Zélus, a Synchronous Language with ODEs [1]

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[1] Joint work with Benveniste, Bourke, Caillaud (INRIA) and Pagano (Esterel-Tech.).

Hybrid Systems Modelers

Program complex discrete systems and
their physical environments in a single language

Many tools exist

- Simulink/Stateflow, LabVIEW, Modelica, Ptolemy, ...

Focus on programming language issues to improve safety

Our proposal

- Build a hybrid modeler on top of a synchronous language
- Recycle existing techniques and tools
- Clarify underlying principles and guide language design/semantics

Reuse existing tools and techniques

Synchronous languages (SCADE/Lustre)

- Widely used for critical systems design and implementation
  - mathematically sound semantics
  - certified compilation (DO178C)
- Expressive language for both discrete controllers and mode changes
- Do not support modelling continuous dynamics

Off-the-shelf ODEs numeric solvers

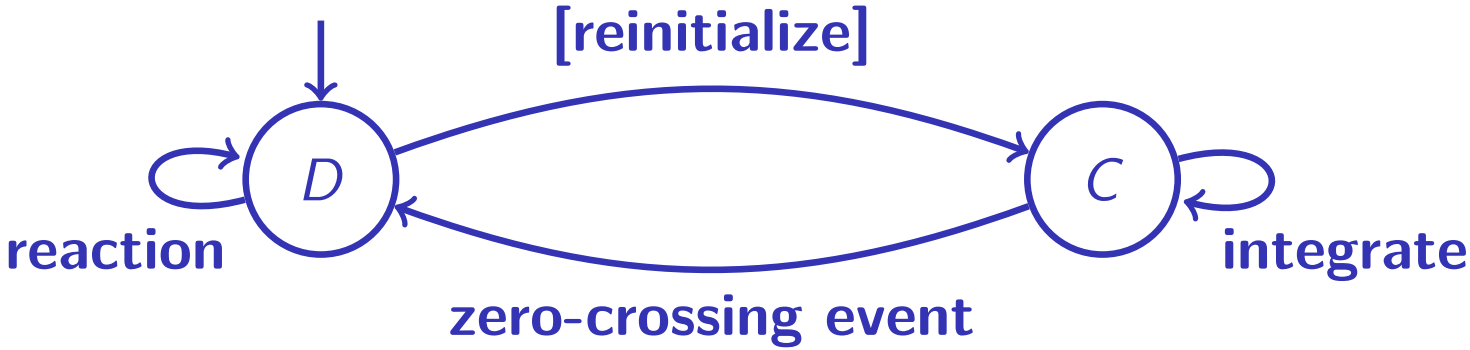
- Sundials CVODE (LLNL) among others, treated as black boxes
- Exploit existing techniques and (variable step) solvers

A conservative extension:

Any synchronous program must be compiled, optimized, and executed as per usual

The Simulation Engine of Hybrid Systems

Alternate discrete steps and integration steps



$$\sigma', y' = d_\sigma(t, y) \qquad upz = g_\sigma(t, y) \qquad \dot{y} = f_\sigma(t, y)$$

Properties of the three functions

- d_σ gathers all discrete changes.
- g_σ defines signals for zero-crossing detection.
- f_σ and g_σ should be free of side effects and, better, continuous.

Causality Issues

Find sufficient conditions for the compiler to generate d, g and f.

The classical solution for difference equations (Lustre)

$$\mathsf{x} = 0 \rightarrow \text{pre } \mathsf{y} \quad \text{and} \quad \mathsf{y} = \text{if } \mathsf{c} \text{ then } \mathsf{x} + 1 \text{ else } \mathsf{x}$$

defines the two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$:

$$x(0) = 0 \qquad y(n) = \texttt{if}\, c(n)\, \texttt{then}\, x(n) + 1\, \texttt{else}\, x(n)$$
$$x(n) = y(n-1)$$

If every feedback loop crosses a delay, elements can be computed sequentially. E.g.,an excerpt of the C code:

```
if (self->v_1) {x = 0;} else {x = self->v_2;};
if (c) {y = x+1;} else {y = x;};
self->v_2 = y; self->v_1 = false;
```

Can we reproduce the very same argument when mixing ODEs and discrete transitions and generate sequential code?

The Non Standard Interpretation of Hybrid Systems

We proposed in [CDC 2010, JCSS 2012] to build the semantics on Non-standard analysis.

der y = z init 4.0 and z = 10.0 − 0.1 ∗ y and k = y + 1.0

defines signals y, z and k, where for all $t \in \mathbb{R}^+$:

$$\frac{dy}{dt}(t) = z(t) \quad y(0) = 4.0 \quad z(t) = 10.0 - 0.1 \cdot y(t) \quad k(t) = y(t) + 1$$

Consider the value that y would have if computed by an ideal solver taking an infinitesimal step of duration ∂.
${}^\star y(n)$ stands for the values of y at instant $n\partial$, with $n \in {}^\star\mathbb{N}$ a non-standard integer.

$$\begin{aligned} {}^\star y(0) &= 4 & {}^\star z(n) &= 10 - 0.1 \cdot {}^\star y(n) \\ {}^\star y(n+1) &= {}^\star y(n) + {}^\star z(n) \cdot \partial & {}^\star k(n) &= {}^\star y(n) + 1 \end{aligned}$$

ODEs with reset

Consider the sawtooth signal $y : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that:

$$\frac{dy}{dt}(t) = 1 \quad y(t) = 0 \text{ if } t \in \mathbb{N}$$

written:

der y = 1.0 init 0.0 reset up(y − 1.0) → 0.0

The ideal non-standard semantics is:

$$\begin{aligned} {}^\star y(0) &= 0 & {}^\star y(n) &= \texttt{if}\ {}^\star z(n)\ \texttt{then}\ 0.0\ \texttt{else}\ {}^\star ly(n) \\ {}^\star ly(n) &= {}^\star y(n-1) + \partial & {}^\star c(n) &= ({}^\star y(n) - 1) \geq 0 \\ {}^\star z(0) &= \texttt{false} & {}^\star z(n) &= {}^\star c(n) \wedge \neg {}^\star c(n-1) \end{aligned}$$

${}^\star y(n)$ depends on itself so this set of equations is not causal.

Acessing the "left limit" of a signal (last y)

Two ways to break this cycle

- consider that the effect of the zero-crossing is delayed by one cycle, that is, the test is made on ${}^\star z(n-1)$ instead of on $z(n)$, or,
- distinguish the current value of ${}^\star y(n)$ from the value it would have had were there no reset, namely ${}^\star ly(n)$.

Testing a zero-crossing of *ly* (instead of *y*)

$$ {}^\star c(n) = ({}^\star ly(n) - 1) \geq 0, $$

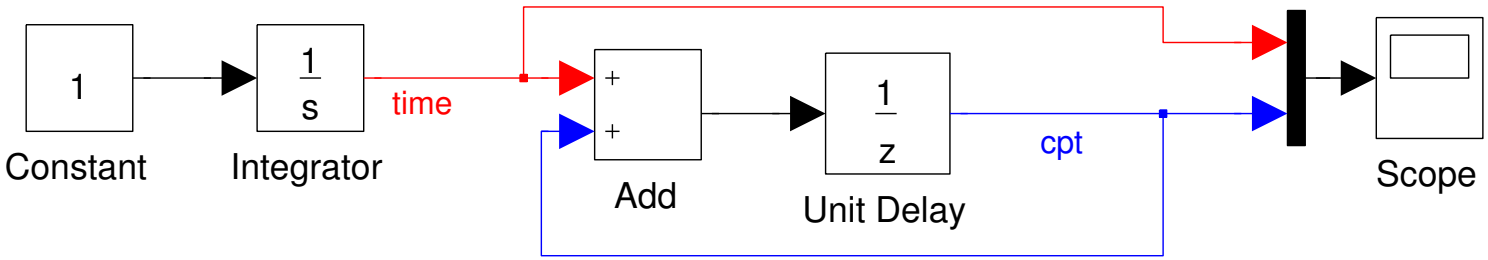
gives a program that is causal since ${}^\star y(n)$ no longer depends instantaneously on itself.

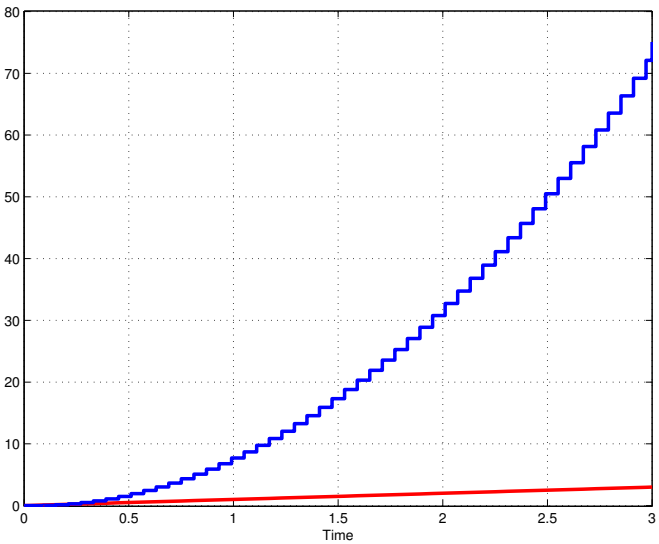
Write

der y = 1.0 init 0.0 reset up(last y − 1.0) → 0.0

Strange beasts. . .

Typing issue: Mixing continuous and discrete components





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- The shape of cpt depends on the steps chosen by the solver.
- Putting another component in parallel can change the result.

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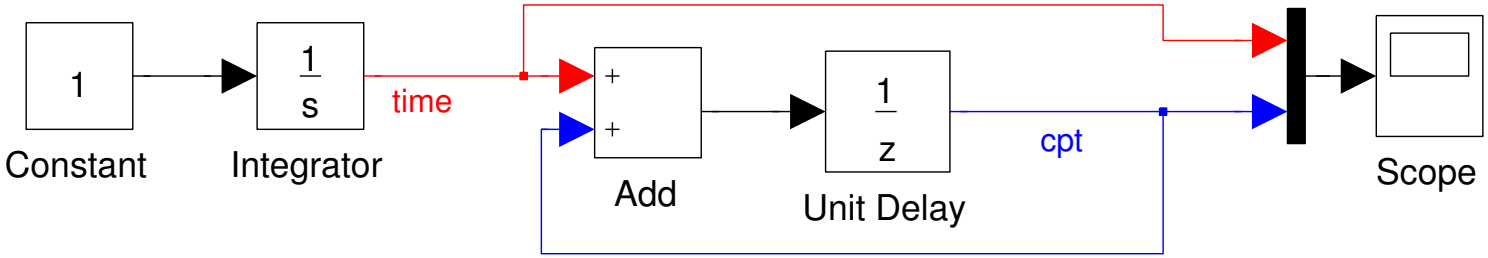

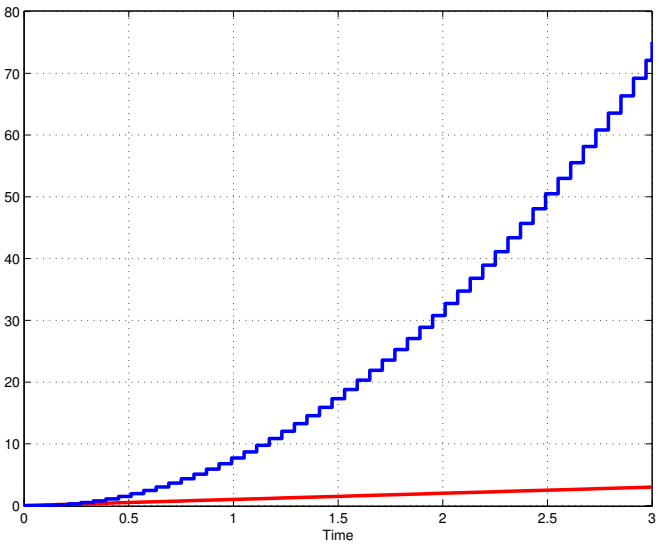

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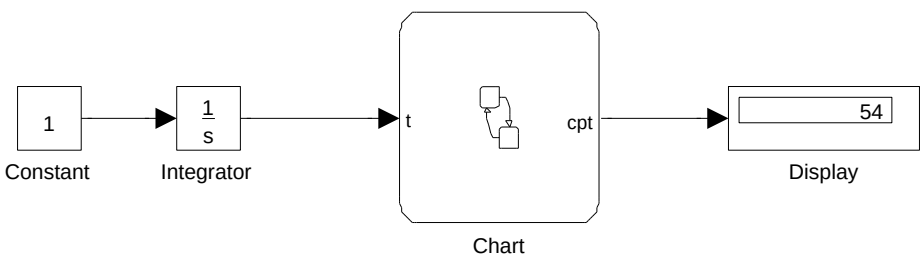
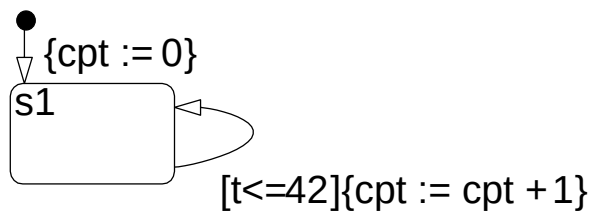
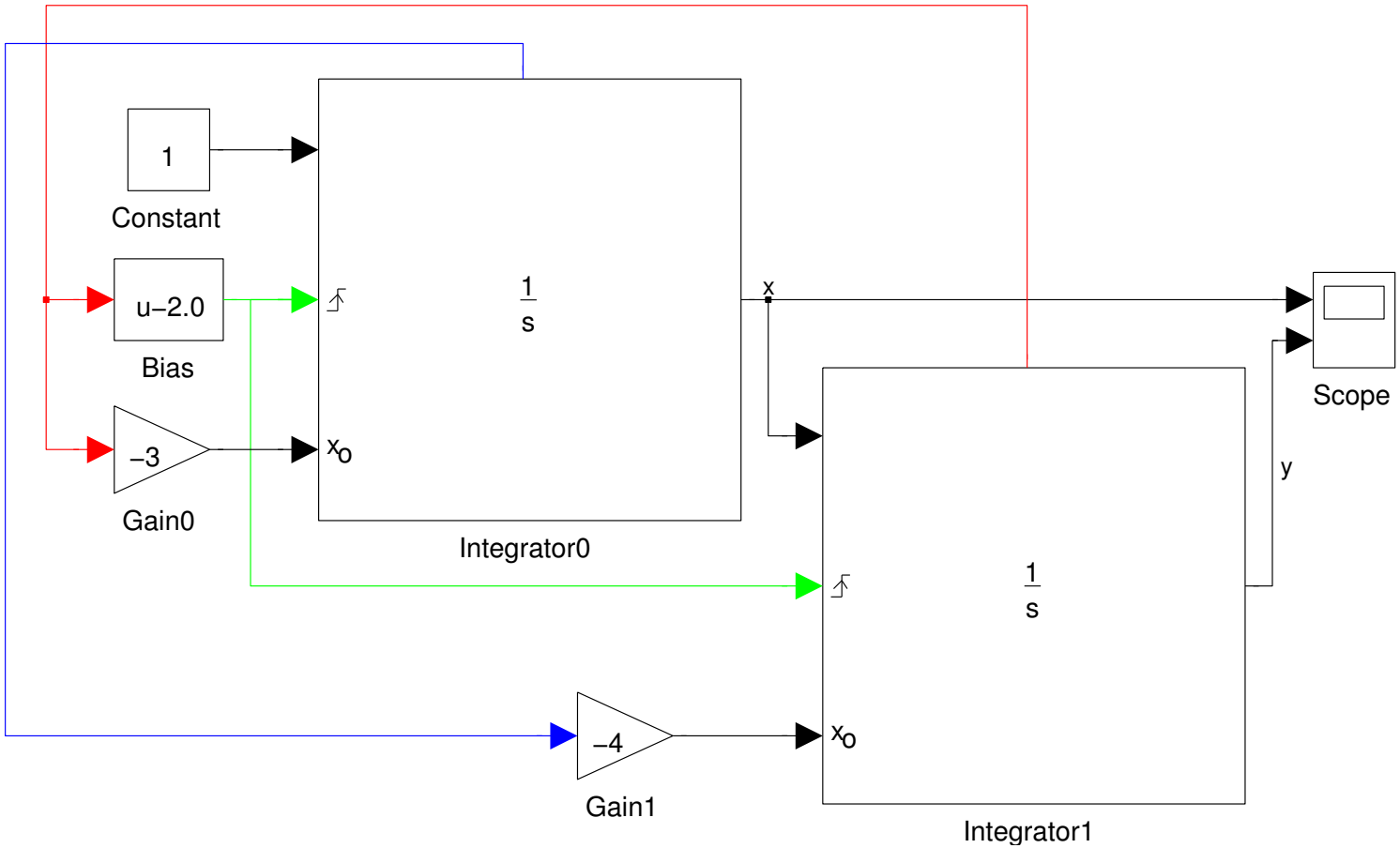
- Similar issues with Stateflow.





Causality issue: the Simulink state port



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–Simulink Reference (2-685)

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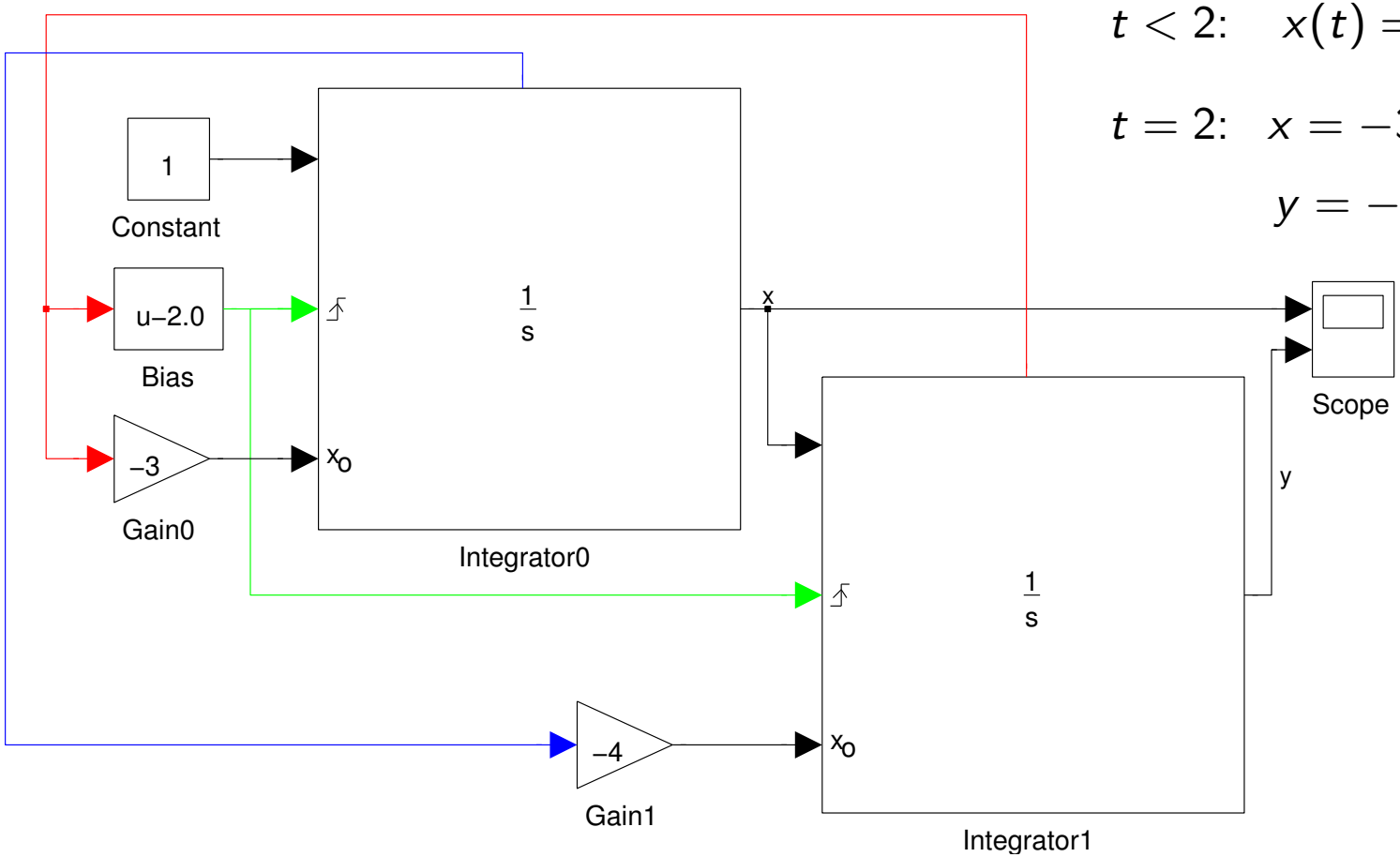
$$t < 2: \quad x(t) = t,\ y(t) = \frac{t^2}{2}$$

$$t = 2: \quad x = -3 \cdot \text{last}\, y = -6,$$

$$y = -4 \cdot \text{last}\, x = -8$$



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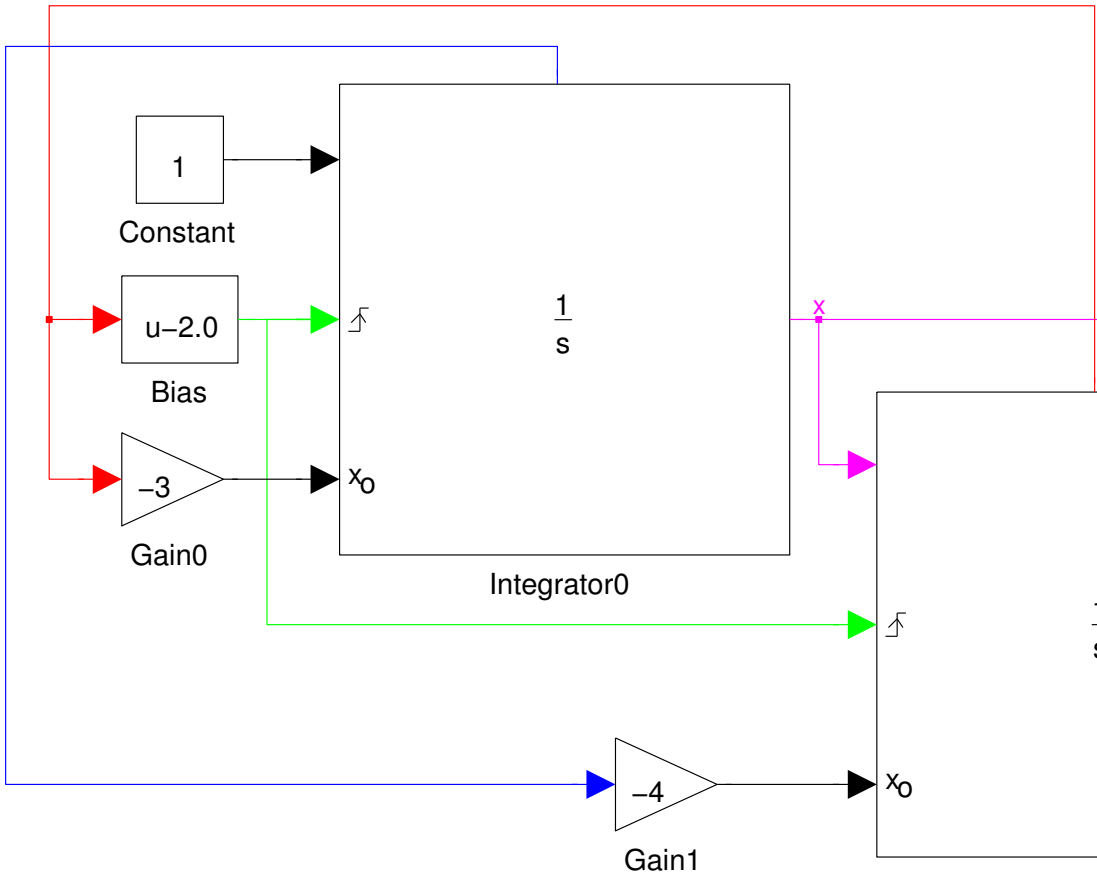

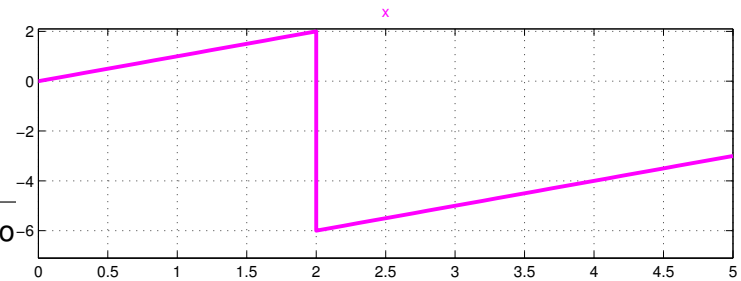

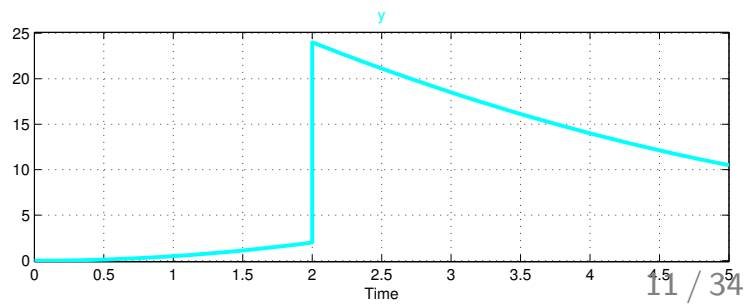

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Excerpt of C code produced by RTW (release R2009)

```
// P_0 = -2.0 P_1 = -3.0 P_2 = -4.0 P_3 = 1.0
static void mdlOutputs(SimStruct * S, int_T tid)
{ _rtX = (ssGetContStates(S));
  ...
  _rtB = (_ssGetBlockIO(S));
  _rtB->B_0_0_0 = _rtX->Integrator1_CSTATE + _rtP->P_0;
  _rtB->B_0_1_0 = _rtP->P_1 * _rtX->Integrator1_CSTATE;
  if (ssIsMajorTimeStep (S))
    { ...
      if (zcEvent || ...)
        { (ssGetContStates (S))->Integrator0_CSTATE =
            _ssGetBlockIO (S))->B_0_1_0;
        }
      ...
  (_ssGetBlockIO (S))->B_0_2_0 =
    (ssGetContStates (S))->Integrator0_CSTATE;
    _rtB->B_0_3_0 = _rtP->P_2 * _rtX->Integrator0_CSTATE;
    if (ssIsMajorTimeStep (S))
    { ...
      if (zcEvent || ...)
       { (ssGetContStates (S))-> Integrator1_CSTATE =
            (ssGetBlockIO (S))->B_0_3_0;
       }
      ...
    } ... }
```

An explanation of the bug

The source program

$$
\begin{aligned}
&\text{der x} = 1.0 \text{ init } 0.0 \text{ reset z} \rightarrow -3.0 * \text{last y} \\
&\text{and der y} = \text{x init } 0.0 \text{ reset z} \rightarrow -4.0 * \text{last x} \\
&\text{and z} = \text{up}(\text{last x} - 2.0)
\end{aligned}
$$

Its non-standard interpretation

$$
\begin{aligned}
{}^\star x(n) &= \texttt{if } {}^\star z(n) \texttt{ then } -3 \cdot {}^\star y(n-1) \texttt{ else } {}^\star x(n-1) + \partial \\
{}^\star y(n) &= \texttt{if } {}^\star z(n) \texttt{ then } -4 \cdot {}^\star x(n-1) \texttt{ else } {}^\star y(n-1) + \partial \cdot {}^\star x(n-1) \\
&\ldots
\end{aligned}
$$

Explanation

- The first two equations are scheduled this way so ${}^\star x(n-1)$ is lost.
- This is a scheduling bug: the sequential code lacks a copy variable.

Zélus

`zelus.di.ens.fr`

Combinatorial and sequential functions

Time is logical as in `Lustre`. A signal is a sequence of values and nothing is said about the actual time to go from one instant to another.

```
let add (x,y) = x + y

let node min_max (x, y) = if x < y then x, y else y, x

let node after (n, t) = (c = n) where
  rec c = 0 → pre(min(tick, n))
  and tick = if t then c + 1 else c
```

When feed into the compiler, we get:

```
val add : int × int  --A--> int
val mix_max : α× α --D--> α× α
val after : int × int  --D--> bool
```

x, y, etc. are infinite sequences of values.

Chronograms

time	=	0	1	2	3	4	5	6
x	=	2	4	2	1	2	3	2
y	=	3	6	5	1	1	9	0
$min\_max(x, y)$	=	$(2, 3)$	$(4, 6)$	$(2, 5)$	$(1, 1)$	$(1, 2)$	$(3, 9)$	$(0, 2)$
$pre(x)$	=	*nil*	2	4	2	1	2	3
$x \rightarrow y$	=	2	6	5	1	1	9	0
t	=	*false*	*true*	*true*	*false*	*false*	*true*	*true*
$after(2, t)$	=	*false*	*false*	*true*	*true*	*true*	*true*	*true*

- A signal is a sequence of values or stream.
- A system is function from streams to streams.
- Operations apply pointwise to their arguments.
- All streams progress synchronously.

The counter can be instantiated twice in a two state automaton,

```
let node blink (n, m, t) = x where
 automaton
 | On → do x = true   until (after(n, t)) then Off
 | Off → do x = false  until (after(m, t)) then On
```

which returns a value for x that alternates between true for n occurrences of t and false for m occurrences of t.

```
let node blink_reset (r, n, m, t) = x where
 reset
   automaton
   | On  → do x = true  until (after(n, t)) then Off
   | Off → do x = false until (after(m, t)) then On
 every r
```

The type signatures inferred by the compiler are:

$$\texttt{val blink} : \texttt{int} \times \texttt{int} \times \texttt{int} \xrightarrow{\texttt{D}} \texttt{bool}$$

$$\texttt{val blink\_reset} : \texttt{int} \times \texttt{int} \times \texttt{int} \times \texttt{int} \xrightarrow{\texttt{D}} \texttt{bool}$$

Examples

Up to syntactic details, these are `Scade 6` or `Lucid Synchrone` programs. Now, a simple heat controller. [2]

```
(* an hysteresis controller for a heater *)
let hybrid heater(active) = temp where
  rec der temp = if active then c −. k ∗. temp else −. k ∗. temp init temp0

let hybrid hysteresis_controller(temp) = active where
  rec automaton
      | Idle →   do active = false until (up(t_min −. temp)) then Active
      | Active → do active = true until (up(temp −. t_max)) then Idle

let hybrid main() = temp where
  rec active = hysteresis_controller(temp)
  and temp = heater(active)
```

[2] This is the hybrid version of one of Nicolas Halbwachs' examples with which he presented Lustre at the Collège de France, in January 2010.

The Bouncing ball

```
let hybrid bouncing(x0,y0,x'0,y'0) = (x,y) where
  der(x) = x' init x0
and
  der(x') = 0.0 init x'0
and
  der(y) = y' init y0
and
  der(y') = −. g init y'0 reset up(−. y) → −. 0.9 *. last y'
```

The type signature is:

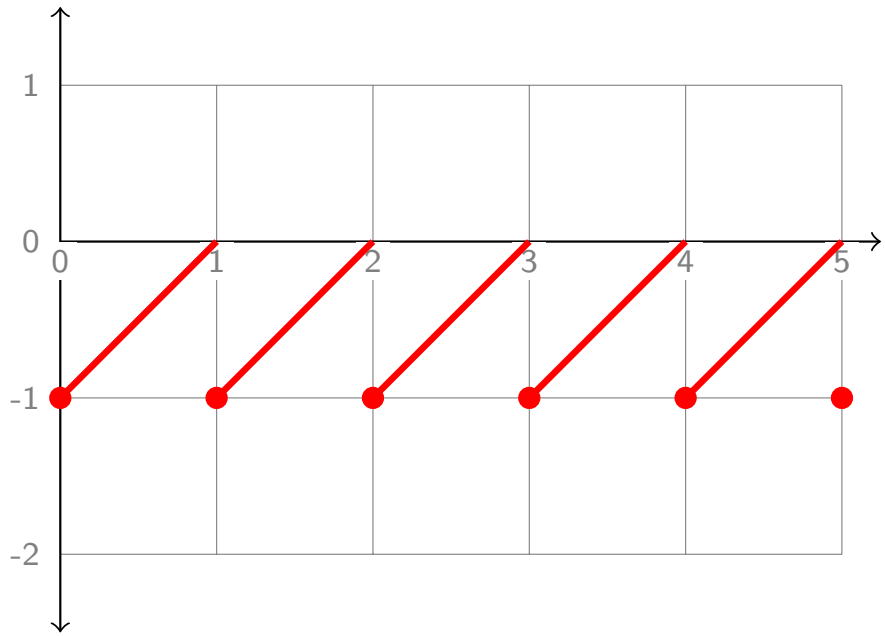
$$\texttt{val bouncing} : \texttt{float} \times \texttt{float} \times \texttt{float} \xrightarrow{c} \texttt{float} \times \texttt{float}$$

- When $-.$ y crosses zero, re-initialize the speed y' with $-. 0.9 * \texttt{last y'}$.
- last y' stands for the previous value of y'.
- As y' is immediately reset, writing last y' is mandatory —otherwise, y' would instantaneously depend on itself.

ODEs and Zero-crossings

E.g., the sawtooth signal, the two-state automaton.

```
let hybrid sawtooth() = t where
  rec der t = 1.0 init −1.0 reset up(last t −. 1.0) → −1.0
```



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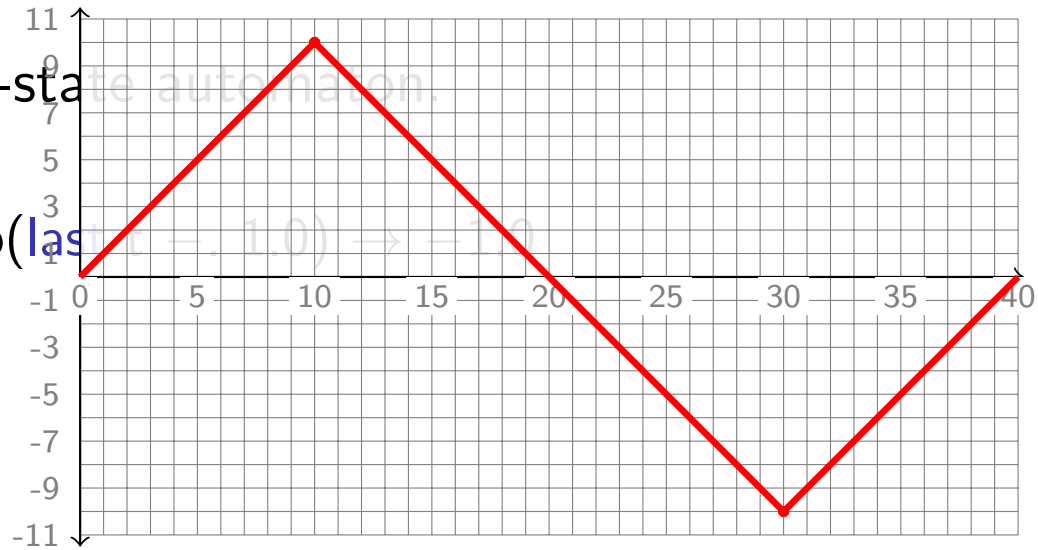


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```
let hybrid fm() = t where
  rec init t = 0.0
  and automaton
      | Up →   do der t = 1.0   until (up(t −. 10.0))   then Down
      | Down → do der t = −1.0 until (up(−10.0 −. t)) then Up
```

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let hybrid fm'() = t where
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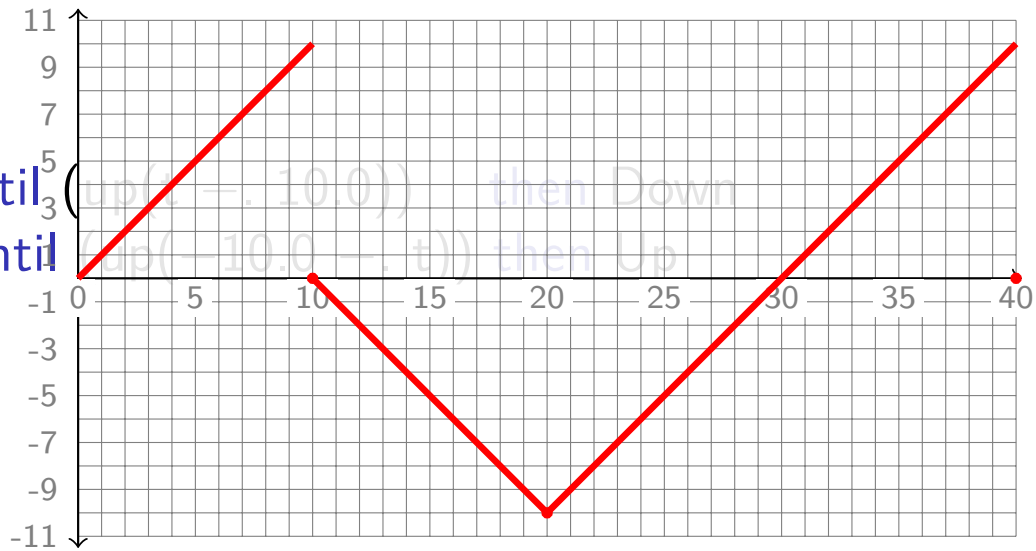


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Two difficulties

Ensure that continuous and discrete time signals interfere correctly

- Discrete time should stay logical and independent on when the solver decides to stop.
- Otherwise, we get the same monsters as `Simulink`/`Stateflow` have.
- Rely on a dedicated type system.

Ensure that fix-point exist and code can be scheduled

- Algebraic loops must be statically detected.
- Introduce the operator last(x) as the "left limit" of a signal.
- Ensure that signals are left-continuous during integration.
- Rely on a dedicated type system.

Mixing discrete (logical) time and continuous time

Given:

```
let node sum(x) = cpt where
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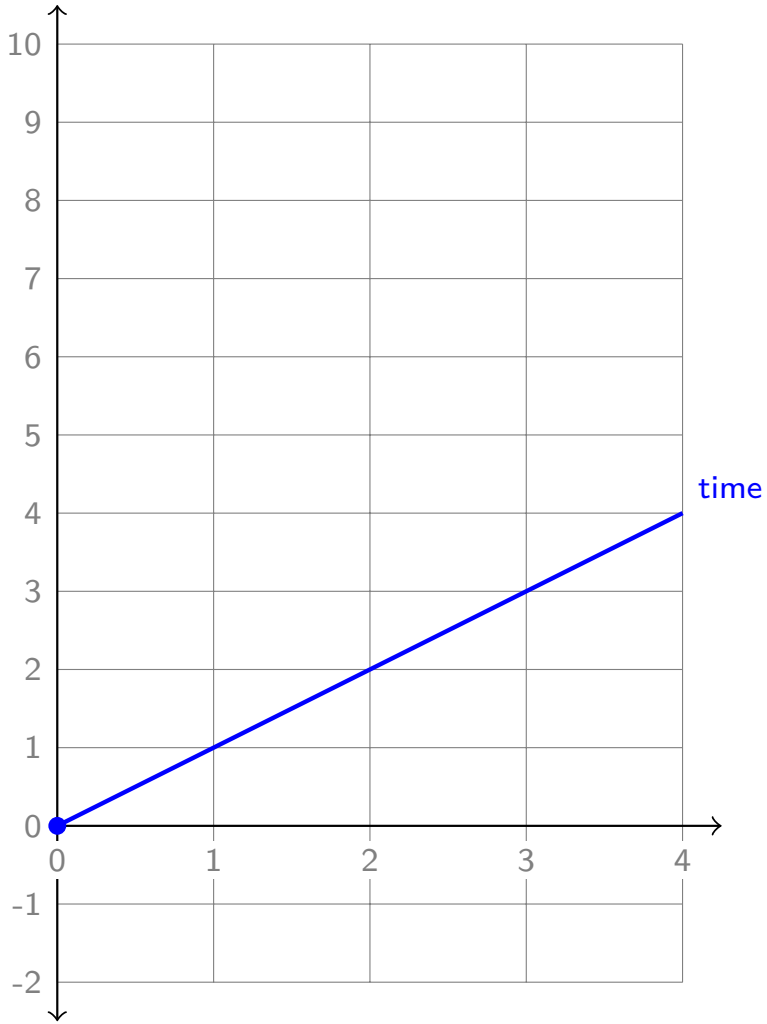
Interpretation:

- Option 1: $\mathbb{N} \subseteq \mathbb{R}$
- Option 2: depends on solver
- Option 3: infinitesimal steps
- Option 4: type and reject



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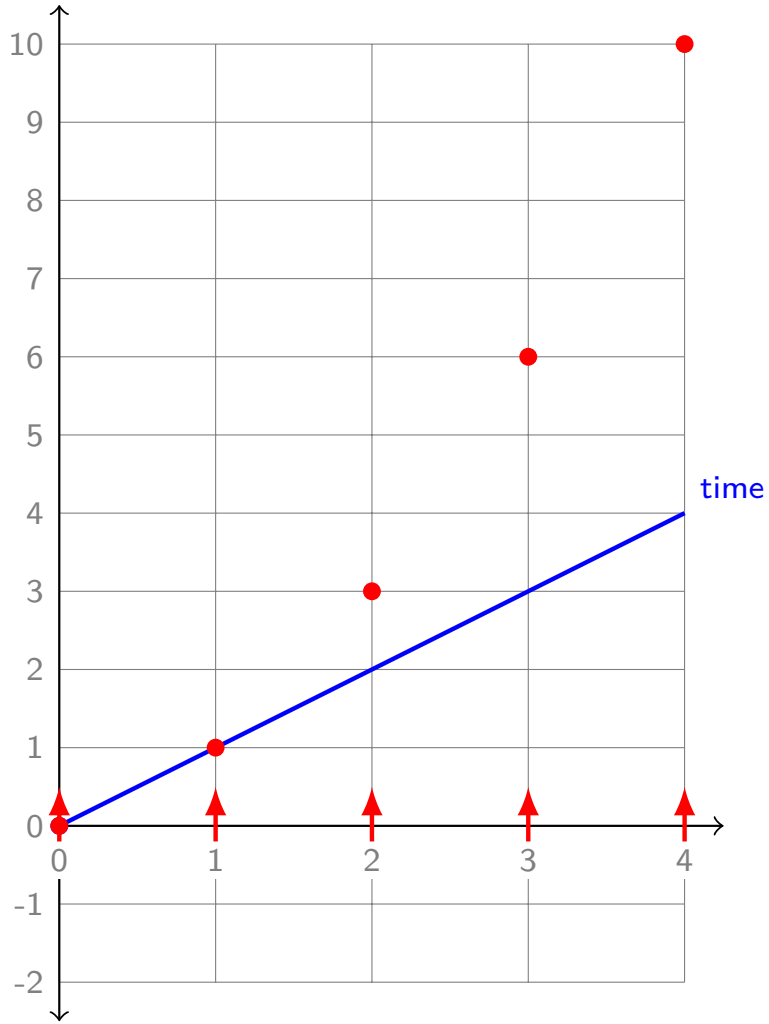

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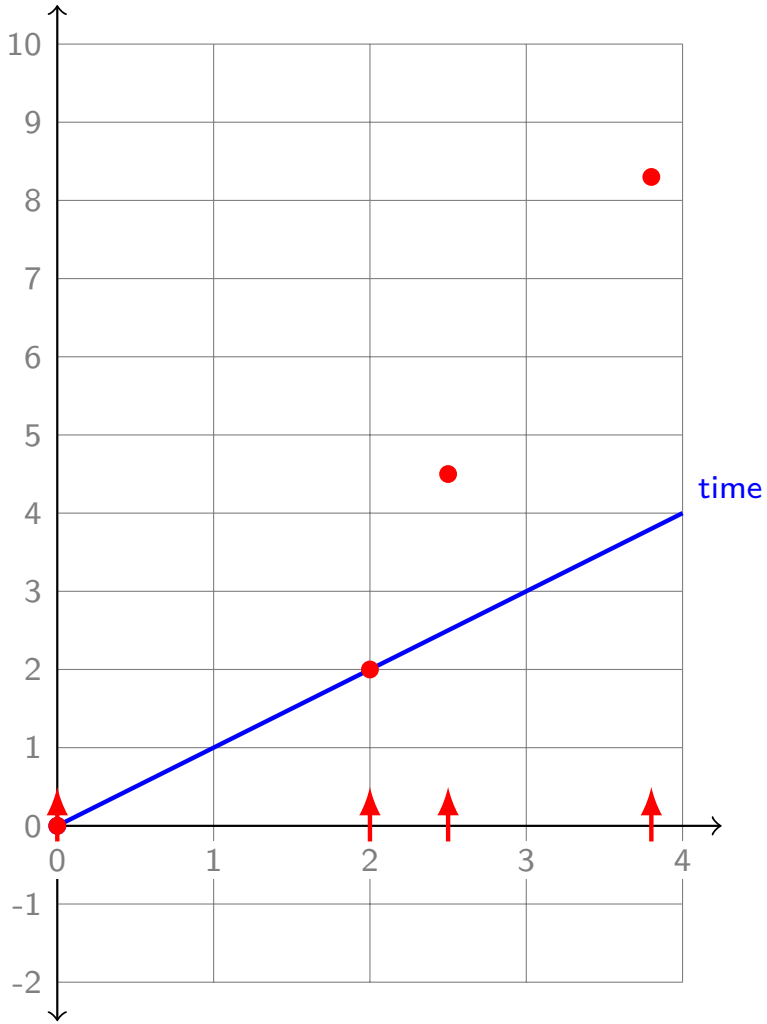

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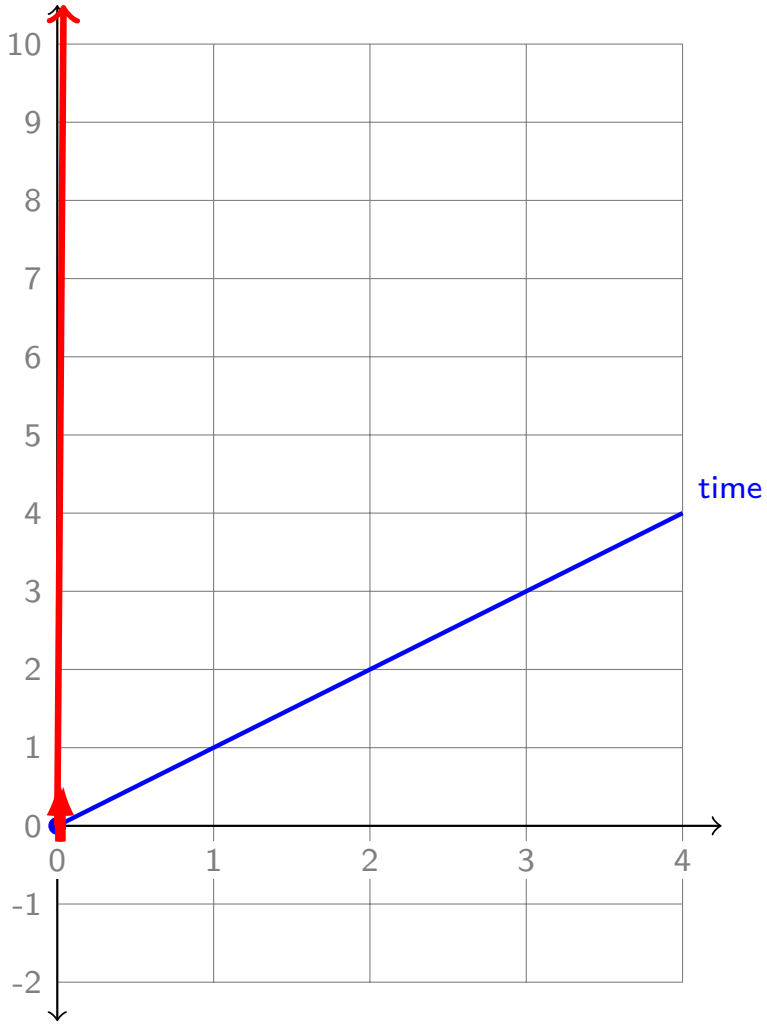

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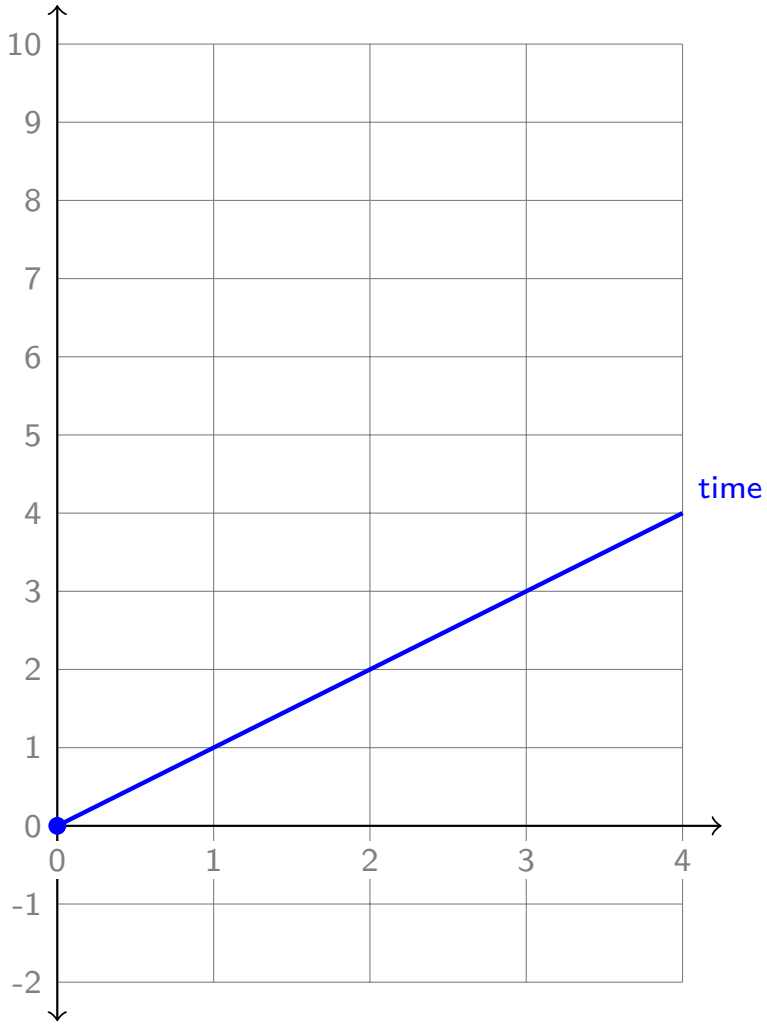
Define:

```
let          wrong () = ()
  where rec
    der time = 1.0 init 0.0
    and y = sum (time)
```



Interpretation:

- Option 1: $\mathbb{N} \subseteq \mathbb{R}$
- Option 2: depends on solver
- Option 3: infinitesimal steps
- Option 4: type and reject

Mixing discrete (logical) time and continuous time

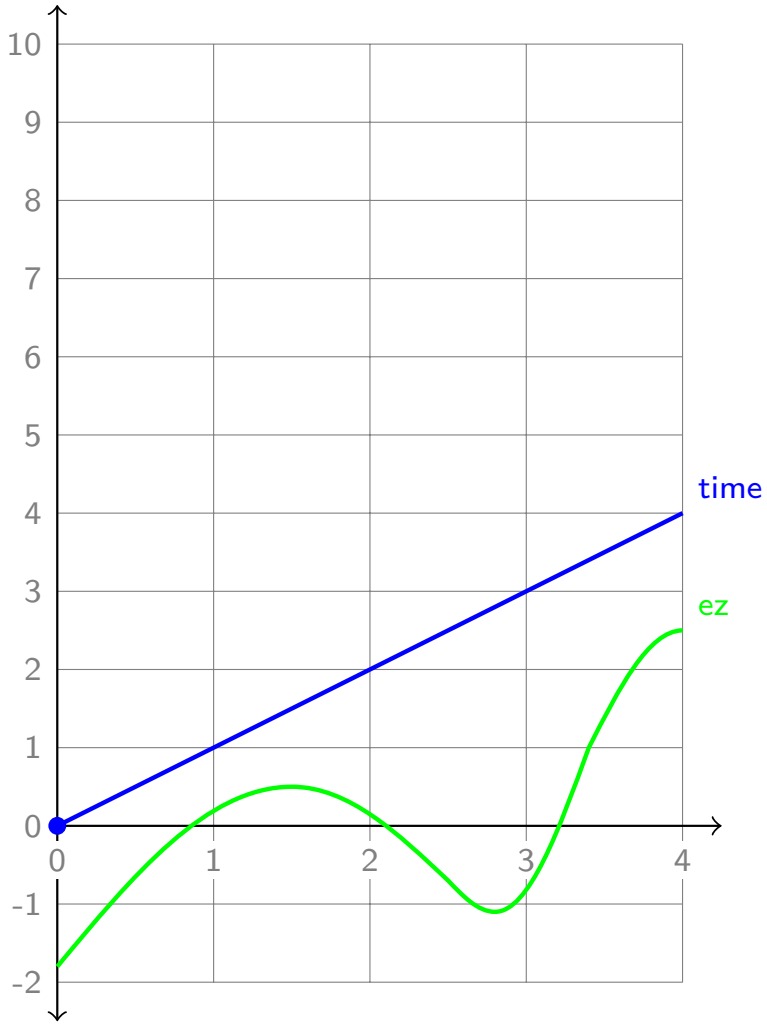
Given:

```
let node sum(x) = cpt where
  rec cpt = (0.0 fby cpt) +. x
```

Define:

```
let hybrid correct () = ()
  where rec
    der time = 1.0 init 0.0
    and y = present up(ez) → sum (time)
                init 0.0
```

- node:
  function acting in discrete time
- hybrid:
  function acting in continuous time



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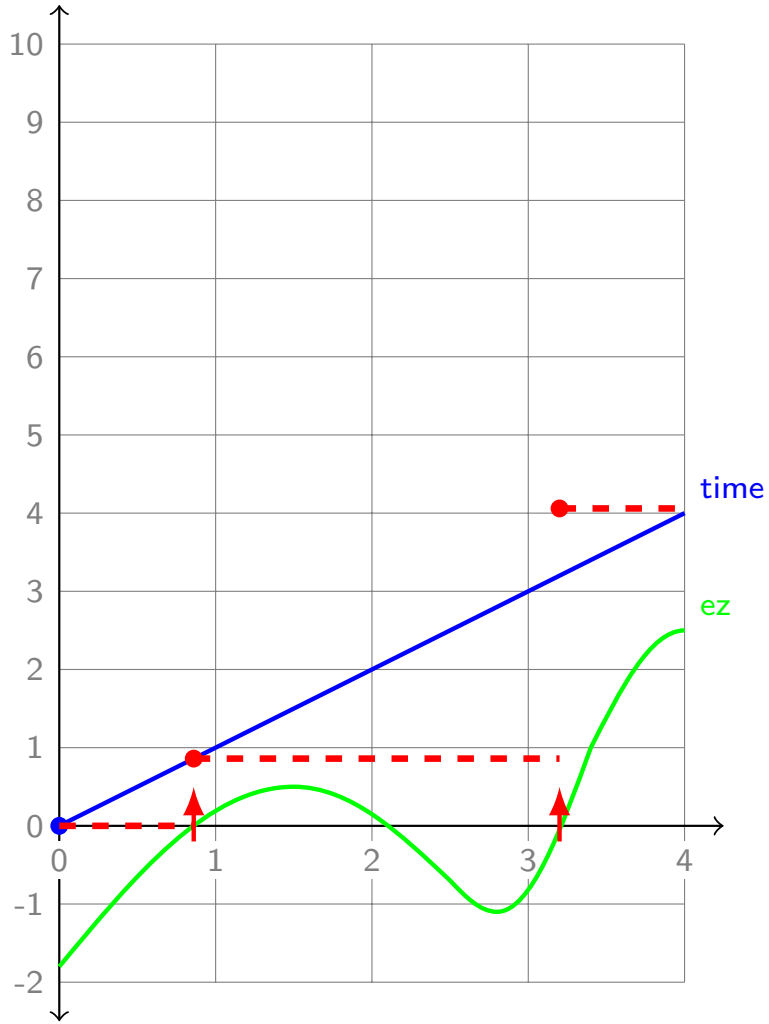


- node:
  function acting in discrete time
- hybrid:
  function acting in continuous time

Explicitly relate simulation and logical time (using zero-crossings)

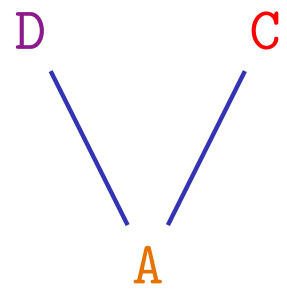
Try to minimize the effects of solver parameters and choices

Basic typing [LCTES'11]

A simple ML type-and-effect system.

The type language

$$
\begin{array}{lcl}
bt & ::= & \texttt{float} \mid \texttt{int} \mid \texttt{bool} \mid \texttt{zero} \\
t & ::= & bt \mid t \times t \mid \beta \\
\sigma & ::= & \forall \beta_1, ..., \beta_n . t \xrightarrow{k} t \\
k & ::= & \texttt{D} \mid \texttt{C} \mid \texttt{A}
\end{array}
$$



Initial conditions

$$
\begin{array}{lcl}
(+) & : & \texttt{int} \times \texttt{int} \xrightarrow{\texttt{A}} \texttt{int} \\
\texttt{if} & : & \forall \beta . \texttt{bool} \times \beta \times \beta \xrightarrow{\texttt{A}} \beta \\
(=) & : & \forall \beta . \beta \times \beta \xrightarrow{\texttt{D}} \texttt{bool} \\
\texttt{pre}(\cdot) & : & \forall \beta . \beta \xrightarrow{\texttt{D}} \beta \\
\cdot \texttt{fby} \cdot & : & \forall \beta . \beta \times \beta \xrightarrow{\texttt{D}} \beta \\
\texttt{up}(\cdot) & : & \texttt{float} \xrightarrow{\texttt{C}} \texttt{zero} \\
\cdot \texttt{on} \cdot & : & \texttt{zero} \times \texttt{bool} \xrightarrow{\texttt{A}} \texttt{zero}
\end{array}
$$

What about continuous automata? [EMSOFT'11]

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Design Considerations for Continuous-Time Modeling in Stateflow Charts

In this section...

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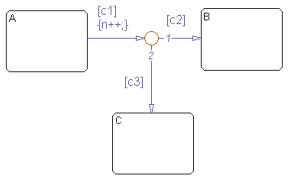
- State `exit` actions, which execute before leaving the state at the beginning of the transition
- Transition actions, which execute during the transition
- State `entry` actions, which execute after entering the new state at the end of the transition
- Condition actions on a transition, but only if the transition directly reaches a state

Consider the following chart.



In this example, the action `{n++}` executes even when conditions `c2` and `c3` are false. In this case, `n` gets updated in a minor time step because there is no state transition.

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This restriction prevents mode changes from occurring between major time steps. When placed in `during` actions, conditions that affect control flow should be governed by discrete variables because they do not change between major time steps.

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The presence of input events makes a chart behave like a triggered subsystem and therefore unable to simulate in continuous-time. For example, the following model generates an error if the chart uses a continuous update method.

• • •

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- [illegible] transition ends at a state

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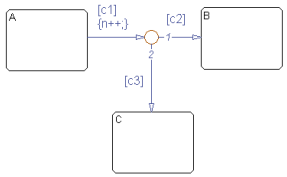

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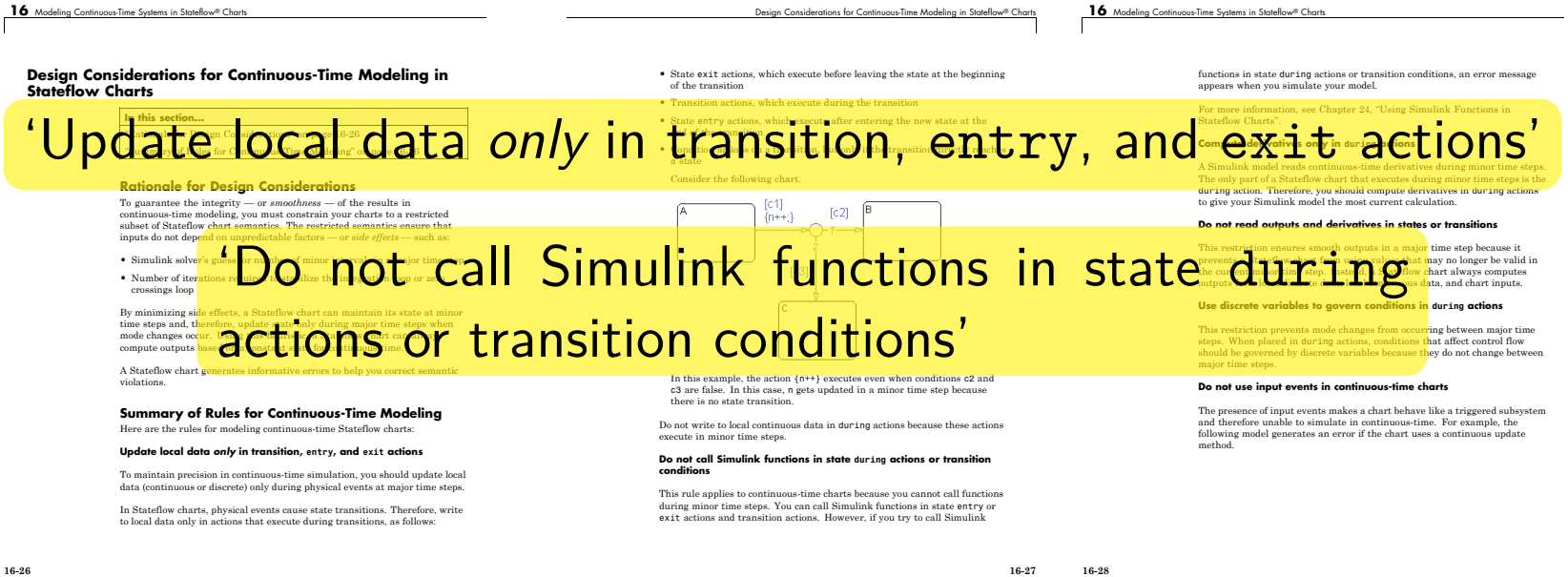

16 Modeling Continuous-Time Systems in Stateflow® Charts

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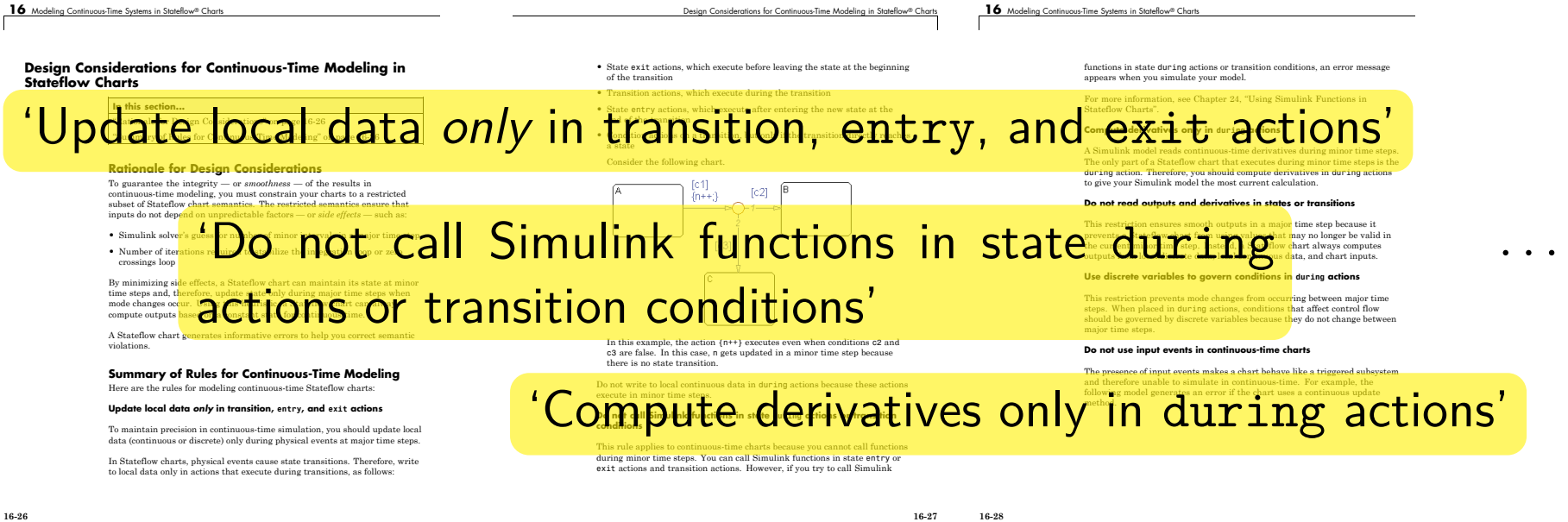

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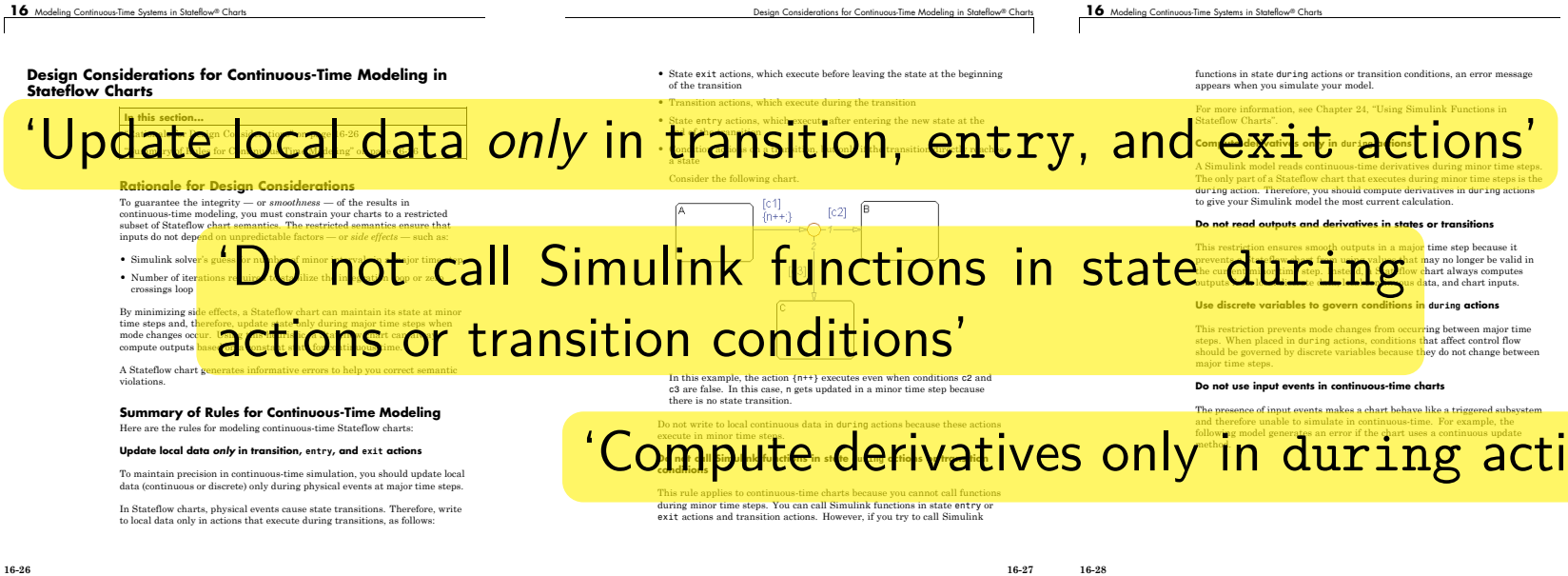

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- 'Restricted subset of Stateflow chart semantics'
  - restricts side-effects to major time steps
  - supported by warnings and errors in tool **(mostly)**
- Our `D`/`C`/`A`/`zero` system extends naturally for the same effect.
- For both discrete (synchronous) and continuous (hybrid) contexts.

Causality issues (feedback loops)

Which programs should we accept?

- OK to reject (no solution).

  rec x = x +. 1.0

- OK as an algebraic constraint (e.g., Simulink and Modelica).

  rec x = 1.0 −. x

- But NOK if sequential code generation is targeted (algebraic loop).
- OK in constructive logic (Esterel)

  rec z1 = if c then z2 else y
  and z2 = if c then x else z1

- But it calls for an expensive boolean analysis.

Can we find a simple and uniform justification for a program mixing continuous-time and discrete-time signals to be causaly correct?

Causality [HSCC'14]

We follow the `Lustre`/`Lucid Synchrone` condition for causality: every feedback loop must cross a delay.
Intuition: associate a time stamp to every expression and ensure that the relation between those time stamps is a partial order.

The type language

$$\begin{array}{lcl} \sigma & ::= & \forall \alpha_1, ..., \alpha_n : C.\ ct \xrightarrow{k} ct \\ ct & ::= & ct \times ct \mid \alpha \\ k & ::= & \mathtt{D} \mid \mathtt{C} \mid \mathtt{A} \end{array}$$

Precedence relation:

$$C ::= \{\alpha_1 < \alpha_1', ..., \alpha_n < \alpha_n'\}$$

$<$ must be a strict partial order. $C \vdash ct_1 < ct_2$ means that ct_1 precedes ct_2 according to C.

Precedence relation

Build the transitive closure of $<$ and lift it to pairs and environments.

$$\frac{}{C + \alpha_1 < \alpha_2 \vdash \alpha_1 < \alpha_2}\ (\textsc{taut})$$

$$\frac{C \vdash ct_1 < ct' \qquad C \vdash ct' < ct_2}{C \vdash ct_1 < ct_2}\ (\textsc{trans})$$

$$\frac{C \vdash ct_1 < ct_1' \qquad C \vdash ct_2 < ct_2'}{C \vdash ct_1 \times ct_2 < ct_1' \times ct_2'}\ (\textsc{pair})$$

$$\frac{\forall i \in \{1, .., n\}, C \vdash ct_i < ct_i'}{C \vdash [x_1 : ct_1; ...; x_n : ct_n] < [x_1 : ct_1'; ...; x_n : ct_n']}\ (\textsc{env})$$

The Type System

Type Judgments

$$\frac{}{C \mid G, H \vdash_k e : ct}\ (\textsc{typ-exp}) \qquad \frac{}{C \mid G, H \vdash_k E : H'}\ (\textsc{typ-env})$$

$$G ::= [\sigma_1/f_1, ..., \sigma_k/f_k] \qquad H ::= [ct_1/x_1, ..., ct_n/x_n]$$

Initial Conditions

$$\begin{array}{lcl}
(+), (-), (*), (/) & : & \forall \alpha.\ \alpha \times \alpha \xrightarrow{\texttt{A}} \alpha \\
\texttt{pre}(\cdot) & : & \forall \alpha_1, \alpha_2 : \{\alpha_2 < \alpha_1\}.\ \alpha_1 \xrightarrow{\texttt{D}} \alpha_2 \\
\cdot\ \texttt{fby}\ \cdot & : & \forall \alpha_1, \alpha_2 : \{\alpha_1 < \alpha_2\}.\ \alpha_1 \times \alpha_2 \xrightarrow{\texttt{D}} \alpha_1 \\
\texttt{up}(\cdot) & : & \forall \alpha_1, \alpha_2 : \{\alpha_2 < \alpha_1\}.\ \alpha_1 \xrightarrow{\texttt{C}} \alpha_2
\end{array}$$

The Typing Rules

$$\frac{C, ct_1 \xrightarrow{k} ct_2 \in \mathit{Inst}(G(f)) \qquad C \mid G, H \vdash_k e : ct_1}{C \mid G, H \vdash_k f(e) : ct_2} \;(\textsc{app})$$

$$C \mid G, H + x : ct \vdash_k x : ct \;(\textsc{var})$$

$$\frac{C \vdash ct_2 < ct_1}{C \mid G, H + x : ct_1 \vdash_{\mathtt{D}} \mathsf{last}(x) : ct_2} \;(\textsc{last})$$

$$\frac{C \mid G, H \vdash_k p : ct \qquad C \mid G, H \vdash_k e : ct}{C \mid G, H \vdash_k p = e : [ct/p]} \;(\textsc{eq})$$

$$\frac{C \mid G, H \vdash_{\mathtt{C}} e : ct_1 \qquad C \vdash ct_2 < ct_1}{C \mid G, H \vdash_{\mathtt{C}} \mathtt{der}\, x \;=\; e : [ct_2/x]} \;(\textsc{der})$$

$$\frac{C \mid G, H \vdash_k e : ct \qquad C \vdash ct < ct'}{C \mid G, H \vdash_k e : ct'} \;(\textsc{sub})$$

Examples

Discrete case

```
let node integr(xi, x') = x where
  rec x = xi → pre x + (pre x' * step)

let node heat(temp0, gain) = temp where
  rec temp = integr(temp0, gain − temp))

let cycle() = (x, y) where rec y = x + 1 and x = y + 2
```

Indeed, taken $x : \alpha_x$ and $y : \alpha_y$, the first equation is correct if both $C \vdash \alpha_x < \alpha_y$ and $C \vdash \alpha_y < \alpha_x$. This means that C must contain $\{\alpha_x < \alpha_y, \alpha_y < \alpha_x\}$ which is cyclic.

Continuous case

```
let hybrid f(x) = o where
  rec der y = 1.0 − x init 0.0 and o = y + 1.0

let hybrid loop(x) = y where rec y = f(y) + x
```

The causality of last

Yet, last x must appear in a discrete context only. In NS semantics:

$$\mathsf{last}(x)(t) = x(^{\bullet}t)$$

last(x) does not necessarily break causality cycles. E.g.:

rec x = last x + 1.0

A more expressive analysis

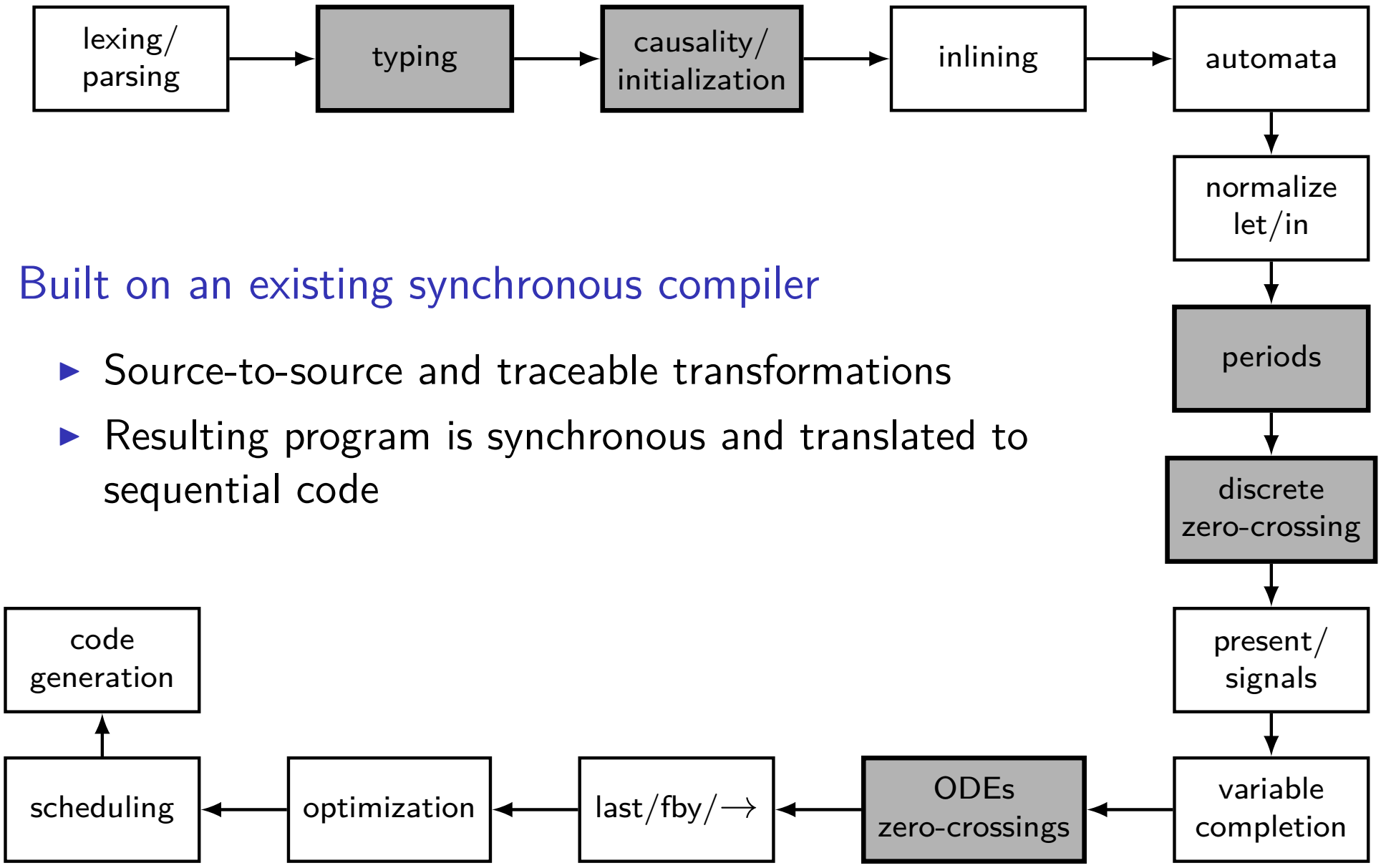
$$\mathsf{last}(x) = \texttt{if}\ d\ \texttt{then}\ \texttt{pre}(x)\ \texttt{else}\ x$$

Add a type $ct_1 + ct_2$ such that $C \mid G, H \vdash_k e : ct_1 + ct_2$ means that:

- During a discrete step, e depends on ct_1;
- During a continuous step e depends on ct_2.

$$\text{(LAST)}\ \frac{C \vdash ct_1' < ct_1}{C, H + [x : ct_1 + ct_2] \vdash \mathsf{last}(x) : ct_1' + ct_2}$$

Compiler architecture



Built on an existing synchronous compiler

- Source-to-source and traceable transformations
- Resulting program is synchronous and translated to sequential code

Comparison with existing tools

Simulink/Stateflow (Mathworks)

- Integrated treatment of automata *vs* two distinct languages
- More rigid separation of discrete and continuous behaviors

Modelica

- Do not handle DAEs
- Our proposal for automata has been integrated into version 3.3

Ptolemy

- More restrictive: A unique computational model (synchronous)
- Everything is compiled to sequential code

Perspectives

Semantics

- Corretness property: well-typed programs do not have discontinuities outside of zero-crossing events.
- This does not ensure the absence of zeno behaviors.
- The synchronous non-standard semantics is useful to prove the correctness theorem.
- Can we do it with the same precision and consision with super-dense time?

DAEs?

- Only ODEs for the moment. DAEs raise several issues (index reduction, etc.)
- Techniques by Acary and Brogliato to model *Non smooth dynamical systems* (e.g., billard balls)