On stability of switched linear hyperbolic conservation laws with reflecting boundaries

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Abstract. We consider stability of an infinite dimensional switching system, posed as a system of linear hyperbolic partial differential equations (PDEs) with reflecting boundaries, where the system parameters and the boundary conditions switch in time. Asymptotic stability of the solution for arbitrary switching is proved under commutativity of the advective velocity matrices and a joint spectral radius condition involving the boundary data.

Problem Formulation. Motivated by applications [2], we consider hybrid dynamics governed by linear hyperbolic PDE systems and a discrete set of modes:

$$\begin{matrix} \partial_t u(t,s) + A^j \partial_s u(t,s) = 0 \\ C_L^j u(t,a) = 0,\ C_R^j u(t,b) = 0 \end{matrix}, \quad j \in \mathcal{Q} \simeq \{1,\dots,N\}, \tag{1}$$

where the matrices $A^j \in \mathbb{R}^{n\times n}$ specify the advective velocities and the matrices $C_L^j \in \mathbb{R}^{(n-m_j)\times n}$ and $C_R^j \in \mathbb{R}^{m_j\times n}$ specify the boundary data for the unknown vector function $u(t,s) = (u^1(t,s),\dots,u^n(t,s))^\top$ on the space-time strip $\Omega([t_1,t_2]) := \{(t,s)\,|\,t\in[t_1,t_2], s\in[a,b]\}$. We assume that

(H)$_1$ the subsystems for fixed j are strictly hyperbolic, i. e. A^j has m_j negative and $(n-m_j)$ positive eigenvalues λ_i^j with n corresponding linearly independent left (right) eigenvectors l_i^j (r_i^j);

(H)$_2$ the switching signals in time $\mathcal{T} = \{t \geq 0\}$ are piecewise constant functions $\sigma(\cdot)\colon \mathcal{T} \to \mathcal{Q}$ with switching times τ_k $(k \in \mathbb{N})$ such that there are only finitely many switches $j \curvearrowright j'$ in each finite time interval of $\mathcal{T}$.

We consider the switched system in the space of piecewise continuously differentiable functions, denoted as $\mathcal{PC}^1 = \mathcal{PC}^1([a,b],\mathbb{R}^n)$, setting $\mathbf{u}(t) := u(t,\cdot)$, and say that for an initial condition $\bar{u}(\cdot) \in \mathcal{PC}^1$, the function $\mathbf{u}(\cdot)\colon \mathcal{T} \to \mathcal{PC}^1$ is a solution of the switched system (1) if

$$\mathbf{u}|_{t=0} = \bar{u} \quad \wedge \quad \begin{cases} \mathbf{u}|_{\tau_k+} := \mathbf{u}|_{\tau_k-} \text{ for all switching times } \tau_k \text{ of } \sigma(\cdot), \\ \mathbf{u}|_{(\tau_k+,\tau_{k+1}-)} \text{ solves (1) with } j = \sigma(t) = \text{const.} \end{cases} \tag{2}$$

Under the above assumptions, it is easy to see that the system is well-posed, if and only if it is well-posed in each mode, i. e., following [1]:

$$\operatorname{rank}\big[(C_L^j)^\top \big| l_1^j \big| \cdots \big| l_{m_j}^j\big] = \operatorname{rank}\big[(C_R^j)^\top \big| l_{m_j+1}^j \big| \cdots \big| l_n^j\big] = n \quad \text{for all } j \in \mathcal{Q}. \tag{3}$$

For a fixed $j \in \mathcal{Q}$, it is convenient to consider system (1) in an equivalent diagonal form. Using the transformation $S_j A^j S_j^{-1}$, where $S_j := \big[l_1^j \big| \cdots \big| l_n^j\big]^\top$, the system (1) can be written in *characteristic coordinates* $\xi := S_j u$

$$\begin{aligned} \partial_t \xi(t,s) + \operatorname{diag}(\Lambda_I^j, \Lambda_{II}^j)\partial_s \xi(t,s) = 0 \\ \xi_{II}(t,a) = G_L^j \xi_I(t,a), \quad \xi_I(t,b) = G_R^j \xi_I(t,b) \end{aligned} \tag{4}$$

where, $\xi_I = (\xi^1, \dots, \xi^m)^\top$, $\xi_{II} = (\xi^{m+1}, \dots, \xi^n)^\top$, $\Lambda_I^j = \operatorname{diag}(\lambda_1^j, \dots, \lambda_{m_j}^j)$, $\Lambda_{II}^j = \operatorname{diag}(\lambda_{m_j+1}^j, \dots, \lambda_n^j)$ and

$$\begin{aligned} G_R^j &= -\big(\big[c_{n-m_j+1}^j \big| \cdots \big| c_n^j\big]^\top \big[r_1^j \big| \cdots \big| r_{m_j}^j\big]\big)^{-1} \big[c_{n-m_j+1}^j \big| \cdots \big| c_n^j\big]^\top \big[r_{m_j+1}^j \big| \cdots \big| r_n^j\big] \\ G_L^j &= -\big(\big[c_1^j \big| \cdots \big| c_{n-m_j}^j\big]^\top \big[r_{m_j+1}^j \big| \cdots \big| r_n^j\big]\big)^{-1} \big[c_1^j \big| \cdots \big| c_{n-m_j}^j\big]^\top \big[r_1^j \big| \cdots \big| r_{m_j}^j\big]. \end{aligned} \tag{5}$$

Thus, the solution (2) of the switched system (1) can equivalently be written as $\mathbf{u}(\cdot) = S_{\sigma(\cdot)}^{-1} \boldsymbol{\xi}(\cdot)$, where $\boldsymbol{\xi}(\cdot)$ satisfies

$$\boldsymbol{\xi}|_{t=0} = S_{\sigma(0)}\bar{u} \quad \wedge \quad \begin{cases} \boldsymbol{\xi}|_{\tau_k+} = S_{\sigma(\tau_k+)} S_{\sigma(\tau_k-)}^{-1} \boldsymbol{\xi}|_{\tau_k-} \text{ for all } \tau_k, \\ \boldsymbol{\xi}|_{(\tau_k+, \tau_{k+1}-)} \text{ solves (4) with } j = \sigma(t) = \text{const.} \end{cases} \tag{6}$$

Note that if all the subsystems are simultaneously diagonalisable, i. e. $S_{j'} = S_j$ for all $j, j' \in \mathcal{Q}$, then (6) shows that the solution of system (1) is constant along its *characteristic paths* that change their slope at switching times.

Main Result. We consider stability of the above switching system, motivated by a simple PDE counterpart to the well known ODE observation [3] that asymptotic stability of all subsystems is *not* sufficient, even for all subsystems in diagonal form (4).

Example 1. $Q = \{1, 2\}$, $A^j = \operatorname{diag}(-1, +1)$, $[a, b] = [0, 1]$, $G_L^j = 1.5(j-1)$, $G_R^j = 1.5(2-j)$. For $\bar{u}(\cdot) \equiv 1$, the solution of the subsystems is 0 for all $t > 2$, but alternating $\sigma(\cdot)$ at $t = 0.5, 1.5, 2.5, \dots$ leads to $\lim_{t\to\infty} \|\mathbf{u}(t)\|_\infty = \infty$. □

Indeed, the non-diagonal system (1) can be shown to blow up under switching of the advective velocity matrices, although its boundaries are un-switched *and* are "dissipative" in the sense of [4]; i. e., the following spectral radius condition holds:

$$\|G(G_L, G_R)\|_{\min} := \inf_{\gamma = \operatorname{diag}\{\gamma_i\}, \gamma_i > 0 (i=1,\dots,n)} \left\| \gamma \begin{pmatrix} 0 & G_R \\ G_L & 0 \end{pmatrix} \gamma^{-1} \right\|_\infty < 1. \tag{7}$$

Moreover, it is easy to see that a switched system in diagonal form even satisfying (7) in each mode can blow up just by alternately changing m_j.

Our goal here is thus to impose sufficient conditions for the switched system to be asymptotically stable under arbitrary switching. For $\mathbf{u}(\cdot) \in \mathcal{PC}^1$, we use the norm $\|\mathbf{u}(\cdot)\| := \max_{i=1,\dots,n;\ s\in[a,b]} |\mathbf{u}^i(s)|$ and, w. l. o. g., we consider $\mathbf{u}(\cdot) \equiv 0$ as the only equilibrium state of the switched system. We say that the switched system is asymptotically stable under arbitrary switching, if for all $\varepsilon > 0$ sufficiently small, there exists a $\delta(\varepsilon) > 0$ such that if $\|\bar{u}(\cdot)\| \leq \delta$, then $\|\mathbf{u}(\cdot)\| \leq \varepsilon$ for all $t \geq 0$ and $\lim_{t\to\infty} \|\mathbf{u}(\cdot)\| = 0$, independently of the switching signal $\sigma(\cdot)$. Our main result is the following.

Theorem 1. *Consider a system* (1) *under hypotheses (H)$_{1,2}$ being well-posed in the sense of* (3) *and suppose that the following conditions hold for all $j, j' \in \mathcal{Q}$*

$$(a)\ m_j = m_{j'} \qquad (b)\ A^j A^{j'} = A^{j'} A^j \qquad (c)\ \|G(G_L^j, G_R^{j'})\|_{\min} < 1 \tag{8}$$

where G_L^j, G_R^j are given as in (5) *and $\|G(\cdot,\cdot)\|_{\min}$ is defined as in* (7)*. Then the system is asymptotically stable under arbitrary switching.*

Proof. Under condition $(8)_b$, the system (1) can be simultaneously diagonalized for all modes to (4) with $S_{j'} = S_j$ for all $j, j' \in \mathcal{Q}$ and we can consider its solution $\boldsymbol{\xi}(\cdot)$ along its characteristic paths, see (6). Then we follow arguments of Li [4] Lemma 2.1, concluding that condition $(8)_c$ implies

$$\begin{aligned}\theta &:= \max_{j,j'\in\mathcal{Q}} \{\||G_L^j||G_R^{j'}|\|_\infty, \||G_R^{j'}||G_L^j|\|_\infty\} \\ &= \max_{\substack{r=1,\dots,m \\ l=m+1,\dots,n \\ j,j'\in\mathcal{Q}}} \left\{ \sum_{p=1}^{m} \sum_{k=m+1}^{n} |g_{rk}^{R,j'}||g_{kp}^{L,j}|, \sum_{k=m+1}^{n} \sum_{p=1}^{m} |g_{lp}^{L,j}||g_{pk}^{R,j'}| \right\} < 1,\end{aligned} \tag{9}$$

where $G_L^j = (g_{pq}^{L,j})$ and $G_R^{j'} = (g_{pq}^{R,j'})$. It suffices to show that for any fixed $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$|\xi(t,s)| := \max_{i=1,\dots,n} |\xi^i(t,s)| \leq \varepsilon \tag{10}$$

for all $0 \leq t < \infty$, $a \leq s \leq b$. Let $T_{\min} := \left(\max_{i=1,\dots,n; j=1,\dots,N} |\lambda_i^j|\right)^{-1}$. By boundedness of $\bar{u}$ and thus $\bar{\xi} := S_{\sigma(0)}\bar{u}$, by continuity of the solution along the characteristic path and by linearity of the boundary conditions for fixed $j \in \mathcal{Q}$, there exists a $\delta(\varepsilon) \leq \varepsilon$ such that

$$|\xi(t,s)| \leq \alpha\varepsilon \quad \text{for all } (t,s) \in \Omega([0,T^\circ)) \tag{11}$$

for some $T^\circ > 0$ sufficiently small (i. e. smaller than $\tau_1 > 0$) and for some $\alpha \leq 1$ to be specified later. Thus, to show (10), it suffices to prove that for any fixed $T > 0$, if (10) holds on $\Omega([0,T])$, then it still holds on domain $\Omega([0, T + T_{\min}])$. So assume (10) holds on $\Omega([0,T])$ and fix some $(t^*, s^*) \in \Omega([T, T + T_{\min}])$. Due to $(8)_a$, let z_r denote the r-th characteristic path passing through (t^*, s^*) $(r = 1, \dots, m)$. Backwards in time, z_r either intersects $t = 0$ before hitting any boundary (case 1) or it intersects the line $s = b$ (case 2). See Figure 1 for an illustration with an example switching configuration. For case 1: Using (2), $\xi^r(t^*, s^*) = \xi^r(0, \tilde{s}_1)$ for some $a \leq \tilde{s}_1 \leq b$. So, $|\xi^r(t^*, s^*)| \leq \delta \leq \varepsilon$ by assumption. For case 2: Again by (2), $\xi^r(t^*, s^*) = \xi^r(t_r, b)$, where $0 \leq t_r \leq t^*$ is the time when the r-th characteristic path hits $s = b$. Thus,

$$|\xi^r(t^*, s^*)| = |\sum_{l=m+1}^{n} g_{rl}^{R,j} \xi^l(t_r, b)| \leq \sum_{l=m+1}^{n} |g_{rl}^{R,j}||\xi^l(t_r, b)| \tag{12}$$

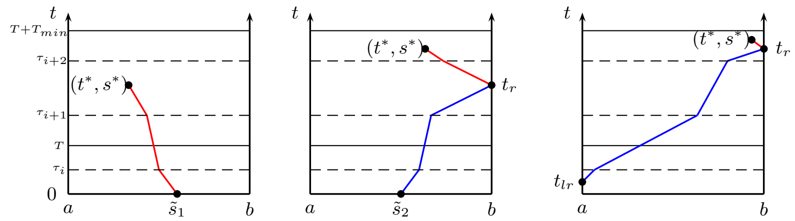

Fig. 1. (a) Case 1, (b) Case 2 (i), (c) Case 2 (ii)

with $j = \sigma(t_r)$. Now, let z_l denote the l-th characteristic path passing though (t_r, b) $(l = m+1, \ldots, n)$. Then, either z_l intersects the line $t = 0$ before hitting the line $s = a$ (case $2(i)$) or it hits $s = a$ (case $2(ii)$). For case $2(i)$, we have $|\xi^l(t_r, b)| = |\xi_l(0, \tilde{s}_2)| \leq \delta \leq \alpha\varepsilon$ for some $a \leq \tilde{s}_2 \leq b$ by assumption. Substituting this in (12), we get $|\xi^r(t^*, s^*)| \leq K_1^j \alpha\varepsilon$ with $K_1^j := \sum_{l=m+1}^n |g_{rl}^{R,j}|$. For case $2(ii)$, we have $|\xi^l(t_r, b)| = |\xi^l(t_{rl}, a)| = |\sum_{p=1}^m g_{lp}^{L,j'} \xi^p(t_{rl}, a)| \leq |\sum_{p=1}^m g_{lp}^{L,j'}||\xi^p(t_{rl}, a)|$ with $0 \leq t_{rl} \leq T_{min}$ is the time when the characteristic path z_l hits $s = a$ and $j' = \sigma(t_{rl})$. Substituting this in (12), we get by assumption and using (9) $|\xi^r(t^*, s^*)| \leq \sum_{l=m+1}^n \sum_{p=1}^m |g_{rl}^{R,j}||g_{lp}^{L,j'}||\xi^p(t_{rl}, a)| \leq \theta\varepsilon \leq \varepsilon$.
Similar estimates can be obtained for $\xi^l(t, s)$ $(l = m+1, \ldots, n)$ with constants $K_2^j := \sum_{p=1}^m |g_{lp}^{L,j}|$. Define $K := \max_{j \in \mathcal{Q}}\{K_1^j, K_2^j\}$. Choosing δ in (11) with $\alpha = \max\{1, \frac{1}{K}\}$ we conclude (10) for all $t \geq 0$ by induction. Essentially the same arguments applied to $\hat{\boldsymbol{\xi}}(t) := \exp(\beta t)\boldsymbol{\xi}(t)$, show that $\|\boldsymbol{\xi}(t)\| \leq \varepsilon \exp(-\beta t)$ for $\beta > 0$ sufficiently small, (see [4], page 185). The system is thus asymptotically stable. □

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