Concurrent Games with Tail Objectives

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Abstract. We study infinite stochastic games played by two-players over a finite state space, with objectives specified by sets of infinite traces. The games are *concurrent* (players make moves simultaneously and independently), *stochastic* (the next state is determined by a probability distribution that depends on the current state and chosen moves of the players) and *infinite* (proceeds for infinite number of rounds). The analysis of concurrent stochastic games can be classified into: *quantitative analysis*, analyzing the optimum value of the game; and *qualitative analysis*, analyzing the set of states with optimum value 1. We consider concurrent games with tail objectives, i.e., objectives that are independent of the finite-prefix of traces, and show that the class of tail objectives are strictly richer than the ω-regular objectives. We develop new proof techniques to extend several properties of concurrent games with ω-regular objectives to concurrent games with tail objectives. We prove the *positive limit-one* property for tail objectives, that states for all concurrent games if the optimum value for a player is positive for a tail objective Φ at some state, then there is a state where the optimum value is 1 for Φ, for the player. We also show that the optimum values of *zero-sum* (strictly conflicting objectives) games with tail objectives can be related to equilibrium values of *nonzero-sum* (not strictly conflicting objectives) games with simpler reachability objectives. A consequence of our analysis presents a polynomial time reduction of the quantitative analysis of tail objectives to the qualitative analysis for the sub-class of one-player stochastic games (Markov decision processes).

1 Introduction

Stochastic games. Non-cooperative games provide a natural framework to model interactions between agents [13, 14]. A wide class of games progress over time and in stateful manner, and the current game depends on the history of interactions. Infinite *stochastic games* [15, 9] are a natural model for such dynamic games. A stochastic game is played over a finite *state space* and is played in rounds. In *concurrent games*, in each round, each player chooses an action from a finite set of available actions, simultaneously and independently of the other player. The game proceeds to a new state according to a probabilistic transition relation (stochastic transition matrix) based on the current state and the joint actions of the players. Concurrent games (also known as Blackwell games) subsume the simpler class of *turn-based games*, where at every state at most one player can choose between multiple actions; and Markov decision processes (MDPs), where only one player can choose between multiple actions at every state. Concurrent games also provide the framework to model synchronous reactive systems [6]. In verification and control of finite state reactive systems such games proceed for infinite rounds, generating an infinite sequence of states, called the *outcome* of the game. The players receive a payoff based on a payoff function that maps every outcome to a real number.

Objectives. Payoffs are generally Borel measurable functions [12]. For example, the payoff set for each player is a Borel set B_i in the Cantor topology on S^ω (where S is the set of states), and player i gets payoff 1 if the outcome of the game is a member of B_i, and 0 otherwise. In verification, payoff functions are usually index sets of *ω-regular languages.* The ω-regular languages generalize the classical regular languages to infinite strings, they occur in low levels of the Borel hierarchy (they are in $\mathbf{\Sigma}_3^0 \cap \mathbf{\Pi}_3^0$), and they form a robust and expressive language for determining payoffs for commonly used specifications. The simplest ω-regular objectives correspond to safety ("closed sets") and reachability ("open sets") objectives.

Zero-sum games, determinacy and nonzero-sum games. Games may be *zero-sum*, where two players have directly conflicting objectives and the payoff of one player is one minus the payoff of the other, or *nonzero-sum*, where each player has a prescribed payoff function based on the outcome of the game. The fundamental question for games is the existence of equilibrium values. For zero-sum games, this involves showing a *determinacy* theorem that states that the expected optimum value obtained by player 1 is exactly one minus the expected optimum value obtained by player 2. For one-step zero-sum games, this is von Neumann's minmax theorem [18]. For infinite games, the existence of such equilibria is not obvious, in fact, by using the axiom of choice, one can construct games for which determinacy does not hold. However, a remarkable result by Martin [12] shows that all stochastic zero-sum games with Borel payoffs are determined. For nonzero-sum games, the fundamental equilibrium concept is a *Nash equilibrium* [10], that is, a strategy profile such that no player can gain by deviating from the profile, assuming the other player continue playing the strategy in the profile.

Qualitative and quantitative analysis. The analysis of zero-sum concurrent games can be broadly classified into: (a) *quantitative analysis* that involves analysis of the optimum values of the games; and (b) *qualitative analysis* that involves simpler analysis of the set of states where the optimum value is 1.

Properties of concurrent games. The result of Martin [12] established the determinacy of zero-sum concurrent games for all Borel objectives. The determinacy result sets forth the problem of study and closer understanding of properties and behaviors of concurrent games with different class of objectives. Several interesting questions related to concurrent games are: (1) characterizing certain zero-one laws for concurrent games; (2) relationship of qualitative and quantitative analysis; (3) relationship of zero-sum and nonzero-sum games. The results of $[6, 7, 1]$ exhibited several interesting properties for concurrent games with ω-regular objectives specified as parity objectives. The result of [6] showed the positive limit-one property, that states if there is a state with positive optimum value, then there is a state with optimum value 1, for concurrent games with parity objectives. The positive limit-one property has been a key property to develop algorithms and improved complexity bound for quantitative analysis of concurrent games with parity objectives [1]. The above properties can possibly be the basic ingredients for the computational complexity analysis of quantitative analysis of concurrent games.

Outline of results. In this work, we consider *tail objectives*, the objectives that do not depend on any finite-prefix of the traces. Tail objectives subsume canonical ω-regular objectives such as parity objectives and Müller objectives, and we show that there exist tail objectives that cannot be expressed as ω-regular objectives. Hence tail objectives are a strictly richer class of objectives than ω-regular objectives. Our results characterize several properties of concurrent games with tail objectives. The results are as follows.

1. We show the positive limit-one property for concurrent games with tail objectives. Our result thus extends the result of [6] from parity objectives to a richer class of objective that lie in the higher levels of Borel hierarchy. The result of [6] follows from a complementation argument of quantitative μ-calculus formula. Our proof technique is completely different: it uses certain strategy construction procedures and a convergence result from measure theory (Lévy's zero-one law). It may be noted that the positive limit-one property for concurrent games with Müller objectives follows from the positive limit-one property for parity objectives and the reduction of Müller objectives to parity objectives [17]. Since Müller objectives are tail objectives, our result presents a direct proof for the positive limit-one property for concurrent games with Müller objectives.
2. We relate the optimum values of zero-sum games with tail objectives with Nash-equilibrium values of non-zero sum games with reachability objectives. This establishes a relationship between the values of concurrent games with complex tail objectives and Nash equilibrium of nonzero-sum games with simpler objectives. From the above analysis we obtain a polynomial time reduction of quantitative analysis of tail objectives to qualitative analysis for the special case of MDPs. The above result was previously known for the sub-class of ω-regular objectives specified as Müller objectives $[4, 5, 2]$. The proof techniques of $[4, 5, 2]$ use different analysis of the structure of MDPs and is completely different from our proof techniques.

2 Definitions

Notation. For a countable set A, a *probability distribution* on A is a function $\delta : A \to [0, 1]$ such that $\sum_{a \in A} \delta(a) = 1$. We denote the set of probability distributions on A by $\mathcal{D}(A)$. Given a distribution $\delta \in \mathcal{D}(A)$, we denote by $\mathrm{Supp}(\delta) = \{x \in A \mid \delta(x) > 0\}$ the *support* of δ.

Definition 1 (Concurrent Games). *A (two-player)* concurrent game struc*ture $G = \langle S, Moves, Mv_1, Mv_2, \delta\rangle$ consists of the following components:*

- *A finite state space S and a finite set $Moves$ of moves.*
- *Two move assignments $Mv_1, Mv_2 \colon S \to 2^{Moves} \setminus \emptyset$. For $i \in \{1, 2\}$, assignment Mv_i associates with each state $s \in S$ the non-empty set $Mv_i(s) \subseteq Moves$ of moves available to player i at s.*
- *A probabilistic transition function $\delta : S \times Moves \times Moves \to \mathcal{D}(S)$, that gives the probability $\delta(s, a_1, a_2)(t)$ of a transition from s to t when player 1 plays move a_1 and player 2 plays move a_2, for all $s, t \in S$ and $a_1 \in Mv_1(s)$, $a_2 \in Mv_2(s)$.* ∎

An important special class of concurrent games are Markov decision processes (MDPs), where at every state s we have $|Mv_2(s)| = 1$, i.e., the set of available moves for player 2 is singleton at every state.

At every state $s \in S$, player 1 chooses a move $a_1 \in Mv_1(s)$, and simultaneously and independently player 2 chooses a move $a_2 \in Mv_2(s)$. The game then proceeds to the successor state t with probability $\delta(s, a_1, a_2)(t)$, for all $t \in S$. A state s is called an *absorbing state* if for all $a_1 \in Mv_1(s)$ and $a_2 \in Mv_2(s)$ we have $\delta(s, a_1, a_2)(s) = 1$. In other words, at s for all choices of moves of the players the next state is always s. We assume that the players act *non-cooperatively*, i.e., each player chooses her strategy independently and secretly from the other player, and is only interested in maximizing her own reward. For all states $s \in S$ and moves $a_1 \in Mv_1(s)$ and $a_2 \in Mv_2(s)$, we indicate by $\mathrm{Dest}(s, a_1, a_2) = \mathrm{Supp}(\delta(s, a_1, a_2))$ the set of possible successors of s when moves a_1, a_2 are selected.

A *path* or a *play* ω of G is an infinite sequence $\omega = \langle s_0, s_1, s_2, \ldots\rangle$ of states in S such that for all $k \geq 0$, there are moves $a_1^k \in Mv_1(s_k)$ and $a_2^k \in Mv_2(s_k)$ with $\delta(s_k, a_1^k, a_2^k)(s_{k+1}) > 0$. We denote by Ω the set of all paths and by Ω_s the set of all paths $\omega = \langle s_0, s_1, s_2, \ldots\rangle$ such that $s_0 = s$, i.e., the set of plays starting from state s.

Strategies. A *selector* ξ for player $i \in \{1, 2\}$ is a function $\xi : S \to \mathcal{D}(Moves)$ such that for all $s \in S$ and $a \in Moves$, if $\xi(s)(a) > 0$, then $a \in Mv_i(s)$. We denote by Λ_i the set of all selectors for player $i \in \{1, 2\}$. A *strategy* for player 1 is a function $\tau : S^+ \to \Lambda_1$ that associates with every finite non-empty sequence of states, representing the history of the play so far, a selector. Similarly we define strategies π for player 2. We denote by Γ and Π the set of all strategies for player 1 and player 2, respectively.

Once the starting state s and the strategies τ and π for the two players have been chosen, the game is reduced to an ordinary stochastic process. Hence the probabilities of events are uniquely defined, where an *event* $\mathcal{A} \subseteq \Omega_s$ is a measurable set of paths. For an event $\mathcal{A} \subseteq \Omega_s$ we denote by $\Pr_s^{\tau,\pi}(\mathcal{A})$ the probability that a path belongs to $\mathcal{A}$ when the game starts from s and the players follow the strategies τ and π. For $i \geq 0$, we also denote by $\Theta_i : \Omega \to S$ the random variable denoting the i-th state along a path.

Objectives. We specify objectives for the players by providing the set of *winning plays* $\Phi \subseteq \Omega$ for each player. Given an objective Φ we denote by $\overline{\Phi} = \Omega \setminus \Phi$, the complementary objective of Φ. A concurrent game with objective Φ_1 for player 1 and Φ_2 for player 2 is *zero-sum* if $\Phi_2 = \overline{\Phi}_1$. A general class of objectives are the Borel objectives [11]. A *Borel objective* $\Phi \subseteq S^\omega$ is a Borel set in the Cantor topology on S^ω. In this paper we consider *ω-regular objectives* [17], which lie in the first $2^1/_2$ levels of the Borel hierarchy (i.e., in the intersection of $\mathbf{\Sigma}_3^0$ and $\mathbf{\Pi}_3^0$) and *tail objectives* which is a strict superset of ω-regular objectives. The ω-regular objectives, and subclasses thereof, and tail objectives are defined below. For a play $\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega$, we define $\mathrm{Inf}(\omega) = \{ s \in S \mid s_k = s \text{ for infinitely many } k \geq 0 \}$ to be the set of states that occur infinitely often in ω.

- *Reachability and safety objectives.* Given a set $T \subseteq S$ of "target" states, the reachability objective requires that some state of T be visited. The set of winning plays is thus $\mathrm{Reach}(T) = \{ \omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega \mid s_k \in T \text{ for some } k \geq 0 \}$. Given a set $F \subseteq S$, the safety objective requires that only states of F be visited. Thus, the set of winning plays is $\mathrm{Safe}(F) = \{ \omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega \mid s_k \in F \text{ for all } k \geq 0 \}$.
- *Büchi and coBüchi objectives.* Given a set $B \subseteq S$ of "Büchi" states, the Büchi objective requires that B is visited infinitely often. Formally, the set of winning plays is $\text{Büchi}(B) = \{ \omega \in \Omega \mid \mathrm{Inf}(\omega) \cap B \neq \emptyset \}$. Given $C \subseteq S$, the coBüchi objective requires that all states visited infinitely often are in C. Formally, the set of winning plays is $\text{coBüchi}(C) = \{ \omega \in \Omega \mid \mathrm{Inf}(\omega) \subseteq C \}$.
- *Parity objectives.* For $c, d \in \mathbb{N}$, we let $[c..d] = \{ c, c+1, \ldots, d \}$. Let $p : S \to [0..d]$ be a function that assigns a *priority* $p(s)$ to every state $s \in S$, where $d \in \mathbb{N}$. The *Even parity objective* is defined as $\mathrm{Parity}(p) = \{ \omega \in \Omega \mid \min\big(p(\mathrm{Inf}(\omega))\big) \text{ is even } \}$, and the *Odd parity objective* as $\mathrm{coParity}(p) = \{ \omega \in \Omega \mid \min\big(p(\mathrm{Inf}(\omega))\big) \text{ is odd } \}$.
- *Muller objectives.* Given a set $\mathcal{M} \subseteq 2^S$ of subset of states, the *Müller objective* is defined as $\text{Müller}(\mathcal{M}) = \{ \omega \in \Omega \mid \mathrm{Inf}(\omega) \in \mathcal{M} \}$.
- *Tail objectives.* Informally the class of tail objectives is the sub-class of Borel objectives that are independent of all finite prefixes. An objective Φ is a tail objective, if the following condition hold: a path $\omega \in \Phi$ if and only if for all $i \geq 0$, $\omega_i \in \Phi$, where ω_i denotes the path ω with the prefix of length i deleted. Formally, let $\mathcal{G}_i = \sigma(\Theta_i, \Theta_{i+1}, \ldots)$ be the σ-field generated by the random variables $\Theta_i, \Theta_{i+1}, \ldots$. The tail σ-field $\mathcal{T}$ is defined as $\mathcal{T} = \bigcap_{i \geq 0} \mathcal{G}_i$. An objective Φ is a tail objective if and only if Φ belongs to the tail σ-field $\mathcal{T}$, i.e., the tail objectives are indicator functions of events $\mathcal{A} \in \mathcal{T}$.

The Müller and parity objectives are canonical forms to represent ω-regular objectives [16]. Observe that Müller and parity objectives are tail objectives. Note that for a priority function $p : S \to \{ 0, 1 \}$, an even parity objective $\mathrm{Parity}(p)$ is equivalent to the Büchi objective $\text{Büchi}(p^{-1}(0))$, i.e., the Büchi set consists of the states with priority 0. Büchi and coBüchi objectives are special cases of parity objectives and hence tail objectives. Reachability objectives are not necessarily tail objectives, but for a set $T \subseteq S$ of states, if every state $s \in T$ is an absorbing state, then the objective Reach(T) is equivalent to Büchi(T) and hence is a tail objective. It may be noted that since σ-fields are closed under complementation, the class of tail objectives are closed under complementation. We give an example to show that the class of tail objectives are richer than ω-regular objectives. [1]

Example 1. Let r be a reward function that maps every state s to a real-valued reward $r(s)$, i.e., $r : S \to \mathbb{R}$. For a constant $c \in \mathbb{R}$ consider the objective Φ_c defined as follows: $\Phi_c = \{\omega \in \Omega \mid \omega = \langle s_1, s_2, s_3, \ldots \rangle, \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r(s_i) \geq c\}$. Intuitively, Φ_c accepts the set of paths such that the "long-run" average of the rewards in the path is at least the constant c. The "long-run" average condition lie in the third-level of the Borel-hierarchy (i.e., in $\mathbf{\Pi}_3^0$ and $\mathbf{\Pi}_3^0$-complete) and cannot be expressed as an ω-regular objective. It may be noted that the "long-run" average of a path is independent of all finite-prefixes of the path. Formally, the class Φ_c of objectives are tail objectives. Since Φ_c are $\mathbf{\Pi}_3^0$-complete objectives, it follows that tail objectives lie in higher levels of Borel hierarchy than ω-regular objectives. ∎

Values. The probability that a path satisfies an objective Φ starting from state $s \in S$, given strategies τ, π for the players is $\Pr_s^{\tau,\pi}(\Phi)$. Given a state $s \in S$ and an objective Φ, we are interested in the maximal probability with which player 1 can ensure that Φ and player 2 can ensure that $\overline{\Phi}$ holds from s. We call such probability the *value of the game* G at s for player $i \in \{1, 2\}$. The value for player 1 and player 2 are given by the functions $\langle\langle 1\rangle\rangle_{val}(\Phi) : S \to [0, 1]$ and $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi}) : S \to [0, 1]$, defined for all $s \in S$ by $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \sup_{\tau\in\Gamma} \inf_{\pi\in\Pi} \Pr_s^{\tau,\pi}(\Phi)$ and $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = \sup_{\pi\in\Pi} \inf_{\tau\in\Gamma} \Pr_s^{\tau,\pi}(\overline{\Phi})$. Note that the objectives of the player are complementary and hence we have a zero-sum game. Concurrent games satisfy a *quantitative* version of determinacy [12], stating that for all Borel objectives Φ and all $s \in S$, we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) + \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = 1$. A strategy τ for player 1 is *optimal* for objective Φ if for all $s \in S$ we have $\inf_{\pi\in\Pi} \Pr_s^{\tau,\pi}(\Phi) = \langle\langle 1\rangle\rangle_{val}(\Phi)(s)$. For $\varepsilon > 0$, a strategy τ for player 1 is ε-*optimal* for objective Φ if for all $s \in S$ we have $\inf_{\pi\in\Pi} \Pr_s^{\tau,\pi}(\Phi) \geq \langle\langle 1\rangle\rangle_{val}(\Phi)(s) - \varepsilon$. We define optimal and ε-optimal strategies for player 2 symmetrically. For $\varepsilon > 0$, an objective Φ for player 1 and $\overline{\Phi}$ for player 2, we denote by $\Gamma_\varepsilon(\Phi)$ and $\Pi_\varepsilon(\overline{\Phi})$ the set of ε-optimal strategies for player 1 and player 2, respectively. Even in concurrent games with reachability objectives optimal strategies need not exist [6], and ε-optimal strategies, for all $\varepsilon > 0$, is the best one can achieve. Note that the quantitative determinacy of concurrent games is equivalent to the existence of ε-optimal strategies for objective Φ for player 1 and $\overline{\Phi}$ for player 2, for all $\varepsilon > 0$, at all states $s \in S$, i.e., for all $\varepsilon > 0$, $\Gamma_\varepsilon(\Phi) \neq \emptyset$ and $\Pi_\varepsilon(\overline{\Phi}) \neq \emptyset$.

We refer to the analysis of computing the *limit-sure winning* states (the set of states s such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = 1$) as the *qualitative* analysis of objective

[1] Our example shows that there are $\mathbf{\Pi}_3^0$-complete objectives that are tail objectives. It is possible that the tail objectives can express objectives in even higher levels of Borel hierarchy than $\mathbf{\Pi}_3^0$, which will make our results stronger.

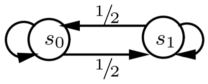

Fig. 1. A simple Markov chain.

Φ. We refer to the analysis of computing the values as the *quantitative* analysis of objective Φ.

3 Positive Limit-one Property

The *positive limit-one* property for concurrent games, for a class $\mathcal{C}$ of objectives, states that for all objectives $\Phi \in \mathcal{C}$, for all concurrent games G, if there is a state s such that the value for player 1 is positive at s for objective Φ, then there is a state s' where the value for player 1 is 1 for objective Φ. The property means if a player can win with positive value from some state, then from some state she can win with value 1. The positive limit-one property was proved for parity objectives in [6] and has been one of the key properties used in the algorithmic analysis of concurrent games with parity objectives [1]. In this section we prove the *positive limit-one* property for concurrent games with tail objectives, and thereby extend the positive limit-one property from parity objectives to a richer class of objectives that subsume several canonical ω-regular objectives. Our proof uses a result from measure theory and certain strategy constructions, whereas the proof for the sub-class of parity objectives [6] followed from complementation arguments of quantitative μ-calculus formula. We first show an example that the positive limit-one property is not true for all objectives, even for simpler class of games.

Example 2. Consider the game shown in Fig 1, where at every state s, we have $Mv_1(s) = Mv_2(s) = \{1\}$ (i.e., the set of moves is singleton at all states). From all states the next state is s_0 and s_1 with equal probability. Consider the objective $\bigcirc(s_1)$ which specifies the next state is s_1; i.e., a play ω starting from state s is winning if the first state of the play is s and the second state (or the next state from s) in the play is s_1. Given the objective $\Phi = \bigcirc(s_1)$ for player 1, we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s_0) = \langle\langle 1\rangle\rangle_{val}(\Phi)(s_1) = 1/2$. Hence though the value is positive at s_0, there is no state with value 1 for player 1. ∎

Notation. In the setting of concurrent games the natural filtration sequence $(\mathcal{F}_n)$ for the stochastic process under any pair of strategies is defined as

$$\mathcal{F}_n = \sigma(\Theta_1, \Theta_2, \ldots, \Theta_n)$$

i.e., the σ-field generated by the random-variables $\Theta_1, \Theta_2, \ldots, \Theta_n$.

Lemma 1 (Lévy's 0-1 law). *Suppose $\mathcal{H}_n \uparrow \mathcal{H}_\infty$, i.e.,$\mathcal{H}_n$ is a sequence of increasing σ-fields and $\mathcal{H}_\infty = \sigma(\cup_n \mathcal{H}_n)$. For all events $\mathcal{A} \in \mathcal{H}_\infty$ we have*

$$\mathrm{E}(\mathbf{1}_{\mathcal{A}} \mid \mathcal{H}_n) = \Pr(\mathcal{A} \mid \mathcal{H}_n) \to \mathbf{1}_{\mathcal{A}} \text{ almost-surely, (i.e., with probability 1),}$$

where $\mathbf{1}_{\mathcal{A}}$ is the indicator function of event $\mathcal{A}$.

The proof of the lemma is available in Durrett (page 262—263) [8]. An immediate consequence of Lemma 1 in the setting of concurrent games is the following lemma.

Lemma 2 (0-1 law in concurrent games). *For all concurrent game structures G, for all events $\mathcal{A} \in \mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$, for all strategies $(\tau, \pi) \in \Gamma \times \Pi$, for all states $s \in S$, we have*

$$\Pr_s^{\tau,\pi}(\mathcal{A} \mid \mathcal{F}_n) \to \mathbf{1}_{\mathcal{A}} \text{ almost-surely.}$$

Intuitively, the lemma means that the probability $\Pr_s^{\tau,\pi}(\mathcal{A} \mid \mathcal{F}_n)$ converges almost-surely (i.e., with probability 1) to 0 or 1 (since indicator functions take values in the range $\{ 0, 1 \}$). Note that the tail σ-field $\mathcal{T}$ is a subset of $\mathcal{F}_\infty$, i.e., $\mathcal{T} \subseteq \mathcal{F}_\infty$, and hence the result of Lemma 2 holds for all $\mathcal{A} \in \mathcal{T}$.

Notation. Given strategies τ and π for player 1 and player 2, a tail objective $\overline{\Phi}$, and a state s, for $\beta > 0$, let

$$H_n^{1,\beta}(\tau, \pi, \overline{\Phi}) = \{ \langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots \rangle \mid \Pr_s^{\tau,\pi}(\overline{\Phi} \mid \langle s_1, s_2, \ldots, s_n \rangle) \geq 1 - \beta \},$$

denote the set of paths ω such that the probability of satisfying $\overline{\Phi}$ given the strategies τ and π, and the prefix of length n of ω is at least $1 - \beta$; and

$$H_n^{0,\beta}(\tau, \pi, \overline{\Phi}) = \{ \langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots \rangle \mid \Pr_s^{\tau,\pi}(\overline{\Phi} \mid \langle s_1, s_2, \ldots, s_n \rangle) \leq \beta \}.$$

denote the set of paths ω such that the probability of satisfying $\overline{\Phi}$ given the strategies τ and π, and the prefix of length n of ω is at most β.

Proposition 1. *For all concurrent game structures G, for all strategies τ and π for player 1 and player 2, respectively, for all tail objectives $\overline{\Phi}$, for all states $s \in S$, for all $\beta > 0$ and $\varepsilon > 0$, there exists n, such that $\Pr_s^{\tau,\pi}(H_n^{1,\beta}(\tau, \pi, \overline{\Phi}) \cup H_n^{0,\beta}(\tau, \pi, \overline{\Phi})) \geq 1 - \varepsilon$.*

Proof. Let us denote $f_n = \Pr_s^{\tau,\pi}(\overline{\Phi} \mid \mathcal{F}_n)$. It follows from Lemma 2 that $f_n \to \overline{\Phi}$ almost-surely as $n \to \infty$. Since almost-sure convergence implies convergence in probability [8], $f_n \to \overline{\Phi}$ in probability. Formally, we have

$$\forall \beta > 0.\ \Pr_s^{\tau,\pi}(|f_n - \overline{\Phi}| > \beta) \to 0 \qquad \text{as } n \to \infty.$$

Equivalently we have

$$\forall \beta > 0.\ \forall \varepsilon > 0.\ \exists n_0.\ \forall n \geq n_0.\ \Pr_s^{\tau,\pi}(|f_n - \overline{\Phi}| \leq \beta) \geq 1 - \varepsilon.$$

Thus we obtain that $\lim_{n\to\infty} \Pr_s^{\tau,\pi}(H_n^{1,\beta}(\tau, \pi, \overline{\Phi}) \cup H_n^{0,\beta}(\tau, \pi, \overline{\Phi})) = 1$; and hence the result follows. ∎

Theorem 1 (Positive limit-one property). *For all concurrent game structures G, for all tail objectives Φ, if there exists a state $s \in S$ such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0$, then there exists a state $s' \in S$ such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s') = 1$.*

Proof. Assume towards contradiction that there exists a state s such that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) > 0$, but for all states s' we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s') < 1$. Let $\alpha = 1 - \langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s)$. Since $0 < \langle\langle 1\rangle\rangle_{val}(\Phi)(s) < 1$, we have $0 < \alpha < 1$. Since $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s') = 1 - \langle\langle 1\rangle\rangle_{val}(\Phi)(s')$ and for all states s' we have $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) < 1$, it follows that $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s') > 0$, for all states s'. Fix η such that $0 < \eta = \min_{s' \in S} \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s')$. Also observe that since $\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = \alpha < 1$, we have $\eta < 1$. Let c be a constant such that $c > 0$, and $\alpha \cdot (1 + c) = \gamma < 1$ (such a constant exists as $\alpha < 1$). Also let $c_1 > 1$ be a constant such that $c_1 \cdot \gamma < 1$ (such a constant exists since $\gamma < 1$); hence we have $1 - c_1 \cdot \gamma > 0$ and $1 - \frac{1}{c_1} > 0$. Fix $\varepsilon > 0$ and $\beta > 0$ such that

$$0 < 2\varepsilon < \min\{ \frac{\eta}{4}, 2c \cdot \alpha, \frac{\eta}{4} \cdot (1 - c_1 \cdot \gamma) \}; \qquad \beta < \min\{ \varepsilon, \frac{1}{2}, 1 - \frac{1}{c_1} \}. \quad (1)$$

Fix ε-optimal strategies τ_ε for player 1 and π_ε for player 2. Let $H_n^{1,\beta} = H_n^{1,\beta}(\tau_\varepsilon, \pi_\varepsilon, \overline{\Phi})$ and $H_n^{0,\beta} = H_n^{0,\beta}(\tau_\varepsilon, \pi_\varepsilon, \overline{\Phi})$. Consider n such that $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cup H_n^{0,\beta}) \geq 1 - \frac{\varepsilon}{4}$ (such n exists by Proposition 1). Also observe that since $\beta < \frac{1}{2}$ we have $H_n^{1,\beta} \cap H_n^{0,\beta} = \emptyset$. Let

$$val = \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}).$$

We have

$$val \leq \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \leq val + \frac{\varepsilon}{4}. \quad (2)$$

The first inequality follows since $H_n^{1,\beta} \cap H_n^{0,\beta} = \emptyset$ and the second inequality follows since $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cup H_n^{0,\beta}) \geq 1 - \frac{\varepsilon}{4}$. Since τ_ε and π_ε are ε-optimal strategies we have $\alpha - \varepsilon \leq \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \leq \alpha + \varepsilon$. This along with (2) yield that

$$\alpha - \varepsilon - \frac{\varepsilon}{4} \leq val \leq \alpha + \varepsilon. \quad (3)$$

Observe that $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \geq 1 - \beta$ and $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \leq \beta$. Let $q = \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta})$. Since $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \geq 1 - \beta$; ignoring the term $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta})$ in val and from the second inequality of (3) we obtain that $(1 - \beta) \cdot q \leq \alpha + \varepsilon$. Since $\varepsilon < c \cdot \alpha$, $\beta < 1 - \frac{1}{c_1}$, and $\gamma = \alpha \cdot (1 + c)$ we have

$$q \leq \frac{\alpha + \varepsilon}{1 - \beta} < \frac{\alpha \cdot (1 + c)}{1 - (1 - \frac{1}{c_1})} = c_1 \cdot \gamma \quad (4)$$

We construct a strategy $\widehat{\pi}_\varepsilon$ as follows: the strategy $\widehat{\pi}_\varepsilon$ follows the strategy π_ε for the first $n - 1$-stages; if a history in $H_n^{1,\beta}$ is generated it follows π_ε, and otherwise it ignores the history and switches to an ε-optimal strategy. Formally, for a history $\langle s_1, s_2, \ldots, s_k\rangle$ we have

$$\widehat{\pi}_\varepsilon(\langle s_1,\ldots,s_k\rangle) = \begin{cases} \pi_\varepsilon(\langle s_1,\ldots,s_k\rangle) & \text{if } k<n; \\ & \text{or } \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1,s_2,\ldots,s_n\rangle) \geq 1-\beta; \\ \widetilde{\pi}_\varepsilon(\langle s_n,\ldots,s_k\rangle) & \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1,s_2,\ldots,s_n\rangle) < 1-\beta, \text{ and} \\ & k\geq n, \text{ where } \widetilde{\pi}_\varepsilon \text{ is an } \varepsilon\text{-optimal strategy} \end{cases}$$

Since $\widehat{\pi}_\varepsilon$ and π_ε coincides for $n-1$-stages we have $\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{1,\beta}) = \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta})$ and $\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{0,\beta}) = \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta})$. Moreover, since $\overline{\Phi}$ is a tail objective that is independent of the prefix of length n; $\eta \leq \min_{s'\in S}\langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s')$ and $\widetilde{\pi}_\varepsilon$ is an ε-optimal strategy, we have $\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \geq \eta-\varepsilon$. Also observe that

$$\begin{aligned}\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \geq (\eta-\varepsilon) &= \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) + (\eta-\varepsilon-\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta})) \\ &\geq \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) + (\eta-\varepsilon-\beta),\end{aligned}$$

since $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \leq \beta$. Hence we have the following inequality

$$\begin{aligned}
\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi}) &\geq \Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{1,\beta}) + \Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{0,\beta}) \\
&= \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(H_n^{0,\beta}) \\
&\geq \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}) \\
&\quad + (\eta-\varepsilon-\beta)\cdot\big(1-q-\frac{\varepsilon}{4}\big) \qquad \big(\text{as } \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}) \geq 1-q-\frac{\varepsilon}{4}\big) \\
&= val + (\eta-\varepsilon-\beta)\cdot(1-q-\frac{\varepsilon}{4}) \\
&\geq \alpha - \frac{5\varepsilon}{4} + (\eta-\varepsilon-\beta)\cdot(1-q-\frac{\varepsilon}{4}) \quad (\text{recall first inequality of (3)}) \\
&> \alpha - \frac{5\varepsilon}{4} + (\eta-2\varepsilon)\cdot(1-q-\frac{\varepsilon}{4}) \qquad (\text{as } \beta<\varepsilon \text{ by (1)}) \\
&> \alpha - \frac{5\varepsilon}{4} + \frac{\eta}{2}\cdot(1-q-\frac{\varepsilon}{4}) \qquad (\text{as } 2\varepsilon < \tfrac{\eta}{2} \text{ by (1)}) \\
&> \alpha - \frac{5\varepsilon}{4} + \frac{\eta}{2}\cdot(1-c_1\cdot\gamma) - \frac{\eta}{2}\cdot\frac{\varepsilon}{4} \qquad (\text{as } q < c_1\cdot\gamma \text{ by (4)}) \\
&> \alpha - \varepsilon - \frac{\varepsilon}{4} + 4\varepsilon - \frac{\varepsilon}{8} \qquad (\text{as } 2\varepsilon < \tfrac{\eta}{4}\cdot(1-c_1\cdot\gamma) \text{ by (1), and } \eta\leq 1) \\
&> \alpha+\varepsilon.
\end{aligned}$$

The first equality follows since for histories in $H_n^{1,\beta}$, the strategies π_ε and $\widehat{\pi}_\varepsilon$ coincide. Hence we have $\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi}) > \alpha+\varepsilon$ and $\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\Phi) < 1-\alpha-\varepsilon$. This is a contradiction to the fact that $\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = 1-\alpha$ and τ_ε is an ε-optimal strategy. The desired result follows. ∎

Notation. We use the following notation for the rest of the paper:

$$W_1^1 = \{\, s \mid \langle\langle 1\rangle\rangle_{val}(\Phi)(s) = 1 \,\}; \qquad W_2^1 = \{\, s \mid \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s) = 1 \,\}.$$

$$W_1^{>0} = \{ s \mid \langle\langle 1 \rangle\rangle_{val}(\Phi)(s) > 0 \}; \qquad W_2^{>0} = \{ s \mid \langle\langle 2 \rangle\rangle_{val}(\overline{\Phi})(s) > 0 \}.$$

By determinacy of concurrent games with tail objectives, we have $W_1^1 = S \setminus W_2^{>0}$ and $W_2^1 = S \setminus W_1^{>0}$. We have the following finer characterization of the sets.

Corollary 1. *For all concurrent game structures G, with tail objectives Φ for player 1, the following assertions hold:*

1. *(a) if $W_1^{>0} \neq \emptyset$, then $W_1^1 \neq \emptyset$; and (b) if $W_2^{>0} \neq \emptyset$, then $W_2^1 \neq \emptyset$.*
2. *(a) if $W_1^{>0} = S$, then $W_1^1 = S$; and (b) if $W_2^{>0} = S$, then $W_2^1 = S$.*

Proof. The first result is a direct consequence of Theorem 1. The second result is derived as follows: if $W_1^{>0} = S$, then by determinacy we have $W_2^1 = \emptyset$. If $W_2^1 = \emptyset$, it follows from part 1 that $W_2^{>0} = \emptyset$, and hence $W_1^1 = S$. The result of part 2 shows that if a player has positive optimum value at every state, then the optimum value is 1 at all states. ∎

4 Zero-sum Tail Games to Nonzero-sum Reachability Games

In this section we relate the values of zero-sum games with tail objectives with the Nash equilibrium values of nonzero-sum games with reachability objectives. The result shows that the values of a zero-sum game with complex objectives can be related to equilibrium values of a nonzero-sum game with simpler objectives. We also show that for MDPs the value function for a tail objective Φ can be computed by computing the maximal probability of reaching the set of states with value 1. As an immediate consequence of the above analysis, we obtain a polynomial time reduction of the quantitative analysis of MDPs with tail objectives to the qualitative analysis. We first prove a *limit-reachability* property of ε-optimal strategies: the property states that for tail objectives, if the players play ε-optimal strategies, for small $\varepsilon > 0$, then the game reaches $W_1^1 \cup W_2^1$ with high probability.

Theorem 2 (Limit-reachability). *For all concurrent game structures G, for all tail objectives Φ for player 1, for all $\varepsilon' > 0$, there exists $\varepsilon > 0$, such that for all states $s \in S$, for all ε-optimal strategies τ_ε and π_ε, we have*

$$\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\mathit{Reach}(W_1^1 \cup W_2^1)) \geq 1 - \varepsilon'.$$

Proof. By determinacy it follows that $W_1^1 \cup W_2^1 = S \setminus (W_1^{>0} \cup W_2^{>0})$. For a state $s \in W_1^1 \cup W_2^1$ the result holds trivially. Consider a state $s \in W_1^{>0} \cup W_2^{>0}$ and let $\alpha = \langle\langle 2 \rangle\rangle_{val}(\overline{\Phi})(s)$. Observe that $0 < \alpha < 1$. Let $\eta_1 = \min_{s \in W_2^{>0}} \langle\langle 1 \rangle\rangle_{val}(\Phi)(s)$ and $\eta_2 = \max_{s \in W_2^{>0}} \langle\langle 2 \rangle\rangle_{val}(\overline{\Phi})(s)$, and let $\eta = \min\{ \eta_1, 1 - \eta_2 \}$, and note that $0 < \eta < 1$. Given $\varepsilon' > 0$, fix ε such that $0 < 2\varepsilon < \min\{ \frac{\eta}{2}, \frac{\eta \cdot \varepsilon'}{12} \}$. Fix any ε-optimal strategies τ_ε and π_ε for player 1 and player 2, respectively. Fix β such that $0 < \beta < \varepsilon$ and $\beta < \frac{1}{2}$. Let $H_n^{1,\beta} = H_n^{1,\beta}(\tau_\varepsilon, \pi_\varepsilon, \overline{\Phi})$ and $H_n^{0,\beta} =$

$H_n^{0,\beta}(\tau_\varepsilon, \pi_\varepsilon, \overline{\Phi})$. Consider n such that $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cup H_n^{0,\beta}) = 1 - \frac{\varepsilon}{4}$ (such n exists by Proposition 1), and also as $\beta < \frac{1}{2}$, we have $H_n^{1,\beta} \cap H_n^{0,\beta} = \emptyset$. Let us denote by

$$\mathit{val} = \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{1,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid H_n^{0,\beta}) \cdot \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}).$$

Similar to inequality (2) of Theorem 1 we obtain that

$$\mathit{val} \le \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \le \mathit{val} + \frac{\varepsilon}{4}$$

Since τ_ε and π_ε are ε-optimal strategies, similar to inequality (3) of Theorem 1 we obtain that $\alpha - \varepsilon - \frac{\varepsilon}{4} \le \mathit{val} \le \alpha + \varepsilon$.

For $W \subseteq S$, let $\mathrm{Reach}^n(W) = \{ \langle s_1, s_2, s_3 \ldots \rangle \mid \exists k \le n.\ s_k \in W \}$ denote the set of paths that reaches W in n-steps. We use the following notations: $\overline{\mathrm{Reach}(W_1^1)} = \Omega \setminus \mathrm{Reach}^n(W_1^1)$, and $\overline{\mathrm{Reach}(W_2^1)} = \Omega \setminus \mathrm{Reach}^n(W_2^1)$. Consider a strategy $\widehat{\tau}_\varepsilon$ defined as follows: for histories in $H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)}$, $\widehat{\tau}_\varepsilon$ ignores the history after stage n and follows an ε-optimal strategy $\widetilde{\tau}_\varepsilon$; and for all other histories it follows τ_ε. Let $z_1 = \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)})$. Since $\eta_2 = \max_{s \in W_2^{>0}} \langle\langle 2 \rangle\rangle_{val}(\overline{\Phi})(s)$, and player 1 switches to an ε-optimal strategy for histories of length n in $H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)}$ and $\overline{\Phi}$ is a tail objective, it follows that for all $\omega = \langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots \rangle \in H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_2^1)}$, we have $\Pr_s^{\widehat{\tau}_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n \rangle) \le \eta_2 + \varepsilon$; where as $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n \rangle) \ge 1 - \beta$. Hence we have

$$\mathit{val}_2 = \Pr_s^{\widehat{\tau}_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) \le \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) - z_1 \cdot (1 - \beta - \eta_2 - \varepsilon) \le \mathit{val} + \frac{\varepsilon}{4} - z_1 \cdot (1 - \beta - \eta_2 - \varepsilon),$$

since with probability z_1 the decrease is at least by $1 - \beta - \eta_2 - \varepsilon$. Since π_ε is an ε-optimal strategy we have $\mathit{val}_2 \ge \alpha - \varepsilon$; and since $\mathit{val} \le \alpha + \varepsilon$, we have the following inequality

$$\begin{aligned} z_1 \cdot (1 - \eta_2 - \beta - \varepsilon) &\le 2\varepsilon + \frac{\varepsilon}{4} < 3\varepsilon \\ &\Rightarrow z_1 < \frac{3\varepsilon}{\eta - \beta - \varepsilon} \qquad (\text{since } \eta \le 1 - \eta_2) \\ &\Rightarrow z_1 < \frac{3\varepsilon}{\eta - 2\varepsilon} < \frac{6\varepsilon}{\eta} < \frac{\varepsilon'}{4} \qquad (\text{since } \beta < \varepsilon; \varepsilon < \frac{\eta}{4}; \varepsilon < \frac{\eta \cdot \varepsilon'}{24}) \end{aligned}$$

Consider a strategy $\widehat{\pi}_\varepsilon$ defined as follows: for histories in $H_n^{0,\beta} \cap \overline{\mathrm{Reach}(W_1^1)}$, $\widehat{\pi}_\varepsilon$ ignores the history after stage n and follows an ε-optimal strategy $\widetilde{\pi}_\varepsilon$; and for all other histories it follows π_ε. Let $z_2 = \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta} \cap \overline{\mathrm{Reach}(W_1^1)})$. Since $\eta_1 = \min_{s \in W_2^{>0}} \langle\langle 2 \rangle\rangle_{val}(\overline{\Phi})(s)$, and player 2 switches to an ε-optimal strategy for histories of length n in $H_n^{0,\beta} \cap \overline{\mathrm{Reach}(W_1^1)}$ and $\overline{\Phi}$ is a tail objective, it follows that for all $\omega = \langle s_1, s_2, \ldots, s_n, s_{n+1}, \ldots \rangle \in H_n^{1,\beta} \cap \overline{\mathrm{Reach}(W_1^1)}$, we have $\Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n \rangle) \ge \eta_1 - \varepsilon$; where as $\Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi} \mid \langle s_1, s_2 \ldots, s_n \rangle) \le \beta$. Hence we have

$$\mathit{val}_1 = \Pr_s^{\tau_\varepsilon,\widehat{\pi}_\varepsilon}(\overline{\Phi}) \ge \Pr_s^{\tau_\varepsilon,\pi_\varepsilon}(\overline{\Phi}) + z_2 \cdot (\eta_1 - \varepsilon - \beta) \ge \mathit{val} + z_2 \cdot (\eta_1 - \varepsilon - \beta),$$

since with probability z_2 the increase is at least by $\eta_1 - \varepsilon - \beta$. Since τ_ε is an ε-optimal strategy we have $val_1 \leq \alpha + \varepsilon$; and since $val \geq \alpha - \varepsilon + \frac{\varepsilon}{4}$, we have the following inequality

$$
\begin{aligned}
z_2 \cdot (\eta_1 - \beta - \varepsilon) &\leq 2\varepsilon + \frac{\varepsilon}{4} < 3\varepsilon \\
\Rightarrow z_2 &< \frac{3\varepsilon}{\eta - \beta - \varepsilon} \qquad (\text{since } \eta \leq \eta_1) \\
\Rightarrow z_2 &< \frac{\varepsilon'}{4} \qquad (\text{similar to the inequality for } z_1 < \frac{\varepsilon'}{4})
\end{aligned}
$$

Hence $z_1 + z_2 \leq \frac{\varepsilon'}{2}$; and then we have

$$
\begin{aligned}
\mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}(W_1^1 \cup W_2^1)) &\geq \mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1 \cup W_2^1) \cap (H_n^{1,\beta} \cup H_n^{0,\beta})) \\
&= \mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1 \cup W_2^1) \cap H_n^{1,\beta}) \\
&\quad + \mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1 \cup W_2^1) \cap H_n^{0,\beta}) \\
&\geq \mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_1^1) \cap H_n^{1,\beta}) \\
&\quad + \mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(\mathrm{Reach}^n(W_2^1) \cap H_n^{0,\beta}) \\
&\geq \mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{1,\beta}) + \mathrm{Pr}_s^{\tau_\varepsilon,\pi_\varepsilon}(H_n^{0,\beta}) - (z_1 + z_2) \\
&\geq 1 - \frac{\varepsilon}{4} + \frac{\varepsilon'}{2} \geq 1 - \varepsilon' \qquad (\text{since } \varepsilon \leq \varepsilon').
\end{aligned}
$$

The result follows. ∎

Theorem 2 proves the limit-reachability property for tail objectives, under ε-optimal strategies, for small ε. We present an example to show that Theorem 2 is not true for all objectives, or for tail objectives with arbitrary strategies.

Example 3. Observe that in the game shown in Example 2, the objective was not a tail objective and we had $W_1^1 \cup W_2^1 = \emptyset$. Hence Theorem 2 need not necessarily hold for all objectives. Also consider the game shown in Fig 2. In the game shown s_1 and s_2 are absorbing state. At s_0 the available moves for the players are as follows: $Mv_1(s_0) = \{ a \}$ and $Mv_2(s_0) = \{ 1, 2 \}$. The transition function is as follows: if player 2 plays move 2, then the next state is s_1 and s_2 with equal probability, and if player 2 plays move 1, then the next state is s_0. The objective of player 1 is $\Phi = \text{Büchi}(\{ s_0, s_1 \})$, i.e., to visit s_0 or s_1 infinitely often. We have $W_1^1 = \{ s_1 \}$ and $W_2^1 = \{ s_2 \}$. Given a strategy π that chooses move 1 always, the set $W_1^1 \cup W_2^1$ of states is reached with probability 0; however π is not an optimal or ε-optimal strategy for player 2 (for $\varepsilon < \frac{1}{2}$). This shows that Theorem 2 need not hold if ε-optimal strategies are not considered. In the game shown, for an optimal strategy for player 2 (e.g., a strategy to choose move 2) the play reaches $W_1^1 \cup W_2^1$ with probability 1. ∎

Lemma 3 is immediate from Theorem 2.

Lemma 3. *For all concurrent game structures G, for all tail objectives Φ for player 1 and $\overline{\Phi}$ for player 2, for all states $s \in S$, we have*

$$
\lim_{\varepsilon \to 0} \sup_{\tau \in \Gamma_\varepsilon(\Phi), \pi \in \Pi_\varepsilon(\overline{\Phi})} \mathrm{Pr}_s^{\tau,\pi}(\mathit{Reach}(W_1^1 \cup W_2^1)) = 1;
$$

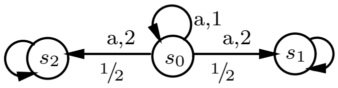

Fig. 2. A game with Büchi objective.

$$\lim_{\varepsilon\to 0} \sup_{\tau\in\Gamma_\varepsilon(\Phi),\pi\in\Pi_\varepsilon(\overline{\Phi})} \Pr_s^{\tau,\pi}(\mathit{Reach}(W_1^1)) = \langle\langle 1\rangle\rangle_{val}(\Phi)(s);$$

$$\lim_{\varepsilon\to 0} \sup_{\tau\in\Gamma_\varepsilon(\Phi),\pi\in\Pi_\varepsilon(\overline{\Phi})} \Pr_s^{\tau,\pi}(\mathit{Reach}(W_2^1)) = \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s).$$

Consider a non-zero sum reachability game G_R such that the states in $W_1^1 \cup W_2^1$ are transformed to absorbing states and the objectives of both players are reachability objectives: the objective for player 1 is $\text{Reach}(W_1^1)$ and the objective for player 2 is $\text{Reach}(W_2^1)$. Note that the game G_R is not zero-sum in the following sense: there are infinite paths ω such that $\omega \notin \text{Reach}(W_1^1)$ and $\omega \notin \text{Reach}(W_2^1)$ and each player gets a payoff 0 for the path ω. We define ε-Nash equilibrium of the game G_R and relate some special ε-Nash equilibrium of G_R with the values of G.

Definition 2 **(ε-Nash equilibrium in G_R).** *A strategy profile $(\tau^*, \pi^*) \in \Gamma\times\Pi$ is an ε-Nash equilibrium at state s if the following two conditions hold:*

$$\Pr_s^{\tau^*,\pi^*}(\mathit{Reach}(W_1^1)) \geq \sup_{\tau\in\Gamma} \Pr_s^{\tau,\pi^*}(\mathit{Reach}(W_1^1)) - \varepsilon$$

$$\Pr_s^{\tau^*,\pi^*}(\mathit{Reach}(W_2^1)) \geq \sup_{\pi\in\Pi} \Pr_s^{\tau^*,\pi}(\mathit{Reach}(W_2^1)) - \varepsilon \ \blacksquare$$

Theorem 3 **(Nash equilibrium of reachability game G_R).** *The following assertion holds for the game G_R.*

1. *For all $\varepsilon > 0$, there is an ε-Nash equilibrium $(\tau_\varepsilon^*, \pi_\varepsilon^*) \in \Gamma_\varepsilon(\Phi) \times \Pi_\varepsilon(\overline{\Phi})$ such that for all states s we have*

$$\lim_{\varepsilon\to 0} \Pr_s^{\tau_\varepsilon^*,\pi_\varepsilon^*}(\mathit{Reach}(W_1^1)) = \langle\langle 1\rangle\rangle_{val}(\Phi)(s)$$

$$\lim_{\varepsilon\to 0} \Pr_s^{\tau_\varepsilon^*,\pi_\varepsilon^*}(\mathit{Reach}(W_2^1)) = \langle\langle 2\rangle\rangle_{val}(\overline{\Phi})(s).$$

Proof. It follows from Lemma 3. ∎

Note that in case of MDPs the strategy for player 2 is trivial, i.e., player 2 has only one strategy. Hence in context of MDPs we drop the strategy π of player 2. A specialization of Theorem 3 in case of MDPs yields Theorem 4.

Theorem 4. *For all MDPs G_M, for all tail objectives Φ, we have*

$$\langle\langle 1\rangle\rangle_{val}(\Phi)(s) = \sup_{\tau\in\Gamma} \Pr_s^{\tau}(\mathit{Reach}(W_1^1)) = \langle\langle 1\rangle\rangle_{val}(\mathit{Reach}(W_1^1))(s)$$

Since the values in MDPs with reachability objectives can be computed in polynomial time (by linear-programming) [3, 9], our result presents a polynomial time reduction of quantitative analysis of tail objectives in MDPs to qualitative analysis. Our results (mainly, Theorem 1 and Theorem 2) can also be used to present simple construction of ε-optimal strategies for ω-regular objectives in concurrent games. These results will be presented in a fuller version of the paper.

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