Constructive Models of Discrete and Continuous Physical Phenomena

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Discrete Physical Phenomena

- Collisions of rigid objects
- Switching in electrical circuits
- Actuation from software controllers

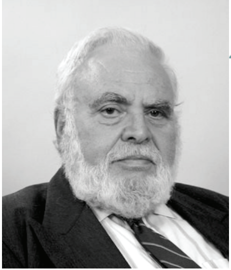
Ultimately, all of these are processes in the continuum of the physical world. But that is rarely the best way to model them.

The Model vs. the Thing Being Modeled

Solomon Wolf Golomb (1932) mathematician and engineer and a professor of electrical engineering at the University of Southern California.

The Model vs. the Thing Being Modeled

You will never strike oil by drilling through the map!



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The value of models depends on the fidelity to what is modeled.

Physical System:

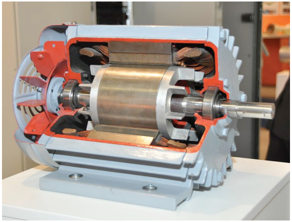
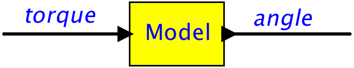


Image: Wikimedia Commons

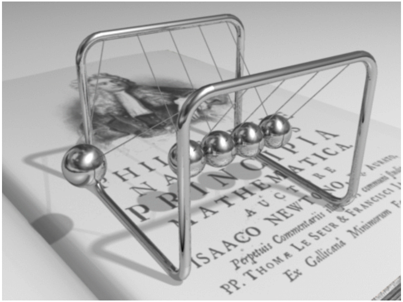
Model:



$$\theta(t) = \theta(0) + \frac{1}{I}\int_0^t \int_0^\tau \mathbf{T}(\alpha)\,d\alpha\,d\tau$$

Differential equations are commonly used to model physical dynamics. Such models are always approximate, but often useful. But they have difficulty with discrete events, such as collisions.

Newton's Cradle – The Model



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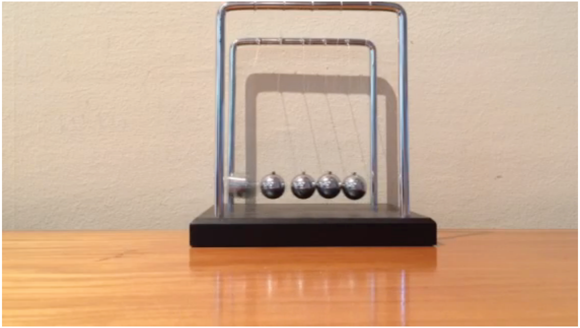
Newton's Cradle – A Physical Realization



Model Fidelity

- In *science*, the thing being modeled is given.
  It is up to the model to be faithful.

- In *engineering*, the thing being modeled is a human construction.
  It bears some responsibility to be faithful to the model.

In *engineering*, model fidelity is a *two-way street*.

Faithful Physical Model of Newton's Cradle?

- localized plastic deformation
- viscous damping
- acoustic wave propagation

If our goal is to understand macroscopic *system* behavior, then such a model may not be a good choice.

Directly Modeling Discrete Phenomena

Difficulties that arise:

- signals are non-differentiable,
- chattering can occur around the discontinuity,
- Zeno behaviors can arise, and
- causality loops can arise.

Discrete physical phenomena are among the most poorly understood.

Mixed Discrete and Continuous Models

Our approach uses:

- superdense model of time,
- generalized functions,
- modal models, and
- constructive semantics.

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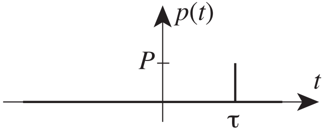
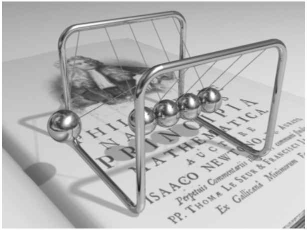
The Standard Model — Real-Valued Time

Let $p\colon \mathbb{R} \to \mathbb{R}$ be a function that represents the momentum of the second ball in Newton's cradle, and let τ be the time of the collision. Then

$$p(t) = \begin{cases} P & \text{if } t = \tau \\ 0 & \text{otherwise} \end{cases}$$

for some constant P and for all $t \in \mathbb{R}$.





The Standard Model — Real-Valued Time

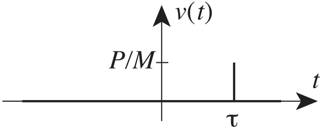
$$p(t) = \begin{cases} P & \text{if } t = \tau \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Momentum is proportional to velocity, so

$$p(t) = Mv(t),$$

where M is the mass of the ball. Hence, combining with (1),

$$v(t) = \begin{cases} P/M & \text{if } t = \tau \\ 0 & \text{otherwise.} \end{cases}$$



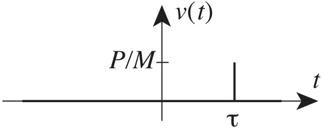
The Standard Model — Real-Valued Time

$$v(t) = \begin{cases} P/M & \text{if } t = \tau \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The position of a mass is the integral of its velocity,

$$x(t) = x(0) + \int_0^t v(\tau) d\tau,$$

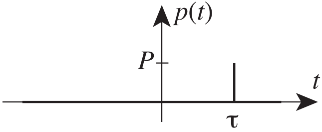
where $x(0)$ is the initial position.
The integral is zero at all t,
so the ball does not move.



The Standard Model — Real-Valued Time

$p \colon \mathbb{R} \to \mathbb{R}$, where

$$p(t) = \begin{cases} P & \text{if } t = \tau \\ 0 & \text{otherwise} \end{cases}$$



Problems with this model:

- Violates conservation of momentum at the time of the collision.
- The model cannot be unambiguously given a discrete representation.

The Standard Model — Real-Valued Time

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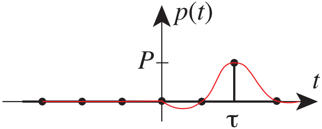

Problems with this model:

- Violates conservation of momentum at the time of the collision.
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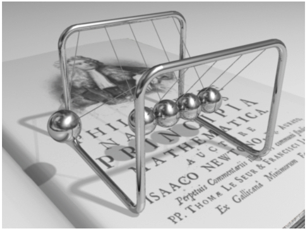
Superdense Time Model

Let $p\colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, where

$$p(t, n) = \begin{cases} P & \text{if } t = \tau \wedge n = 1 \\ 0 & \text{otherwise} \end{cases}$$

I.e., at the time τ of the collision,

- $p(\tau, 0) = 0$
- $p(\tau, 1) = P$
- $p(\tau, 2) = 0$



Summary: Superdense Time

- time stamp: (t, n)
- model time: $t \in \mathbb{R}$
- microstep (or index): $n \in \mathbb{N}$

Two time stamps (t_1, n_1) and (t_2, n_2) are

- weakly simultaneous it $t_1 = t_2$
- strongly simultaneous it $t_1 = t_2$ and $n_1 = n_2$

Superdense time has been used by Manna and Pnueli (1993); Maler et al. (1992); Lee and Zheng (2005).

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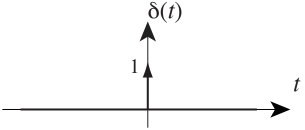
Dirac Delta Function

The Dirac delta function is a function $\delta\colon \mathbb{R} \to \mathbb{R}^+$ given by

$$\forall\, t \in \mathbb{R},\ t \neq 0, \quad \delta(t) = 0, \quad \text{and}$$

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1.$$

That is, the signal value is zero everywhere except at $t = 0$, but its integral is unity.



Impulsive Events with Dirac Delta Functions

Suppose x has a Dirac delta function with weight K occurring at t_1 as follows,

$$x(t) = x_1(t) + K\delta(t - t_1),$$

where x_1 is an ordinary continuous-time signal. Then

$$\int_{-\infty}^{t} x(\tau)d\tau = \begin{cases} \int_{-\infty}^{t} x_1(\tau)d\tau & t < t_1 \\ K + \int_{-\infty}^{t} x_1(\tau)d\tau & t \geq t_1 \end{cases}$$

The Dirac delta causes an instantaneous increment in the integral by K at time $t = t_1$.

Direct Feedthrough

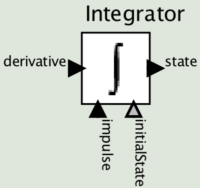
At time t, the *state* output is

$$x(t) = x_0 + \int_{t_0}^{t} \dot{x}(\tau) d\tau,$$

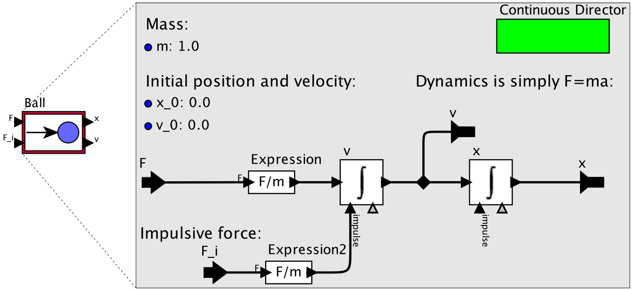
If the *impulse* input is present, then it adds immediately to $x(t)$.

The output at time t depends on the *impulse* input at time t, but not on the *derivative* input.

Ptolemy II Integrator has "impulse" input:



Ball Model



The output v depends immediately on the input F_i, if it is present.

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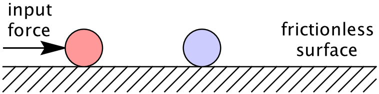
Collisions



Conservation of momentum:

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2.$$

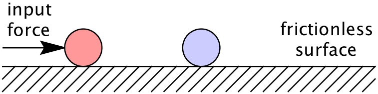
Conservation of kinetic energy:

$$\frac{m_1 (v_1')^2}{2} + \frac{m_2 (v_2')^2}{2} = \frac{m_1 (v_1)^2}{2} + \frac{m_2 (v_2)^2}{2}.$$

We have two equations and two unknowns, v_1' and v_2'.

After a Collision

Quadratic problem has two solutions.



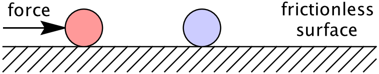
Solution 1: $v_1' = v_1$, $v_2' = v_2$
(ignore collision).

Solution 2:

$$
\begin{aligned}
v_1' &= \frac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2} \\
v_2' &= \frac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2}.
\end{aligned}
$$

Note that if $m_1 = m_2$, then the two masses simply exchange velocities (Newton's cradle).

Coefficient of Restitution

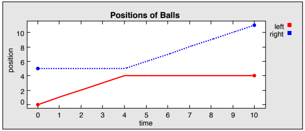


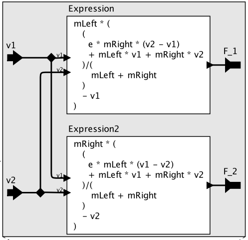
$$v_1' = \frac{em_2(v_2 - v_1) + m_1 v_1 + m_2 v_2}{m_1 + m_2} \quad (3)$$

$$v_2' = \frac{em_1(v_1 - v_2) + m_1 v_1 + m_2 v_2}{m_1 + m_2}. \quad (4)$$

The coefficient of restitution e is determined experimentally for a particular pair of materials and must lie in the range $0 \leq e \leq 1$.

Collision Model



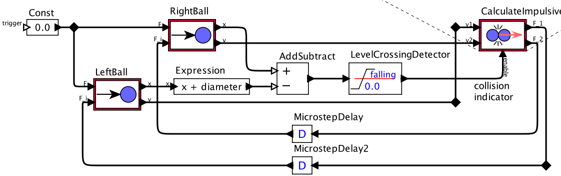


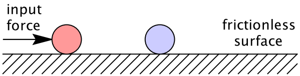
Model of two balls colliding.

Parameters:

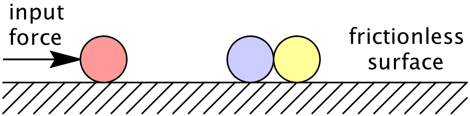
- mLeft: 1.0 left mass
- mRight: 1.0 right mass
- e: 1.0 coefficient of restitution
- diameter: 1.0

Continuous Director





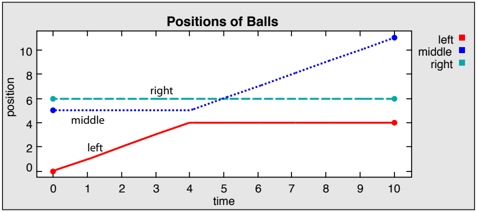
Simultaneous, Causal Collisions



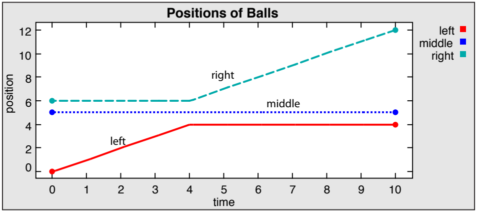
The middle ball instantly collides with the right ball, and the middle ball should not move.

Collision Detection

Detecting discrete events is difficult. No zero crossing occurs. Failure to detect results in the upper plot.

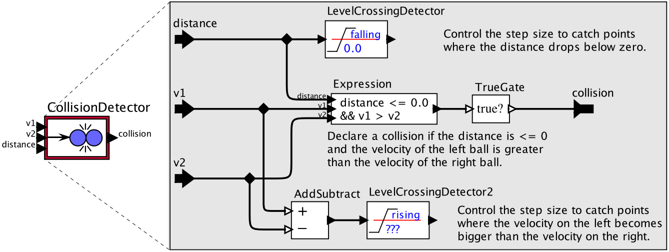


(a)



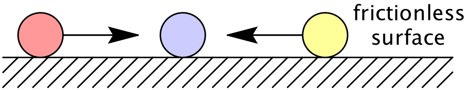
(b)

Better Collision Detector



The LevelCrossingDetector actors regulate step sizes.

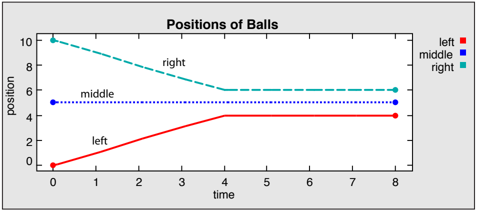
Simultaneous, Noncausal Collisions



Simultaneous collisions where one collision does not cause the other.

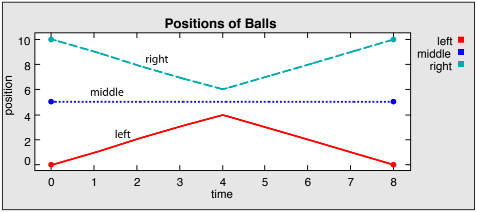
Simultaneous Collisions

Newton's hypothesis superimposes impulsive forces, which cancel.



(a)

Poisson's hypothesis separates a compression phase and a restitution phase.

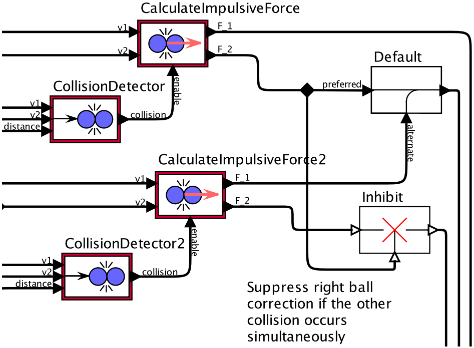


(b)

Arbitrary Interleaving

Recall that one solution that conserves energy and momentum is to ignore a collision.

Here, if two collisions occur simultaneously, the right-middle collision is ignored.

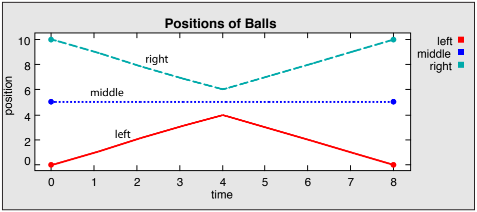


CollisionSimultaneous.xml

Arbitrary Interleaving

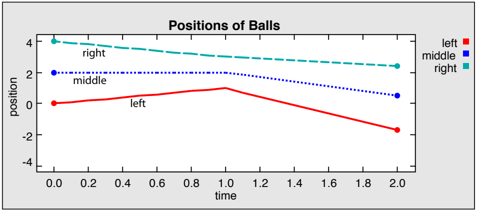
But then, one superdense index later, a second collision occurs, where the middle ball and the right ball collide, traveling towards each other.

Result is like the Poisson solution.

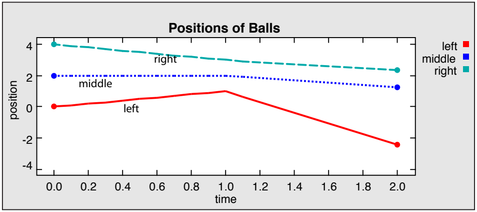


CollisionSimultaneous.xml

Arbitrary Interleaving Yields Nondeterminism



(a)

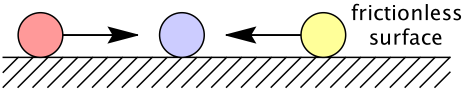


(b)

If the masses are different, the behavior depends on which collision is handled first!

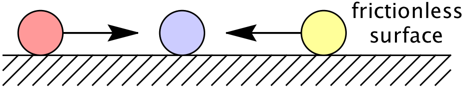
Heisenberg Uncertainty Principle

We cannot simultaneously know the position and momentum of an object to arbitrary precision. But the reaction to these collisions depends on knowing position and momentum precisely.



Heisenberg Uncertainty Principle

Arbitrary interleaving and nondeterministic results appears to be defensible on physical grounds.



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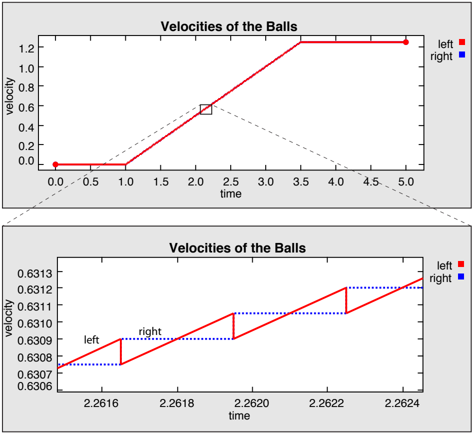
Pushing vs. Collisions

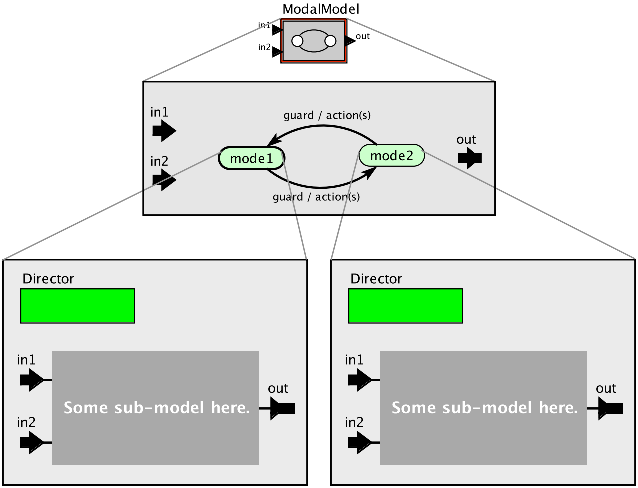
Consider two balls that start out touching, and from time 1 to 3.5 we apply a constant pushing force.

Treating these as collisions is very expensive.

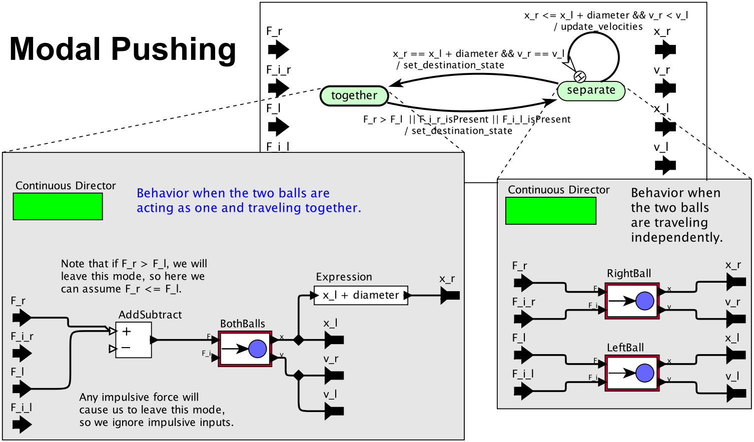


Collision5.xml

Modal Models

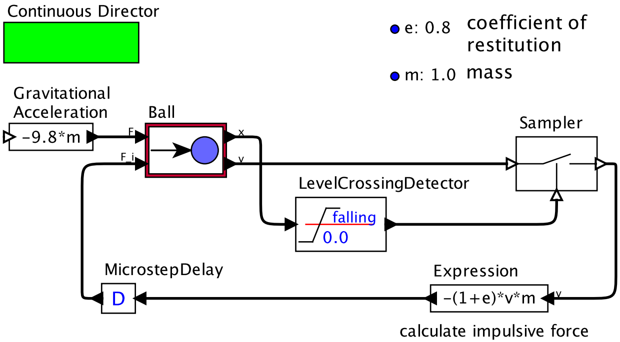


Modal Pushing



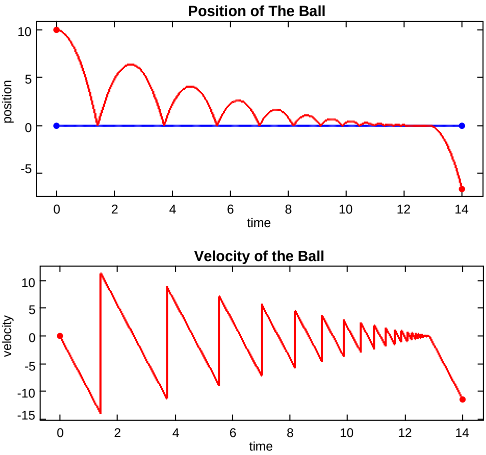
Bouncing Ball — Zeno Conditions

Impulsive force is fed back to model the bounce.



Zeno

Zero-crossing detection ultimately depends on numerical approximations to real numbers. It can only be done up to some precision.



BouncingBall.xml

Zeno

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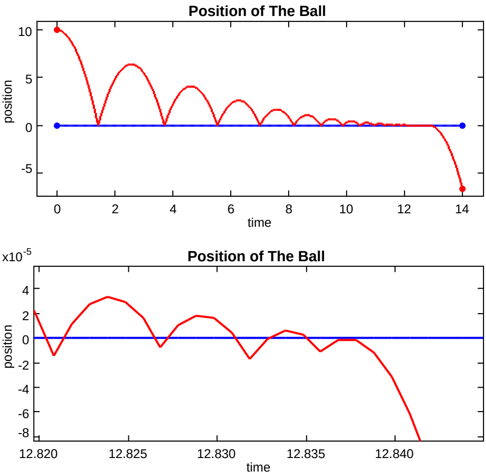
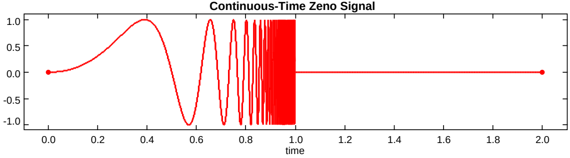
Maybe tunneling is nature's way of resolving Zeno conditions.



Continuous Zeno Behavior



Zeno behaviors are not limited to discrete events.

Modal Bouncing Ball

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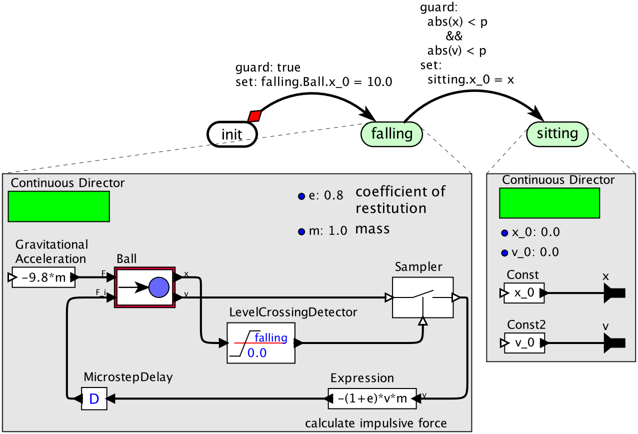
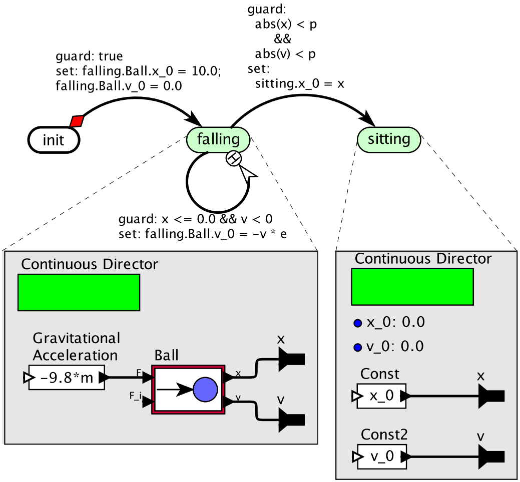


Alternative Modal Bouncing Ball

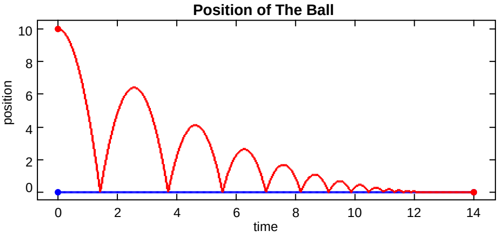
Mode transition handles the discrete change of energy and momentum.



BouncingBallModal2.xml

Regimes of Models

Mode transition changes the model when the state of the system moves outside the regime over which the model is reasonable.



Note that *all* models have a limited regime over which they are reasonable.

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- constructive semantics.

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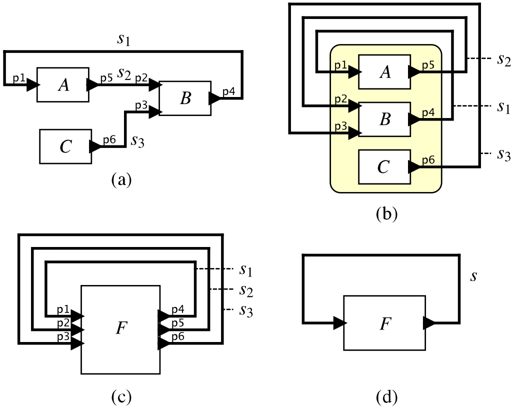
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html

Constructive Fixed-Point Semantics

Every model reduces to a fixed point problem. For each $t \in \mathbb{R} \times \mathbb{N}$, find $s(t)$ s.t.

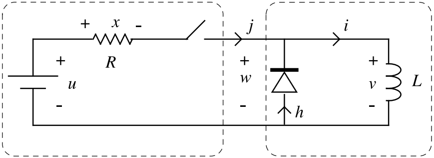
$$F(s(t)) = s(t).$$

With constructive semantics, solution is unique. See report.

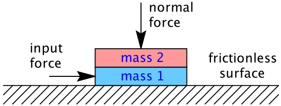


Other Examples Considered in the Paper

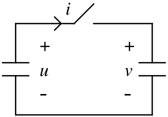
switches and diodes:



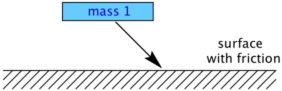
friction:



circuits with no solution:



collisions and friction:



Mixed Discrete and Continuous Models

- superdense model of time,
- generalized functions,
- modal models, and
- constructive semantics.

Detailed report is here:
http://www.eecs.berkeley.edu/Pubs/TechRpts/2014/EECS-2014-15.html

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Questions?