Constructive Models of Discrete and Continuous Physical Phenomena

Edward A. Lee, UC Berkeley — EECS
Summer Simulation Multi-Conference, July 6 - 10, 2014, Monterey, CA
Discrete Physical Phenomena

- Collisions of rigid objects
- Switching in electrical circuits
- Actuation from software controllers

Ultimately, all of these are processes in the continuum of the physical world. But that is rarely the best way to model them.
The Model vs. the Thing Being Modeled

Solomon Wolf Golomb (1932) mathematician and engineer and a professor of electrical engineering at the University of Southern California.
The Model vs. the Thing Being Modeled

You will never strike oil by drilling through the map!

Solomon Wolf Golomb (1932) mathematician and engineer and a professor of electrical engineering at the University of Southern California.
The value of models depends on the fidelity to what is modeled.

Physical System: 

\[ \theta(t) = \theta(0) + \frac{1}{I} \int_0^t \int_0^\tau T(\alpha) d\alpha d\tau \]

Differential equations are commonly used to model physical dynamics. Such models are always approximate, but often useful. But they have difficulty with discrete events, such as collisions.
Newton’s Cradle – The Model

Image by Dominique Toussaint, GNU Free Documentation License, Version 1.2 or later.
Newton’s Cradle – A Physical Realization
**Model Fidelity**

- In *science*, the thing being modeled is given. It is up to the model to be faithful.

- In *engineering*, the thing being modeled is a human construction. It bears some responsibility to be faithful to the model.

In *engineering*, model fidelity is a *two-way street*.
Faithful Physical Model of Newton’s Cradle?

- localized plastic deformation
- viscous damping
- acoustic wave propagation

If our goal is to understand macroscopic system behavior, then such a model may not be a good choice.
Directly Modeling Discrete Phenomena

Difficulties that arise:

- signals are non-differentiable,
- chattering can occur around the discontinuity,
- Zeno behaviors can arise, and
- causality loops can arise.

Discrete physical phenomena are among the most poorly understood.
Mixed Discrete and Continuous Models

Our approach uses:

• superdense model of time,

• generalized functions,

• modal models, and

• constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html
Mixed Discrete and Continuous Models

Our approach uses:

- superdense model of time,
- generalized functions,
- modal models, and
- constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html
The Standard Model — Real-Valued Time

Let $p: \mathbb{R} \rightarrow \mathbb{R}$ be a function that represents the momentum of the second ball in Newton’s cradle, and let $\tau$ be the time of the collision. Then

$$p(t) = \begin{cases} P & \text{if } t = \tau \\ 0 & \text{otherwise} \end{cases}$$

for some constant $P$ and for all $t \in \mathbb{R}$.
The Standard Model — Real-Valued Time

\[ p(t) = \begin{cases} 
P & \text{if } t = \tau \\
0 & \text{otherwise} 
\end{cases} \quad (1) \]

Momentum is proportional to velocity, so

\[ p(t) = M v(t), \]

where \( M \) is the mass of the ball. Hence, combining with (1),

\[ v(t) = \begin{cases} 
\frac{P}{M} & \text{if } t = \tau \\
0 & \text{otherwise}. 
\end{cases} \]
The Standard Model — Real-Valued Time

\[ v(t) = \begin{cases} P/M & \text{if } t = \tau \\ 0 & \text{otherwise.} \end{cases} \]  

(2)

The position of a mass is the integral of its velocity,

\[ x(t) = x(0) + \int_0^t v(\tau) d\tau, \]

where \( x(0) \) is the initial position. The integral is zero at all \( t \), so the ball does not move.
The Standard Model — Real-Valued Time

\[ p: \mathbb{R} \to \mathbb{R}, \text{ where} \]

\[ p(t) = \begin{cases} P & \text{if } t = \tau \\ 0 & \text{otherwise} \end{cases} \]

Problems with this model:

- Violates conservation of momentum at the time of the collision.
- The model cannot be unambiguously given a discrete representation.
The Standard Model — Real-Valued Time

\( p: \mathbb{R} \rightarrow \mathbb{R} \), where

\[
p(t) = \begin{cases} 
P & \text{if } t = \tau \\ 
0 & \text{otherwise} 
\end{cases}
\]

Problems with this model:

- Violates conservation of momentum at the time of the collision.
- The model cannot be unambiguously given a discrete representation.
Superdense Time Model

Let $p: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$, where

$$p(t, n) = \begin{cases} P & \text{if } t = \tau \land n = 1 \\ 0 & \text{otherwise} \end{cases}$$

i.e., at the time $\tau$ of the collision,

- $p(\tau, 0) = 0$
- $p(\tau, 1) = P$
- $p(\tau, 2) = 0$
Summary: Superdense Time

- time stamp: \((t, n)\)
- model time: \(t \in \mathbb{R}\)
- microstep (or index): \(n \in \mathbb{N}\)

Two time stamps \((t_1, n_1)\) and \((t_2, n_2)\) are

- weakly simultaneous \(t_1 = t_2\)
- strongly simultaneous \(t_1 = t_2\) and \(n_1 = n_2\)

Superdense time has been used by Manna and Pnueli (1993); Maler et al. (1992); Lee and Zheng (2005).
Mixed Discrete and Continuous Models

Our approach uses:

- superdense model of time,
- generalized functions,
- modal models, and
- constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html
Dirac Delta Function

The Dirac delta function is a function $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ given by

$$\forall t \in \mathbb{R}, \; t \neq 0, \; \delta(t) = 0,$$

and

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1.$$ 

That is, the signal value is zero everywhere except at $t = 0$, but its integral is unity.

![Graph of Dirac Delta Function](image-url)
Impulsive Events with Dirac Delta Functions

Suppose $x$ has a Dirac delta function with weight $K$ occurring at $t_1$ as follows,

$$x(t) = x_1(t) + K\delta(t - t_1),$$

where $x_1$ is an ordinary continuous-time signal. Then

$$\int_{-\infty}^{t} x(\tau)d\tau = \begin{cases} \int_{-\infty}^{t} x_1(\tau)d\tau & t < t_1 \\ K + \int_{-\infty}^{t} x_1(\tau)d\tau & t \geq t_1 \end{cases}$$

The Dirac delta causes an instantaneous increment in the integral by $K$ at time $t = t_1$. 

Edward A. Lee, UC Berkeley — EECS
Direct Feedthrough

At time $t$, the state output is

$$x(t) = x_0 + \int_{t_0}^{t} \dot{x}(\tau) d\tau,$$

If the impulse input is present, then it adds immediately to $x(t)$.

The output at time $t$ depends on the impulse input at time $t$, but not on the derivative input.
The output $\nu$ depends immediately on the input $F_i$, if it is present.
Mixed Discrete and Continuous Models

- superdense model of time,
- generalized functions,
- modal models, and
- constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html
Collisions

Conservation of momentum:

\[ m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2. \]

Conservation of kinetic energy:

\[ \frac{m_1 (v_1')^2}{2} + \frac{m_2 (v_2')^2}{2} = \frac{m_1 (v_1)^2}{2} + \frac{m_2 (v_2)^2}{2}. \]

We have two equations and two unknowns, \( v_1' \) and \( v_2' \).
After a Collision

Quadratic problem has two solutions.

Solution 1: \( v'_1 = v_1, v'_2 = v_2 \)

(ignoring collision).

Solution 2:

\[
\begin{align*}
  v'_1 &= \frac{v_1(m_1 - m_2) + 2m_2v_2}{m_1 + m_2} \\
  v'_2 &= \frac{v_2(m_2 - m_1) + 2m_1v_1}{m_1 + m_2}.
\end{align*}
\]

Note that if \( m_1 = m_2 \), then the two masses simply exchange velocities
(newton’s cradle).
Coefficient of Restitution

\[ v_1' = \frac{em_2(v_2 - v_1) + m_1v_1 + m_2v_2}{m_1 + m_2} \]  \hspace{1cm} (3)

\[ v_2' = \frac{em_1(v_1 - v_2) + m_1v_1 + m_2v_2}{m_1 + m_2}. \]  \hspace{1cm} (4)

The coefficient of restitution $e$ is determined experimentally for a particular pair of materials and must lie in the range $0 \leq e \leq 1$. 
Collision Model

Model of two balls colliding.

Parameters:
- mLeft: 1.0  left mass
- mRight: 1.0  right mass
- e: 1.0  coefficient of restitution
- diameter: 1.0

Expressions:
- Expression 1:
  \[
  \frac{e \times mRight \times (v2 - v1) + mLeft \times v1 + mRight \times v2}{mLeft + mRight} - v1
  \]
- Expression 2:
  \[
  \frac{e \times mLeft \times (v1 - v2) + mLeft \times v1 + mRight \times v2}{mLeft + mRight} - v2
  \]
Simultaneous, Causal Collisions

The middle ball instantly collides with the right ball, and the middle ball should not move.
Collision Detection

Detecting discrete events is difficult. No zero crossing occurs. Failure to detect results in the upper plot.
Better Collision Detector

The LevelCrossingDetector actors regulate step sizes.
Simultaneous, Noncausal Collisions

Simultaneous collisions where one collision does not cause the other.
Simultaneous Collisions

Newton’s hypothesis superimposes impulsive forces, which cancel.

Poisson’s hypothesis separates a compression phase and a restitution phase.
Arbitrary Interleaving

Recall that one solution that conserves energy and momentum is to ignore a collision.

Here, if two collisions occur simultaneously, the right-middle collision is ignored.
Arbitrary Interleaving

But then, one superdense index later, a second collision occurs, where the middle ball and the right ball collide, traveling towards each other.

Result is like the Poisson solution.
Arbitrary Interleaving Yields Nondeterminism

If the masses are different, the behavior depends on which collision is handled first!
Heisenberg Uncertainty Principle

We cannot simultaneously know the position and momentum of an object to arbitrary precision. But the reaction to these collisions depends on knowing position and momentum precisely.
Heisenberg Uncertainty Principle

Arbitrary interleaving and nondeterministic results appears to be defensible on physical grounds.
Mixed Discrete and Continuous Models

Our approach uses:

- superdense model of time,
- generalized functions,
- modal models, and
- constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html
Pushing vs. Collisions

Consider two balls that start out touching, and from time 1 to 3.5 we apply a constant pushing force.

Treating these as collisions is very expensive.
Modal Models

Modal Model

Director

Some sub-model here.

Director

Some sub-model here.

Edward A. Lee, UC Berkeley — EECS
Modal Pushing

Continuous Director

Behavior when the two balls are acting as one and traveling together.

Note that if \( F_r > F_I \), we will leave this mode, so here we can assume \( F_r = F_I \).

Any impulsive force will cause us to leave this mode, so we ignore impulsive inputs.
Bouncing Ball — Zeno Conditions

Impulsive force is fed back to model the bounce.
Zeno

Zero-crossing detection ultimately depends on numerical approximations to real numbers. It can only be done up to some precision.
Zeno

Zero-crossing detection ultimately depends on numerical approximations to real numbers. It can only be done up to some precision.

Maybe tunneling is nature’s way of resolving Zeno conditions.
Continuous Zeno Behavior

Zeno behaviors are not limited to discrete events.
Modal Bouncing Ball

Impulsive force is fed back to model the bounce.
Alternative Modal Bouncing Ball

Mode transition handles the discrete change of energy and momentum.
Regimes of Models

Mode transition changes the model when the state of the system moves outside the regime over which the model is reasonable.

Note that all models have a limited regime over which they are reasonable.
Mixed Discrete and Continuous Models

Our approach uses:

• superdense model of time,
• generalized functions,
• modal models, and
• constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html
Constructive Fixed-Point Semantics

Every model reduces to a fixed point problem. For each \( t \in \mathbb{R} \times \mathbb{N} \), find \( s(t) \) s.t.

\[
F(s(t)) = s(t).
\]

With constructive semantics, solution is unique. See report.
Other Examples Considered in the Paper

switches and diodes:

friction:

circuits with no solution:

collisions and friction:
Mixed Discrete and Continuous Models

- superdense model of time,
- generalized functions,
- modal models, and
- constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html
References


Questions?