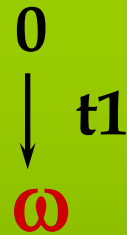
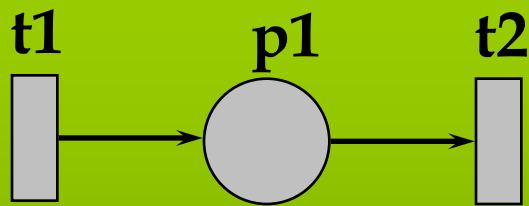
$$(0) \rightleftharpoons (1) \rightleftharpoons (2) \dots$$



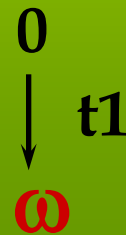
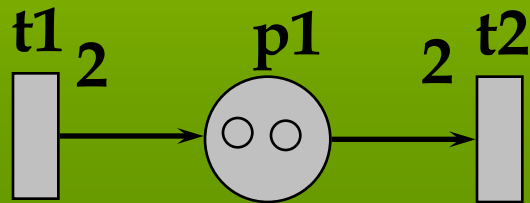
$$(0) \longrightarrow (2) \longrightarrow (0) \dots$$

Coverability Tree

- Is (1) reachable from (0)?



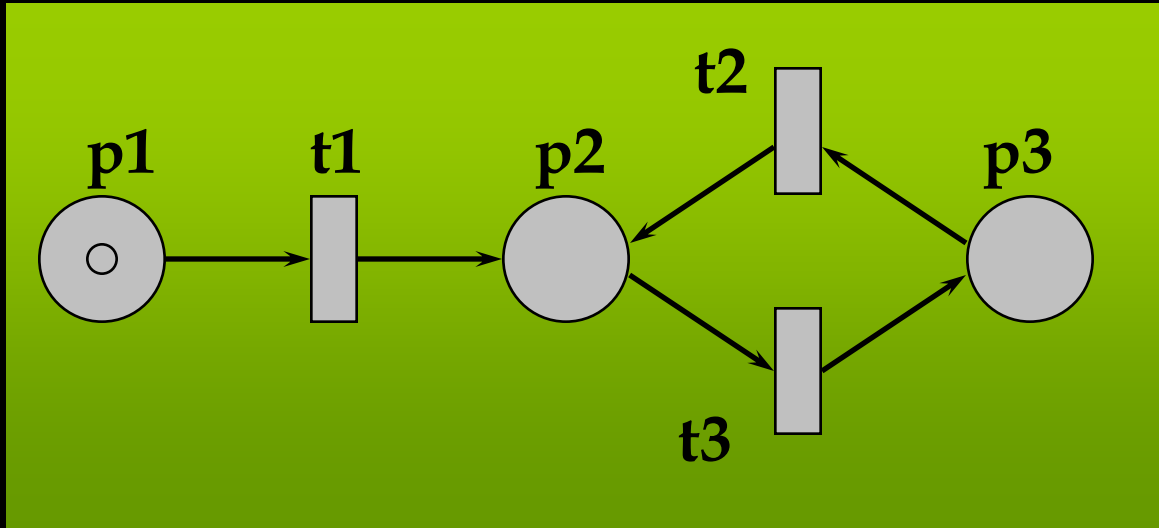
$(0) \longrightarrow (1) \longrightarrow (2) \dots$



$(0) \longrightarrow (2) \longrightarrow (0) \dots$

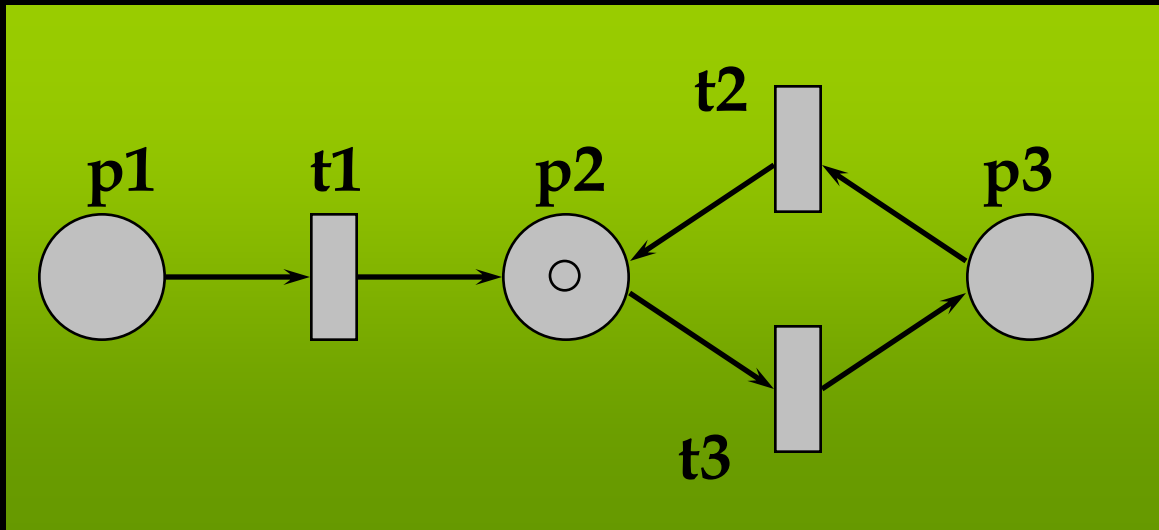
- Cannot solve the reachability problem

Reachability graph



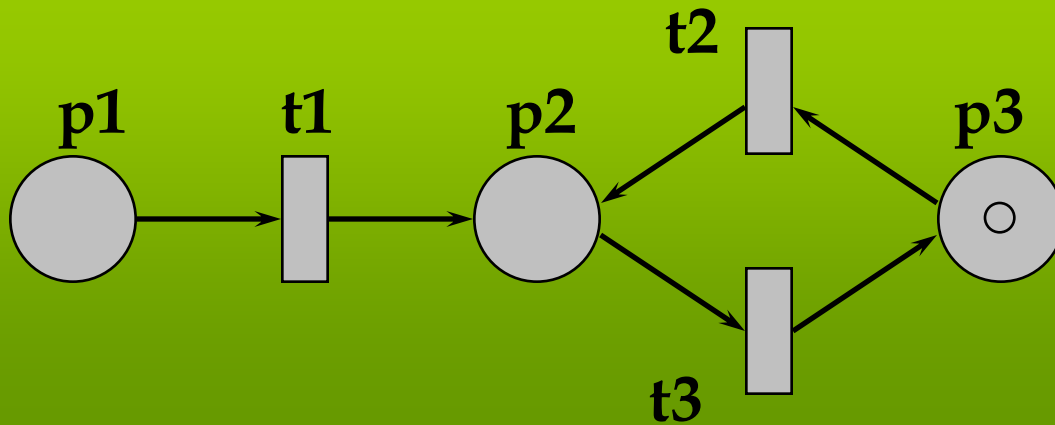
- For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings

Reachability graph



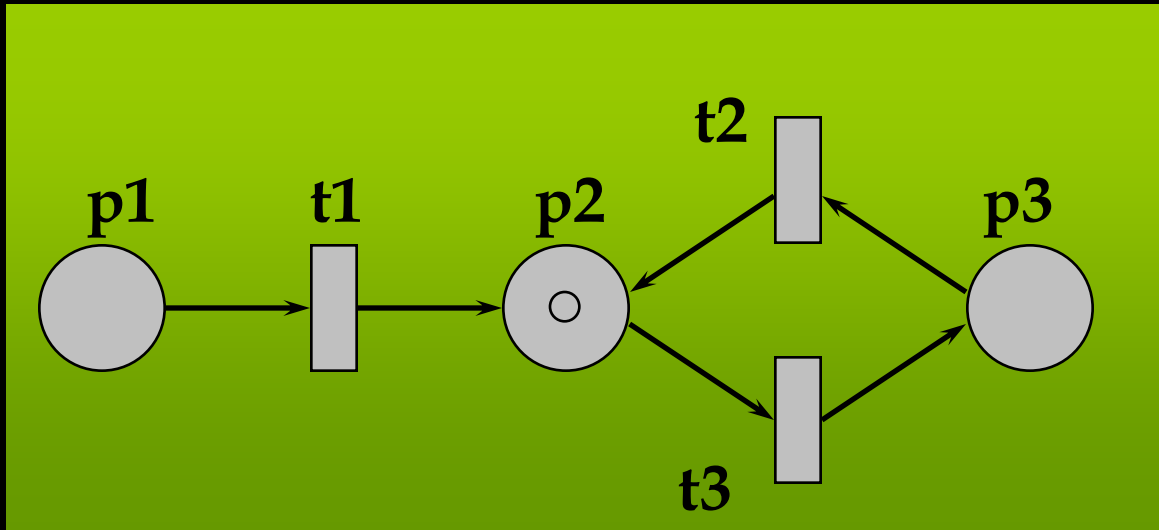
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Reachability graph



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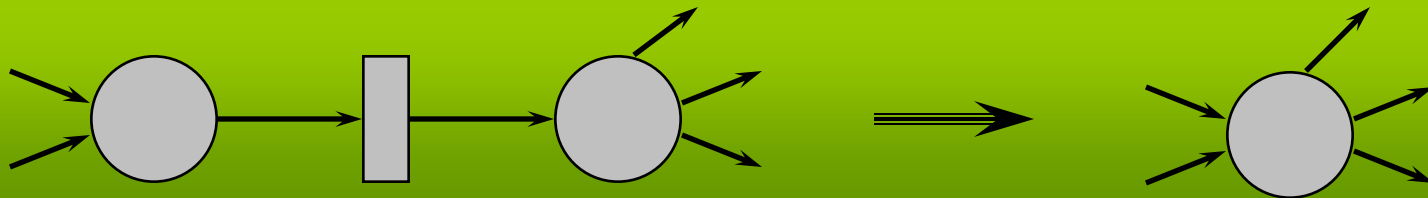
Reachability graph



- For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings

Subclasses of Petri nets

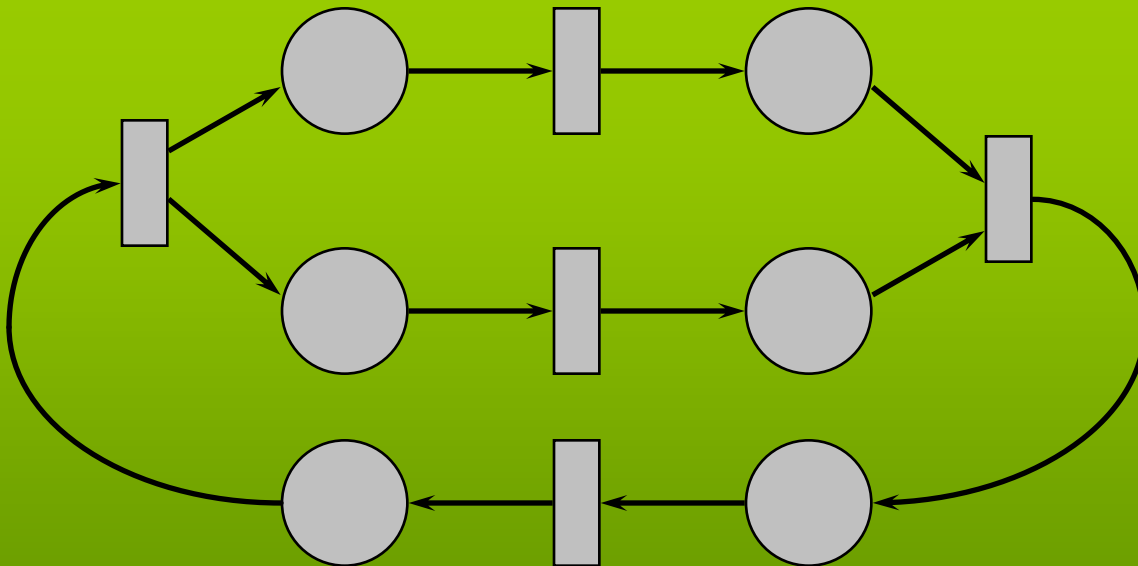
- Reachability analysis is too expensive
- State equations give only partial information
- Some properties are preserved by **reduction rules**
e.g. for liveness and safeness



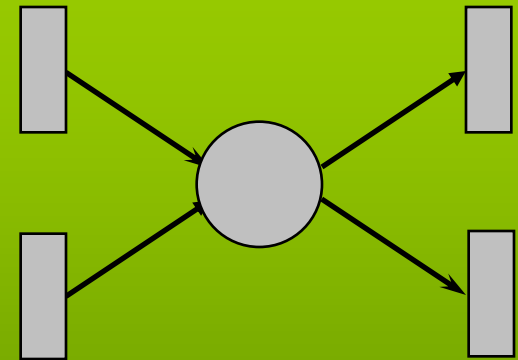
- Even reduction rules only work in some cases
- Must restrict class in order to prove stronger results

Marked Graphs

- Every place has at most 1 predecessor and 1 successor transition
- Models only **causality** and **concurrency** (no conflict)



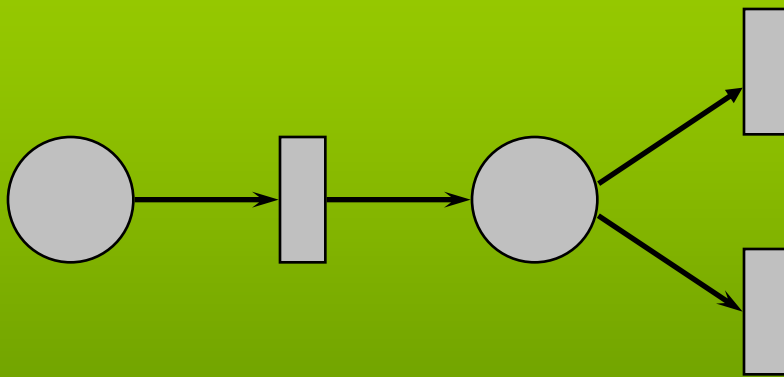
YES



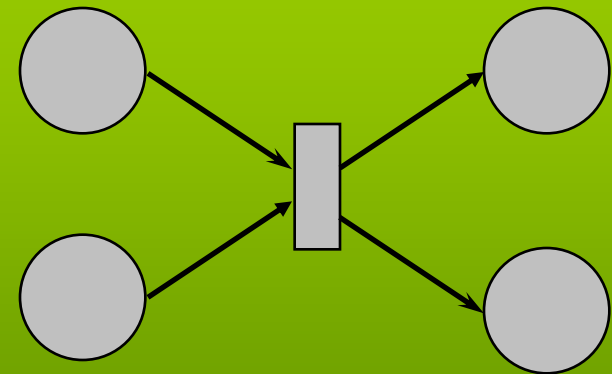
NO

State Machines

- Every transition has at most 1 predecessor and 1 successor place
- Models only **causality** and **conflict**
 - (no concurrency, no synchronization of parallel activities)

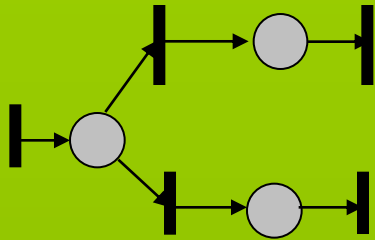


YES



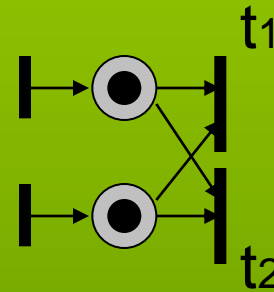
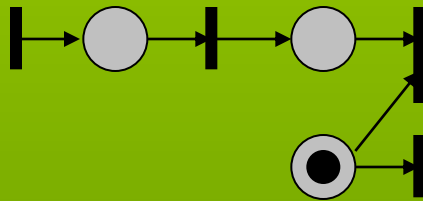
NO

Free-Choice Petri Nets (FCPN)



Free-Choice (FC)

every transition after choice has **exactly 1** predecessor place



Confusion (not-Free-Choice) Extended Free-Choice

Free-Choice: the outcome of a choice depends on the value of a token (abstracted non-deterministically) rather than on its arrival time.



Free-Choice nets

- Introduced by Hack ('72)
- Extensively studied by Best ('86) and Desel and Esparza ('95)
- Can express concurrency, causality and choice **without confusion**
- Very strong structural theory
 - necessary and sufficient conditions for liveness and safeness, based on **decomposition**
 - exploits **duality** between MG and SM

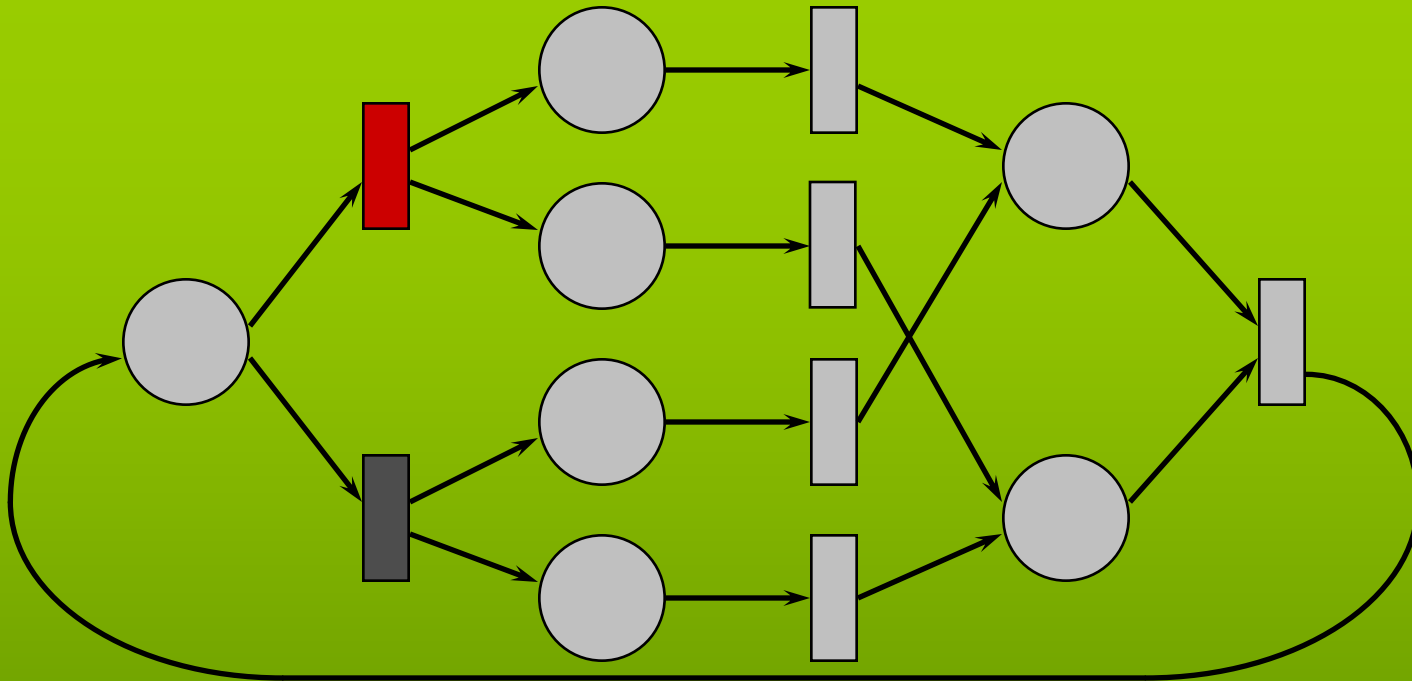


MG (& SM) decomposition

- An **Allocation** is a control function that chooses which transition fires among several conflicting ones ($A: P \rightarrow T$).
- Eliminate the subnet that would be inactive if we were to use the allocation...
- **Reduction Algorithm**
 - Delete all unallocated transitions
 - Delete all places that have all input transitions already deleted
 - Delete all transitions that have at least one input place already deleted
- Obtain a **Reduction** (one for each allocation) that is a conflict free subnet

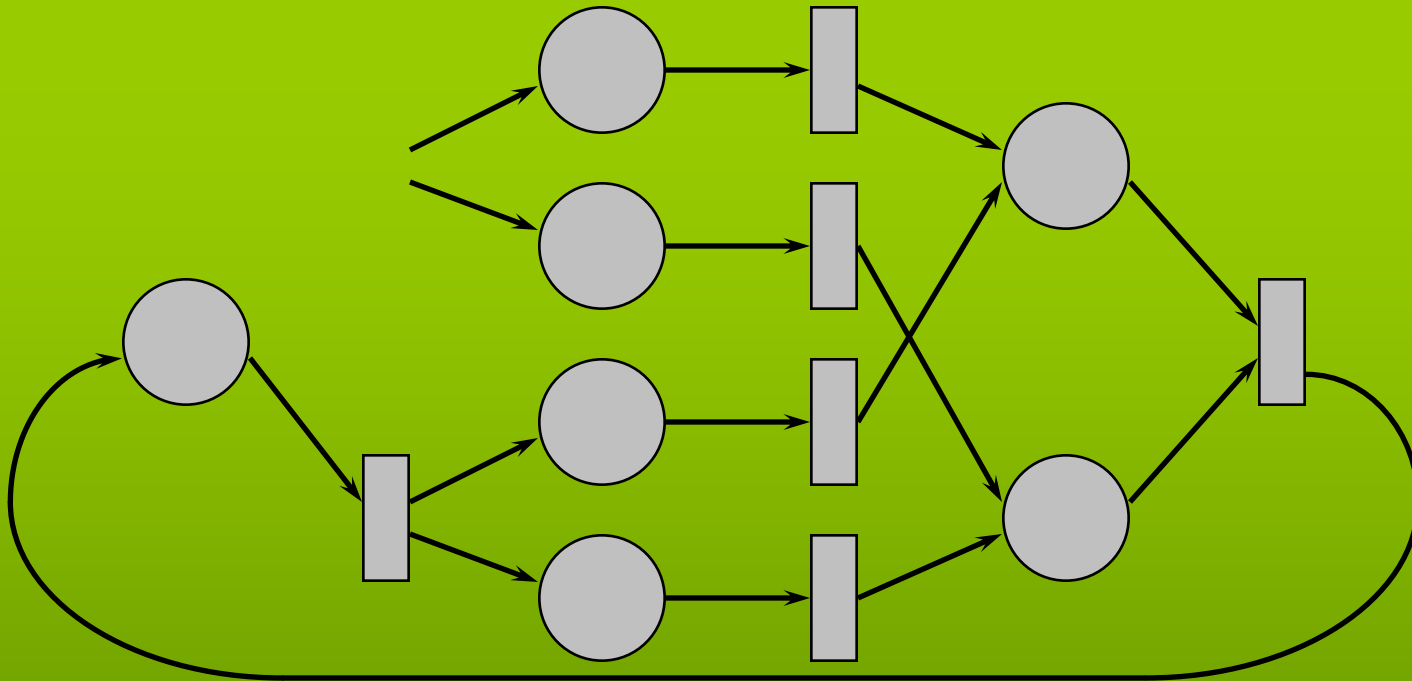
MG reduction and cover

- Choose one successor for each conflicting place:



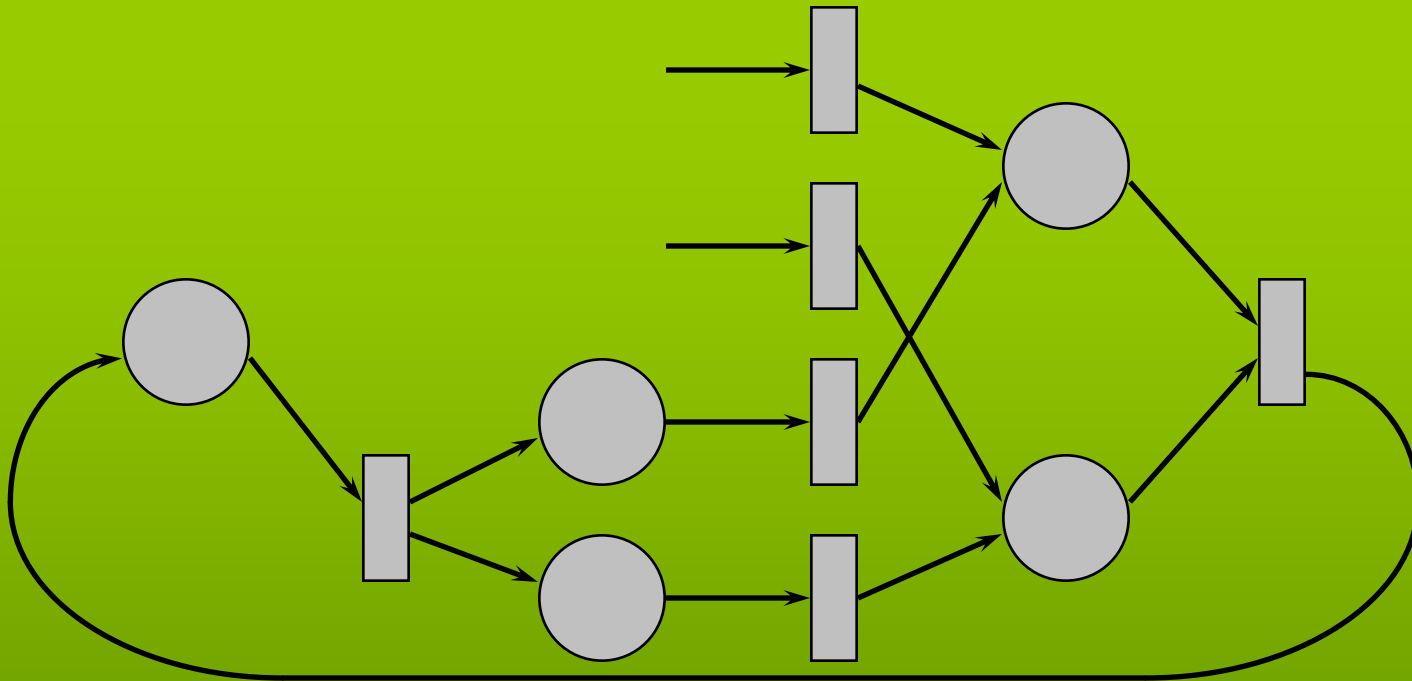
MG reduction and cover

- Choose one successor for each conflicting place:



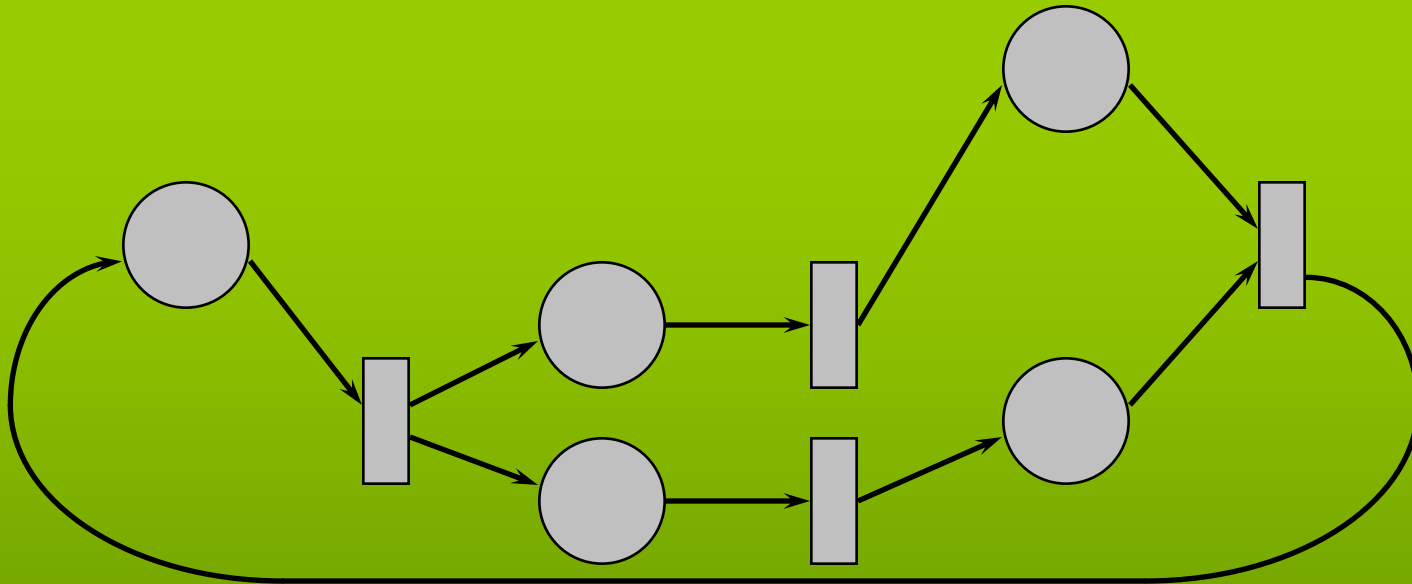
MG reduction and cover

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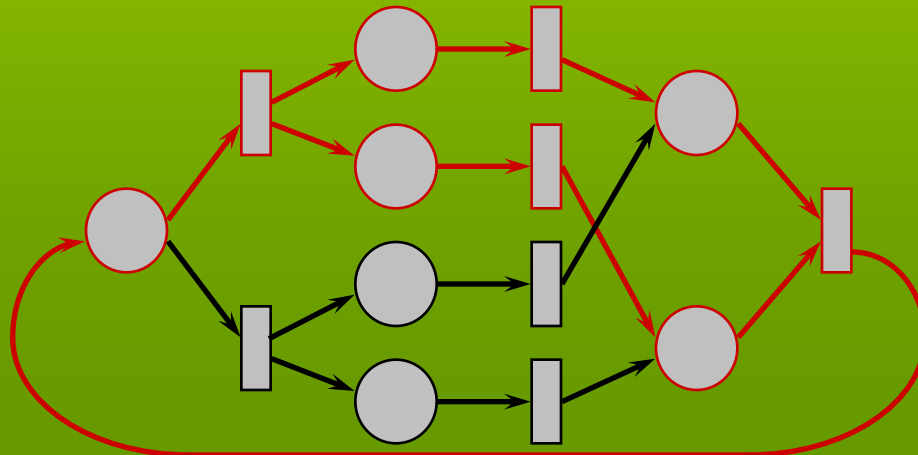
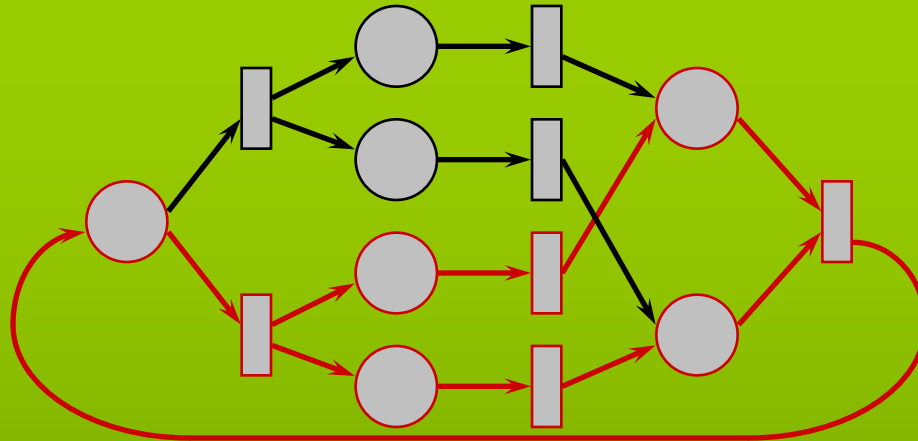
MG reduction and cover

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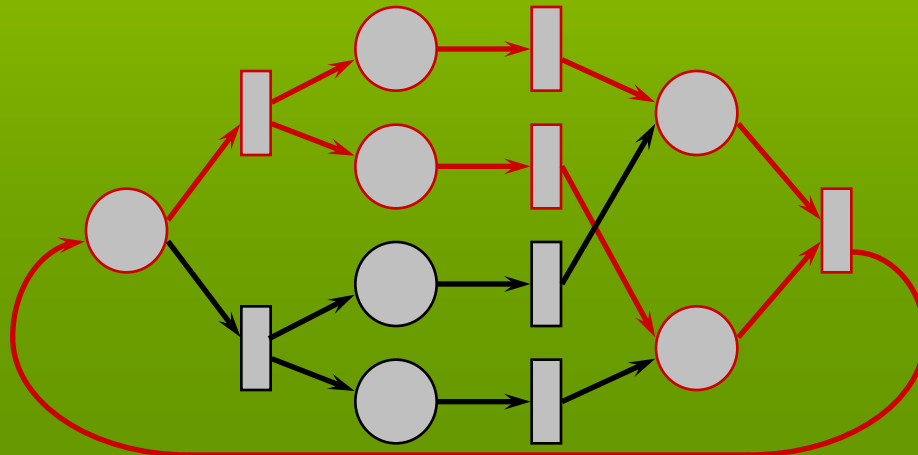
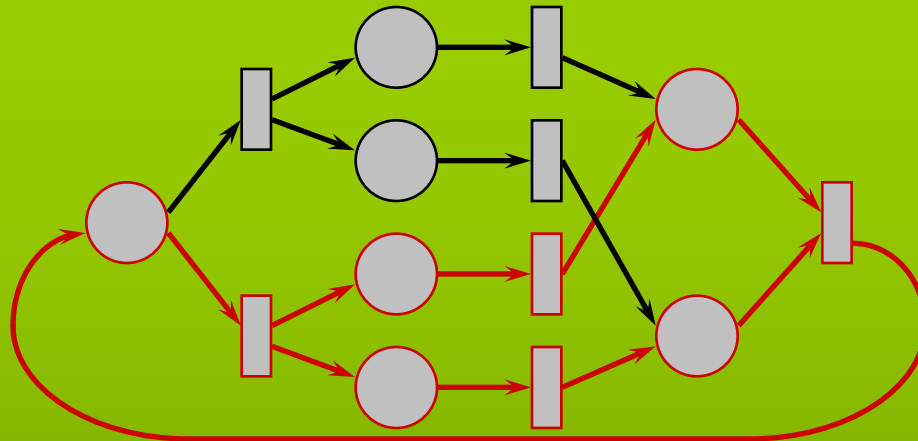
MG reductions

- The set of all reductions yields a **cover of MG components** (T-invariants)



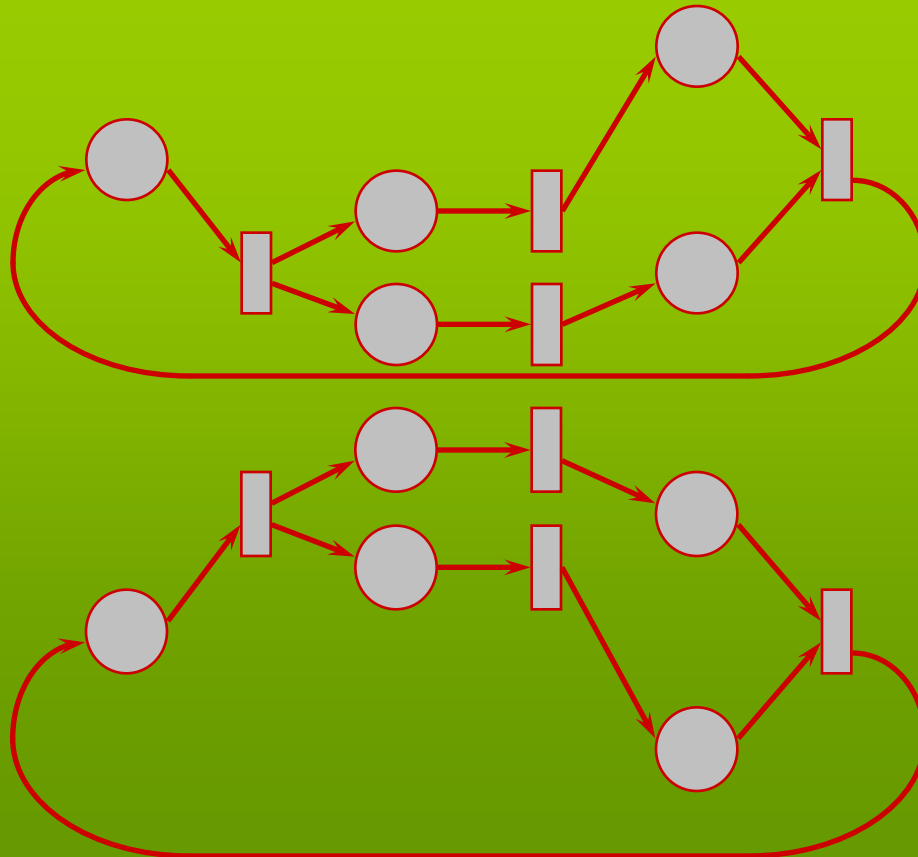
MG reductions

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MG reductions

- The set of all reductions yields a **cover of MG components** (T-invariants)



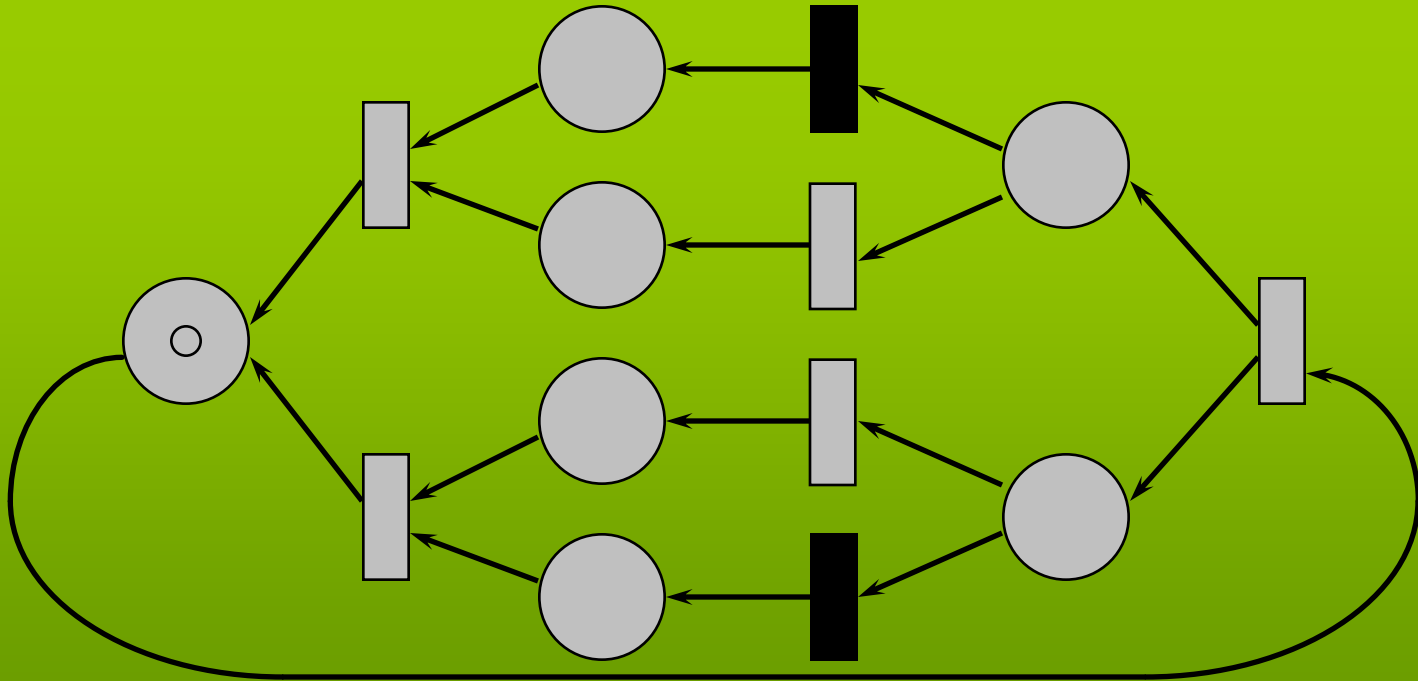


Hack's theorem ('72)

- Let N be a Free-Choice PN:
 - N has a live and safe initial marking (well-formed)**if and only if**
 - every MG reduction is strongly connected and not empty, and the set of all reductions covers the net
 - every SM reduction is strongly connected and not empty, and the set of all reductions covers the net

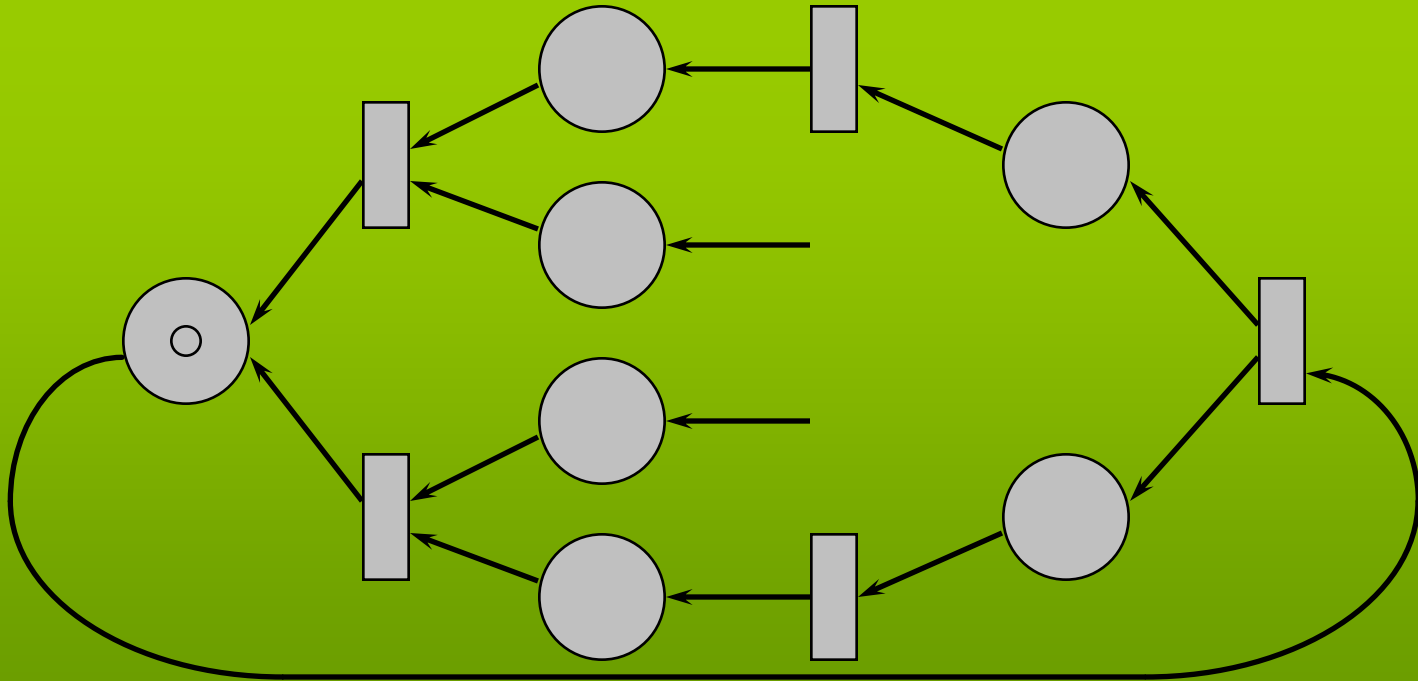
Hack's theorem

- Example of non-live (but safe) FCN



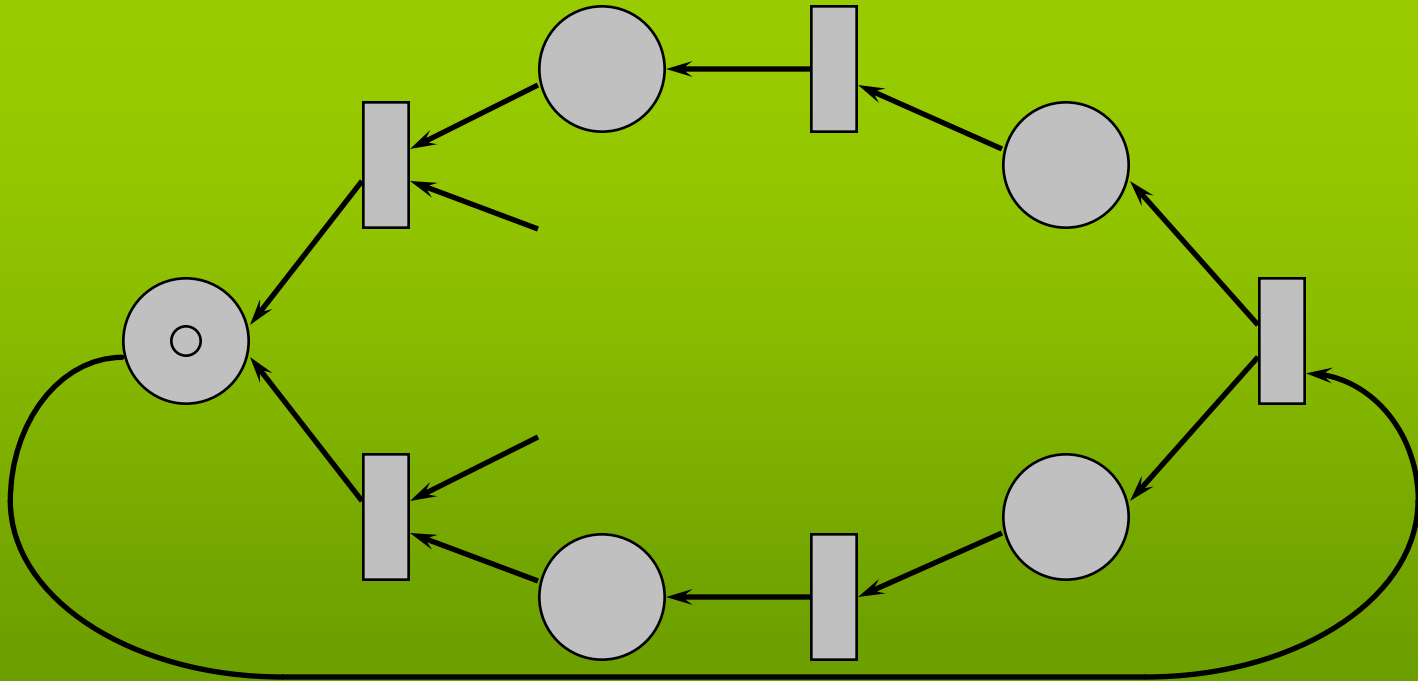
Hack's theorem

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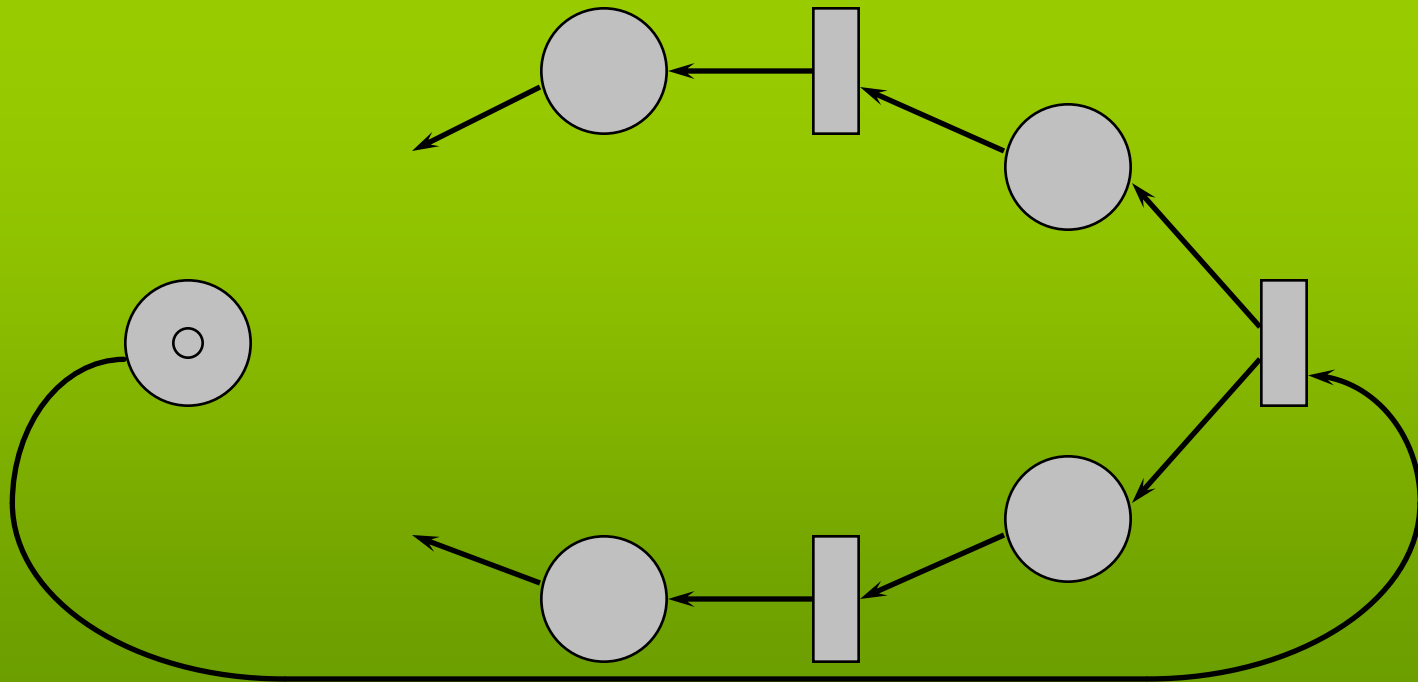
Hack's theorem

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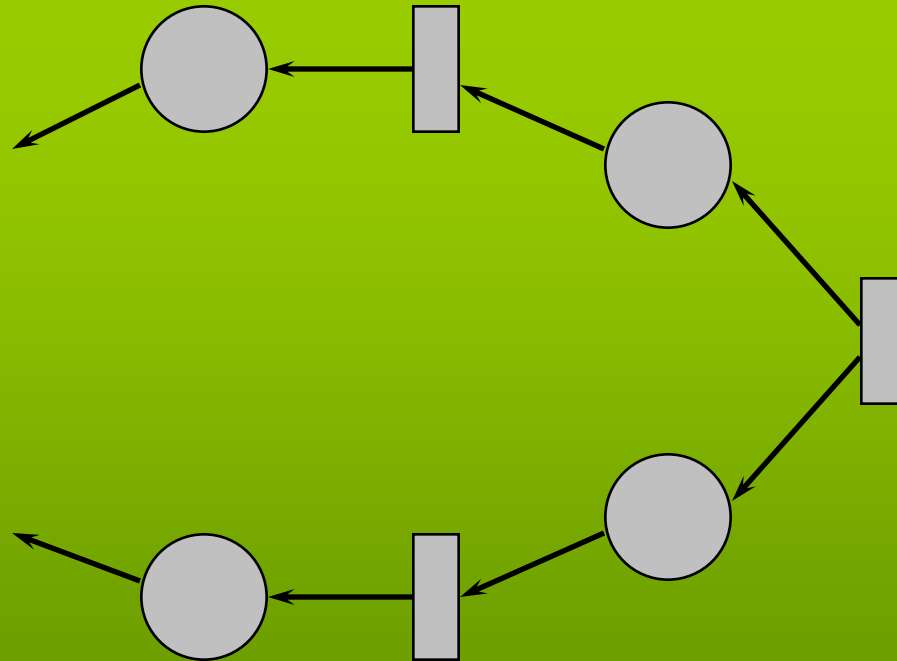
Hack's theorem

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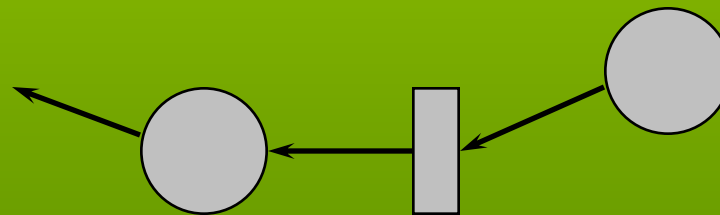
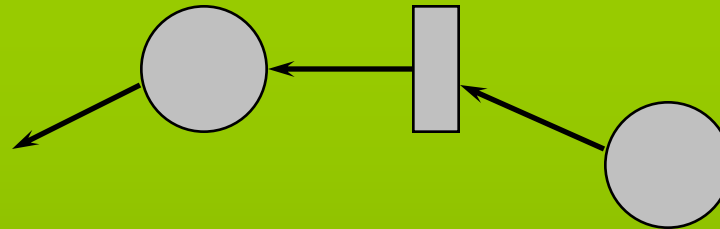
Hack's theorem

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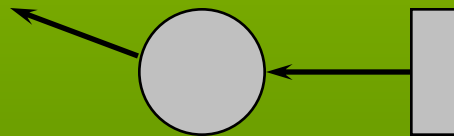
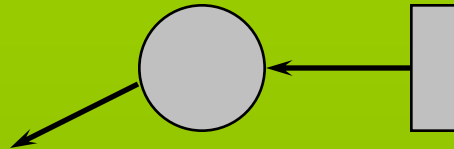
Hack's theorem

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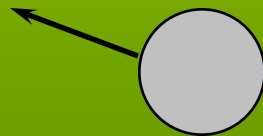
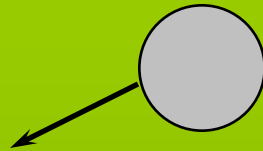
Hack's theorem

- Example of non-live (but safe) FCN



Hack's theorem

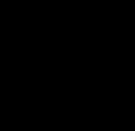
- **Example of non-live (but safe) FCN**





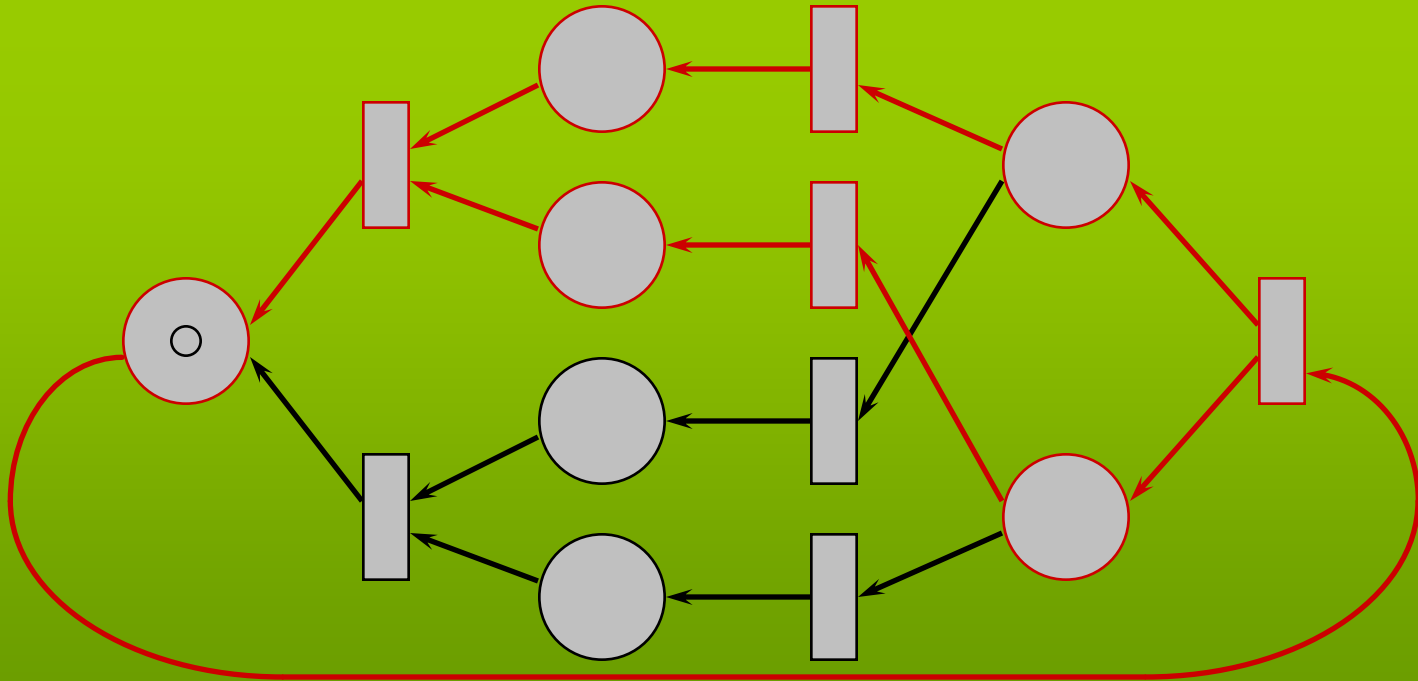
Hack's theorem

- **Example of non-live (but safe) FCN**



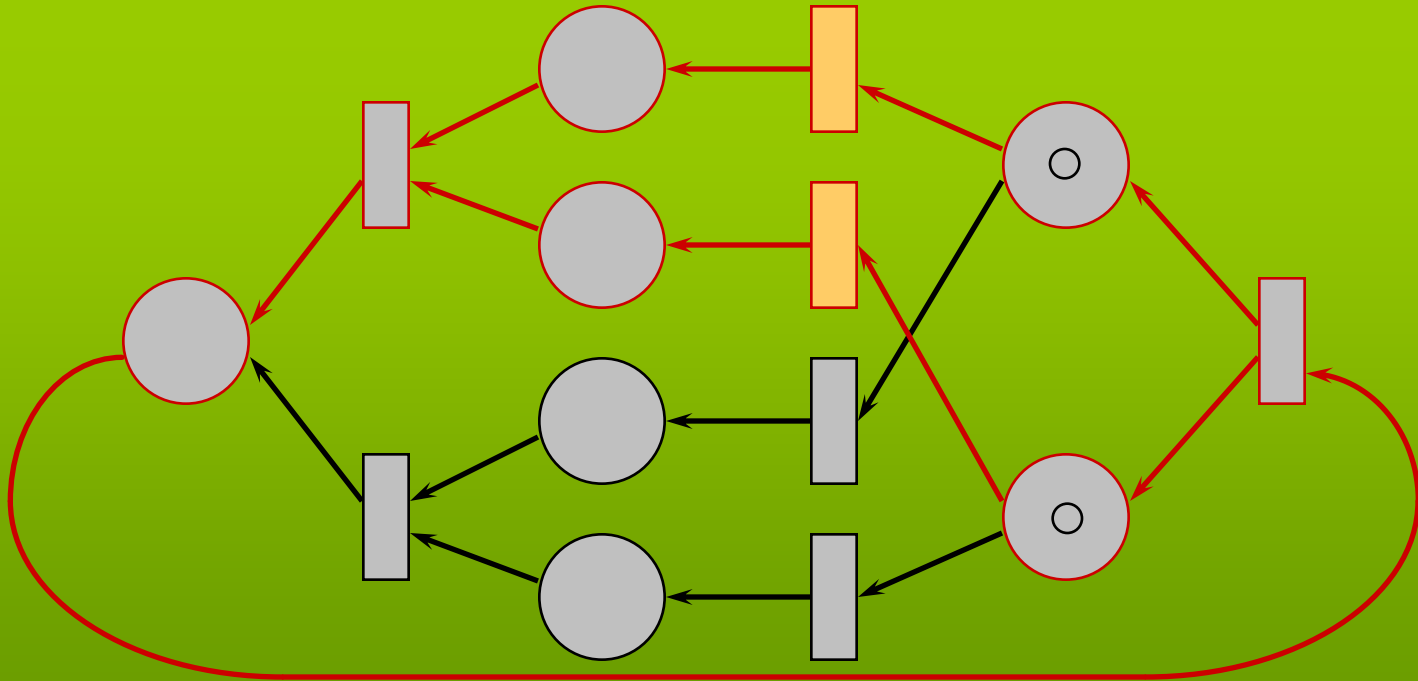
Hack's theorem

- Example of non-live (but safe) FCN



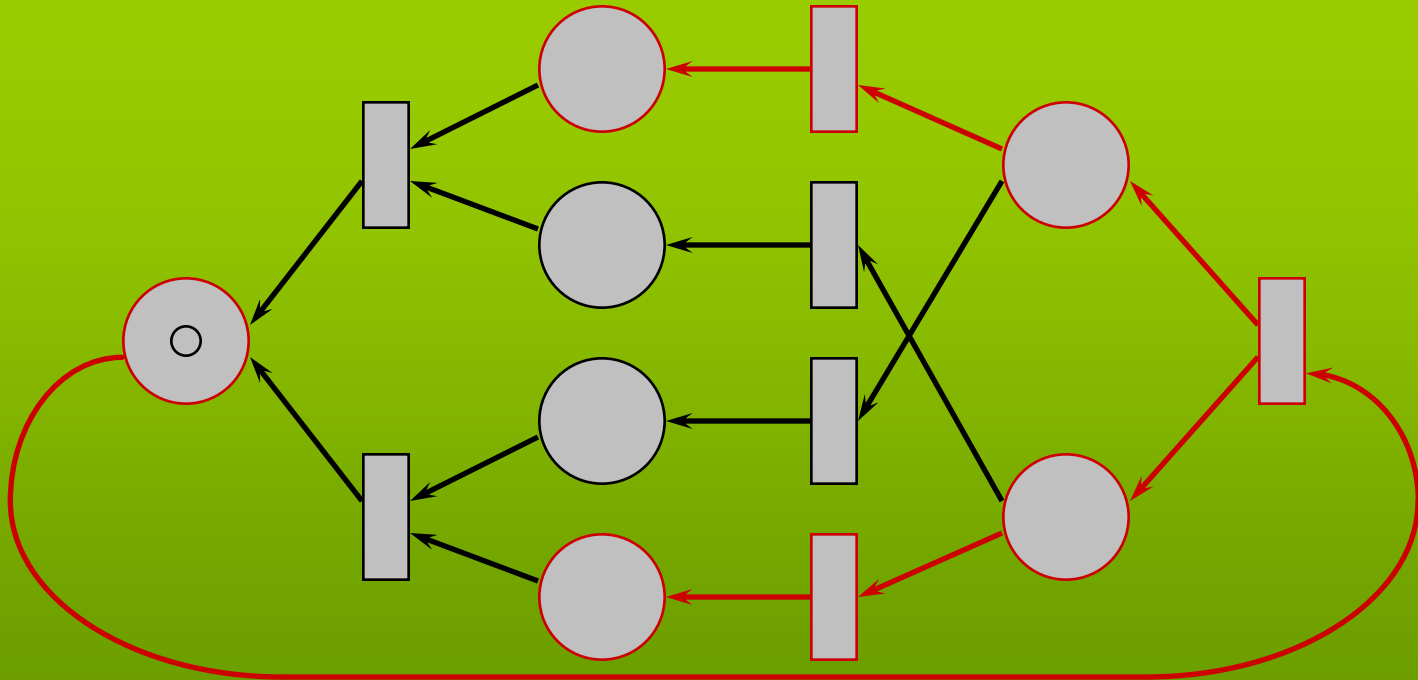
Hack's theorem

- Example of non-live (but safe) FCN



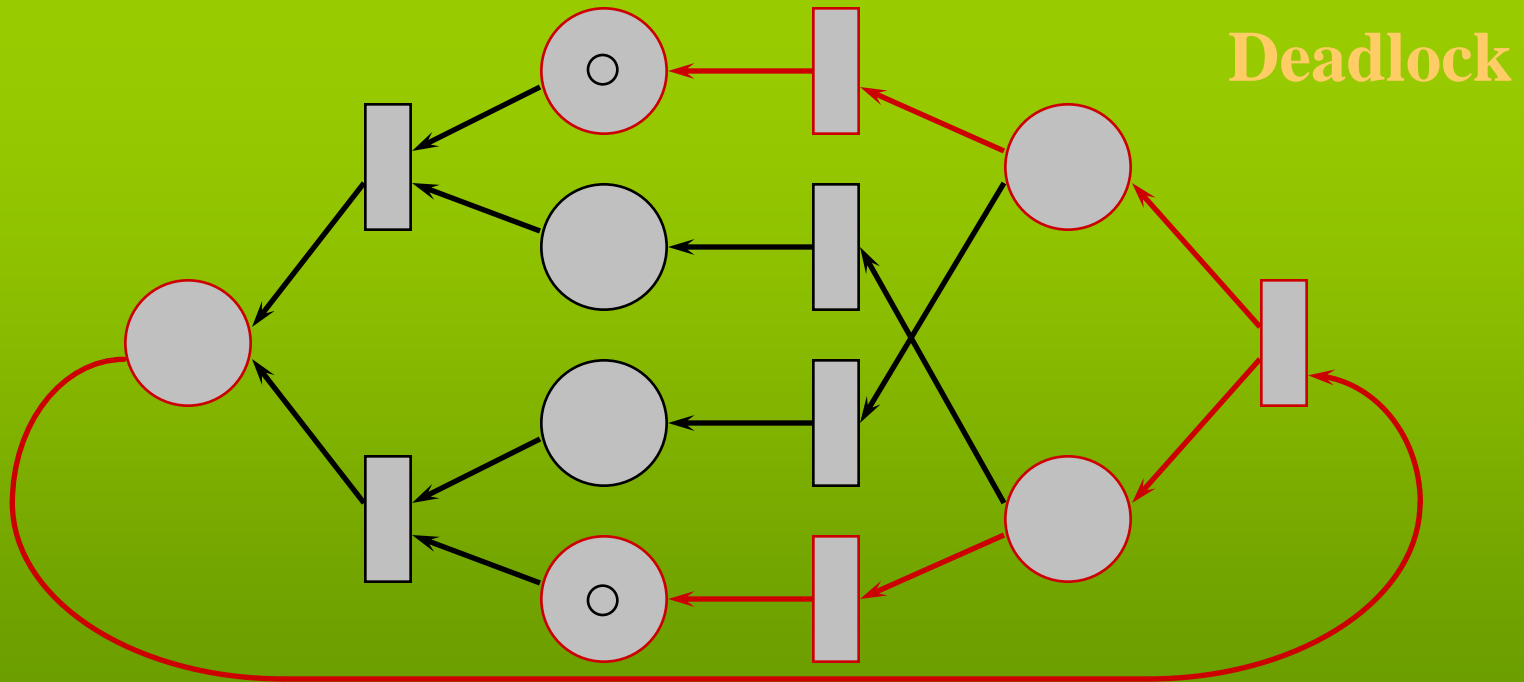
Hack's theorem

- Example of non-live (but safe) FCN



Hack's theorem

- Example of non-live (but safe) FCN





Summary of LSFC nets

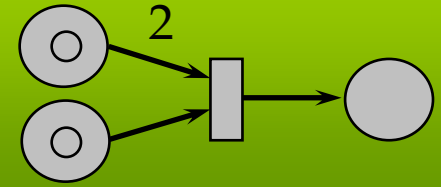
- **Largest class for which structural theory really helps**
- **Structural component analysis may be expensive**
(exponential number of MG and SM components in the worst case)
- **But...**
 - number of MG components is **generally** small
 - FC restriction simplifies characterization of behavior



Petri Net extensions

- **Add interpretation to tokens and transitions**
 - Colored nets (tokens have value)
- **Add time**
 - Time/timed Petri Nets (deterministic delay)
 - type (duration, delay)
 - where (place, transition)
 - Stochastic PNs (probabilistic delay)
 - Generalized Stochastic PNs (timed and immediate transitions)
- **Add hierarchy**
 - Place Charts Nets

PNs Summary



- **PN Graph: places (buffers), transitions (actions), tokens (data)**
- **Firing rule: transition enabled if there are enough tokens in each input place**
- **Properties**
 - **Structural (consistency, structural boundedness...)**
 - **Behavioral (reachability, boundedness, liveness...)**
- **Analysis techniques**
 - **Structural (only CN or CS): State equations, Invariants**
 - **Behavioral: coverability tree**
- **Reachability**
- **Subclasses: Marked Graphs, State Machines, Free-Choice PNs**

References



- T. Murata **Petri Nets: Properties, Analysis and Applications**
- <http://www.daimi.au.dk/PetriNets/>