Controller Synthesis with Budget Constraints

Krishnendu Chatterjee [1], Rupak Majumdar [3], and Thomas A. Henzinger [1,2]

[1] EECS, UC Berkeley, [2] CCS, EPFL, [3] CS, UC Los Angeles

Abstract. We study the controller synthesis problem under budget constraints. In this problem, there is a cost associated with making an observation, and a controller can make only a limited number of observations in each round so that the total cost of the observations does not exceed a given fixed budget. The controller must ensure some ω-regular requirement subject to the budget constraint. Budget constraints arise in designing and implementing controllers for resource-constrained embedded systems, where a controller may not have enough power, time, or bandwidth to obtain data from all sensors in each round. They lead to games of imperfect information, where the unknown information is not fixed a priori, but can vary from round to round, based on the choices made by the controller how to allocate its budget.

We show that the budget-constrained synthesis problem for ω-regular objectives is complete for exponential time. In addition to studying synthesis under a fixed budget constraint, we study the budget optimization problem, where given a plant, an objective, and observation costs, we have to find a controller that achieves the objective with minimal average accumulated cost (or minimal peak cost). We show that this problem is reducible to a game of imperfect information where the winning objective is a conjunction of an ω-regular condition and a long-run average condition (or a least max-cost condition), and this again leads to an exponential-time algorithm.

Finally, we extend our results to games over infinite state spaces, and show that the budget-constrained synthesis problem is decidable for infinite state games with stable quotients of finite index. Consequently, the discrete time budget-constrained synthesis problem is decidable for rectangular hybrid automata.

1 Introduction

The *controller synthesis* problem asks, given a model for a plant, to construct a controller that observes the states of the plant and provides inputs to the plant such that the parallel composition of the plant and the controller is guaranteed to satisfy a given specification, provided, e.g., as an ω-regular set [4, 1, 14, 13]. Controller synthesis reduces to solving two-player games on graphs between a controller and the plant [1, 13, 5], where a winning strategy of the controller player for the specification gives a controller.

In constructing the controller, the usual assumption is that the controller can observe the system state completely. This assumption, called *perfect information*, may not hold in many settings of practical interest. For example, an embedded controller may only observe signals up to a finite precision, and a discrete control process may only observe the global state of other processes, not their private variables. Under such observability restrictions, a more relevant model is a game of *imperfect information*, where the controller only observes a part of the state space, and must construct a winning strategy based only on the observed state.

Games with imperfect information have been studied extensively [15, 13, 10, 11, 2]. Usually, the solution to a game of imperfect information proceeds with a subset construction that reduces the imperfect-information game to a game with perfect information (although on an exponentially larger state space). However, so far, most algorithms make the assumption of *fixed* partial information. Roughly, it is assumed that of n state bits, the controller can only observe the first $k < n$ bits, and must come up with a strategy that makes its decisions based on this limited observation. In the context of embedded control systems, especially in low-power settings such as embedded sensor and actuator networks [16], there is often a different kind of partial information. Instead of a fixed set of bits that are visible to the controller in every round of interaction, the partial information can be due to a cost in sensing each bit, and global constraints on the budget available to the controller. For example, in an embedded control system, the controller is free to sense any signal from the system, however, the act of sensing carries a cost (e.g., cost incurred by the energy consumed to sense, or time taken to run the sensing task, or bandwidth required to transmit the sensed value). Thus, in each round, the controller has to make a choice in allocating its resources (energy, time, or bandwidth) to sensing the most crucial data. Moreover, the controller is allowed to select which bits to sense in each round, so the set of bits sensed in one round may be different from the set sensed in the next.

We introduce and study a model of controller synthesis under budget constraints to study imperfect information of this kind. Our model adds a notion of cost associated with controller moves, and the winning conditions constrain possible controls by imposing budgets on the moves either in each round (modeling, e.g., upper bounds on available resources) or in the long run (modeling, e.g., the desire to minimize average cost, or maximize lifetime). In the first model, in each round, the controller may choose to sense a set I of state signals, as long as the total cost of sensing all the signals in I is bounded by B. Practically, the budget represents, e.g., bounds on available energy or bandwidth limitations of the system. Given a two player game with a cost for every state signal, a budget constraint B, and an ω-regular control objective, we construct a B-*restricted* control strategy that satisfies the control objective while always using at most B cost units at any round, if possible. In the second model, we construct a B-*long-run* control strategy that satisfies the control objective while maintaining the long-run average cost of sensing below B. Practically, this represents, e.g., control subject to available battery power. With embedded resource-scarce control problems becoming more and more common, our model presents a realistic generalization of classically studied supervisory control problems.

Dually, we study the *budget optimization problem*, where given the sensing costs for each state signal, we want to find out the minimum budget with which a controller can achieve its goals. Here, we study two different optimization criteria: the first aims to minimize the maximum sensing cost at any single round, the second aims to minimize the long-run average cost of the controller. Optimizations of the first type may be required to find out minimal power or bandwidth requirements for the system: the battery must be able to provide at least this power in order for the controller to effectively satisfy the control objective. Optimizations of the second type are required to maximize the lifetime of the controller.

Technically, there are two steps in our algorithms. For the budget constrained synthesis problem, we construct, from the budget-constrained game, a game of perfect information by a subset construction such that the controller has a winning strategy in the game of perfect information iff it has a winning strategy in the original game. For the budget optimization problem, we perform a similar subset construction, however, the winning objectives on the transformed games are a combination of ω-regular objectives (from the original game) as well as a *quantitative* requirement to reduce either the maximum cost along the path (corresponding to the first optimization criterion) or the long-run average cost along the path (corresponding to the second optimization criterion). From our reduction and solutions of games of perfect information we obtain that both the budget synthesis and the optimization problem are EXPTIME-complete for ω-regular objectives specified as parity conditions (a canonical form to express ω-regular objectives).

We develop the theory both for finite-state, discrete control problems, as well as for discrete time control for rectangular hybrid automata. In the latter, infinite state case, we show that the control problem can be solved by reducing the system to its stable (bisimulation) quotient. Using known results about stable partitions of rectangular automata [6], it follows that the budget constrained synthesis problem is decidable for rectangular automata, and indeed, for any infinite state control problem with a stable quotient of finite index.

2 Definitions

A *game structure* (*of imperfect information*) is a tuple $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$, where L is a finite set of states, $l_0 \in L$ is the initial state, Σ is a finite alphabet, $\Delta \subseteq L \times \Sigma \times L$ is a set of labeled transitions, $\mathcal{O}$ is a finite set of observations, and $\gamma : \mathcal{O} \to 2^L \backslash \emptyset$ maps each observation to the set of states that it represents. We require the following two properties on G: (i) for all $\ell \in L$ and all $\sigma \in \Sigma$, there exists $\ell' \in L$ such that $(\ell, \sigma, \ell') \in \Delta$; and ($ii$) the set $\{\gamma(o) \mid o \in \mathcal{O}\}$ partitions L. We say that G is a game structure of *perfect information* if $\mathcal{O} = L$ and $\gamma(\ell) = \{\ell\}$ for all $\ell \in L$. We omit $(\mathcal{O}, \gamma)$ in the description of games of perfect information. For $\sigma \in \Sigma$ and $s \subseteq L$, let $\mathsf{Post}^G_\sigma(s) = \{\ell' \in L \mid \exists \ell \in s : (\ell, \sigma, \ell') \in \Delta\}$.

In a game structure, in each turn, Player 1 chooses a letter in Σ, and Player 2 resolves nondeterminism by choosing the successor state. A *play* in G is an infinite sequence $\pi = \ell_0 \sigma_0 \ell_1 \dots \sigma_{n-1} \ell_n \sigma_n \dots$ such that (i) $\ell_0 = l_0$, and (ii) for all $i \geq 0$, we have $(\ell_i, \sigma_i, \ell_{i+1}) \in \Delta$. The *prefix up to* ℓ_n of the play π is denoted by $\pi(n)$; its *length* is $|\pi(n)| = n+1$; and its *last element* is $\mathsf{Last}(\pi(n)) = \ell_n$. The *observation sequence* of π is the unique infinite sequence $\gamma^{-1}(\pi) = o_0\sigma_0 o_1 \dots \sigma_{n-1} o_n \sigma_n \dots$ such that for all $i \geq 0$, we have $\ell_i \in \gamma(o_i)$. Similarly, the *observation sequence* of $\pi(n)$ is the prefix up to o_n of $\gamma^{-1}(\pi)$. The set of infinite plays in G is denoted $\mathsf{Plays}(G)$, and the set of corresponding finite prefixes is denoted $\mathsf{Prefs}(G)$. A state $\ell \in L$ is *reachable* in G if there exists a prefix $\rho \in \mathsf{Prefs}(G)$ such that $\mathsf{Last}(\rho) = \ell$. The *knowledge* associated with a finite observation sequence $\tau = o_0\sigma_0 o_1\sigma_1 \dots \sigma_{n-1} o_n$ is the set $\mathsf{K}(\tau)$ of states in which a play can be after this sequence of observations, that is, $\mathsf{K}(\tau) = \{\mathsf{Last}(\rho) \mid \rho \in \mathsf{Prefs}(G) \text{ and } \gamma^{-1}(\rho) = \tau\}$.

Lemma 1. *Let $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$ be a game structure. For $\sigma \in \Sigma$, $\ell \in L$, and $\rho, \rho' \in \mathsf{Prefs}(G)$ with $\rho' = \rho \cdot \sigma \cdot \ell$, let $o_\ell \in \mathcal{O}$ be the unique observation such that $\ell \in \gamma(o_\ell)$. Then $\mathsf{K}(\gamma^{-1}(\rho')) = \mathsf{Post}^G_\sigma(\mathsf{K}(\gamma^{-1}(\rho))) \cap \gamma(o_\ell)$.*

Strategies. A *strategy* in G for Player 1 is a function $\alpha : \mathsf{Prefs}(G) \to \Sigma$. A strategy α for Player 1 is *observation-based* if for all prefixes $\rho, \rho' \in \mathsf{Prefs}(G)$, if $\gamma^{-1}(\rho) = \gamma^{-1}(\rho')$, then $\alpha(\rho) = \alpha(\rho')$. In games of imperfect information we are interested in the existence of observation-based strategies for Player 1. A *strategy* in G for Player 2 is a function $\beta : \mathsf{Prefs}(G) \times \Sigma \to L$ such that for all $\rho \in \mathsf{Prefs}(G)$ and all $\sigma \in \Sigma$, we have $(\mathsf{Last}(\rho), \sigma, \beta(\rho, \sigma)) \in \Delta$. We denote by $\mathcal{A}_G$, $\mathcal{A}^O_G$, and $\mathcal{B}_G$ the set of all Player-1 strategies, the set of all observation-based Player-1 strategies, and the set of all Player-2 strategies in G, respectively.

The *outcome* of two strategies α (for Player 1) and β (for Player 2) in G is the play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots \in \mathsf{Plays}(G)$ such that for all $i \geq 0$, we have $\sigma_i = \alpha(\pi(i))$ and $\ell_{i+1} = \beta(\pi(i), \sigma_i)$. This play is denoted $\mathsf{outcome}(G, \alpha, \beta)$. The *outcome* of a strategy α for Player 1 in G is the set $\mathsf{Outcome}_1(G, \alpha)$ of plays π such that there exists a strategy β for Player 2 with $\pi = \mathsf{outcome}(G, \alpha, \beta)$. The outcome sets for Player 2 are defined symmetrically.

Qualitative objectives. A *qualitative objective* for G is a set ϕ of infinite sequences of observations and input letters, that is, $\phi \subseteq (\mathcal{O} \times \Sigma)^\omega$. A play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots \in \mathsf{Plays}(G)$ *satisfies* the objective ϕ, denoted $\pi \models \phi$, if $\gamma^{-1}(\pi) \in \phi$. We assume objectives are Borel measurable, that is, a qualitative objective is a Borel set in the Cantor topology on $(\mathcal{O} \times \Sigma)^\omega$ [9]. Observe that by definition, for all objectives ϕ, if $\pi \models \phi$ and $\gamma^{-1}(\pi) = \gamma^{-1}(\pi')$, then $\pi' \models \phi$.

We specifically consider *parity objectives* [5, 17]. Parity objectives are a canonical form to express all ω-regular objectives [17] and lie in the intersection $\Sigma_3 \cap \Pi_3$ of the third levels of the Borel hierarchy. For a play $\pi = \ell_0\sigma_0\ell_1 \dots$, we write $\mathsf{Inf}(\pi)$ for the set of observations that appear infinitely often in $\gamma^{-1}(\pi)$, that is, $\mathsf{Inf}(\pi) = \{o \in \mathcal{O} \mid \ell_i \in \gamma(o) \text{ for infinitely many } i\text{'s}\}$. For $d \in \mathbb{N}$, let $p : \mathcal{O} \to \{\, 0, 1, \dots, d \,\}$ be a *priority function*, which maps each observation to a nonnegative integer priority. The *parity* objective $\mathsf{Parity}(p)$ requires that the minimum priority that appears infinitely often be even. Formally, $\mathsf{Parity}(p) = \{\, \pi \mid \min\{\, p(o) \mid o \in \mathsf{Inf}(\pi) \,\} \text{ is even} \,\}$.

Quantitative objectives. In addition to parity (ω-regular) objectives, our algorithms will require solving games with quantitative objectives. A *quantitative* *objective* for G is a Borel measurable function f on infinite sequences of observations and input letters to reals, that is, $f : (\mathcal{O} \times \Sigma)^\omega \to \mathbb{R} \cup \{ \infty, -\infty \}$. We specifically consider mean-payoff, mean-payoff parity and min-parity objectives. Let $r : \Sigma \to \mathbb{R}$ be a reward-function that maps every input letter σ to a real-valued reward $r(\sigma)$, and let $p : \mathcal{O} \to \{ 0, 1, \dots, d \}$ be a priority function. We define the mean-payoff, mean-payoff parity and min-parity objectives as follows.

1. *Mean-payoff objectives.* For a play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots$ the mean-payoff objective is the long-run average of the rewards of the input letters [19]. Formally, for a reward function $r : \Sigma \to \mathbb{R}$, the mean-payoff objective is a function $\mathsf{M}(r)$ from plays to reals that maps the play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots$ to $\mathsf{M}(r)(\pi) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} r(\sigma_i)$.
2. *Mean-payoff parity objectives.* For a play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots$ the mean-payoff parity objective is the long-run average of the rewards of the input letters if the parity objective is satisfied and $-\infty$ otherwise. Formally, for a reward function $r : \Sigma \to \mathbb{R}$ and a priority function p, the mean-payoff parity objective is a function $\mathsf{MP}(r, p)$ defined on plays as follows: for a play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots$ we have $\mathsf{MP}(p, r)(\pi) = \mathsf{M}(\pi)$ if $\pi \in \mathsf{Parity}(p)$, and $\mathsf{MP}(p, r)(\pi) = -\infty$ otherwise.
3. *Min-parity objectives.* For a play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots$ the min-parity objective is the minimum of the rewards of the input letters if the parity objective is satisfied and $-\infty$ otherwise. Formally, for a reward function $r : \Sigma \to \mathbb{R}$ and a priority function p, the min-parity objective is a function $\mathsf{MinP}(r, p)$ defined on plays as follows: for a play $\pi = \ell_0\sigma_0\ell_1 \dots \sigma_{n-1}\ell_n\sigma_n \dots$ we have $\mathsf{MinP}(p, r)(\pi) = \min\{ r(\sigma_i) \mid i \geq 0 \}$ if $\pi \in \mathsf{Parity}(p)$, and $\mathsf{MinP}(p, r)(\pi) = -\infty$ otherwise.

Sure winning and optimal winning. A strategy λ_i for Player i in G is *sure winning* for a qualitative objective ϕ if for all $\pi \in \mathsf{Outcome}_i(G, \lambda_i)$, we have $\pi \models \phi$. A strategy λ_i for Player i in G is *optimal* for a quantitative objective f if for all strategies λ for Player i we have $\inf_{\pi\in\mathsf{Outcome}_i(G,\lambda_i)} f(\pi) \geq \inf_{\pi\in\mathsf{Outcome}_i(G,\lambda)} f(\pi)$. The following theorem from Martin [12] states that perfect-information games with (qualitative or quantitative) Borel objectives are *determined*: from each state, either Player 1 or Player 2 wins (for qualitative objectives), or a value can be defined (for quantitative objectives).

Theorem 1 (Determinacy). [12] *(1) For all perfect-information game structures G and all qualitative Borel objectives ϕ, either there exists a sure-winning strategy for Player 1 for the objective ϕ, or there exists a sure-winning strategy for Player 2 for the complementary objective* $\mathsf{Plays}(G) \setminus \phi$. *(2) For all perfect-information game structures G and all quantitative Borel objectives f, we have* $\sup_{\alpha\in\mathcal{A}} \inf_{\pi\in\mathsf{Outcome}(G,\alpha)} f(\pi) = \inf_{\beta\in\mathcal{B}} \sup_{\pi\in\mathsf{Outcome}(G,\beta)} f(\pi)$.

3 Imperfect-information to Perfect-information Games

First, we use the results of [2] to show that a game structure G of imperfect information can be encoded by a game structure G^{K} of perfect information such that for every qualitative Borel objective ϕ, there is an observation-based sure-winning strategy for Player 1 in G for ϕ if and only if there is a sure-winning strategy for Player 1 in G^{K} for ϕ. We then show that the same construction works for quantitative Borel objectives. We obtain G^{K} using a subset construction. Each state in G^{K} is a set of states of G representing the knowledge of Player 1. In the worst case, the size of G^{K} is exponentially larger than the size of G.

Given a game structure of imperfect information $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$, we define the *knowledge-based subset construction* of G as the following game structure of perfect information: $G^{\mathsf{K}} = \langle \mathcal{L}, \{l_0\}, \Sigma, \Delta^{\mathsf{K}} \rangle$, where $\mathcal{L} = 2^L \backslash \{\emptyset\}$, and $(s_1, \sigma, s_2) \in \Delta^{\mathsf{K}}$ iff there exists an observation $o \in \mathcal{O}$ such that $s_2 = \mathsf{Post}_\sigma^G(s_1) \cap \gamma(o)$ and $s_2 \neq \emptyset$. Notice that for all $s \in \mathcal{L}$ and all $\sigma \in \Sigma$, there exists a set $s' \in \mathcal{L}$ such that $(s, \sigma, s') \in \Delta^{\mathsf{K}}$. Given a game structure of imperfect information G we refer to the game structure G^{K} as $\mathsf{Pft}(G)$.

Lemma 2 ([2]). *For all sets $s \in \mathcal{L}$ that are reachable in G^{K}, and all observations $o \in \mathcal{O}$, either $s \subseteq \gamma(o)$ or $s \cap \gamma(o) = \emptyset$.*

By an abuse of notation, we define the *observation sequence* of a play $\pi = s_0 \sigma_0 s_1 \dots \sigma_{n-1} s_n \sigma_n \dots \in \mathsf{Plays}(G^{\mathsf{K}})$ as the infinite sequence $\gamma^{-1}(\pi) = o_0 \sigma_0 o_1 \dots \sigma_{n-1} o_n \sigma_n \dots$ of observations such that for all $i \geq 0$, we have $s_i \subseteq \gamma(o_i)$. Since the observations partition the states, and by Lemma 2, this sequence is unique. The play π *satisfies* an objective $\phi \subseteq (\mathcal{O} \times \Sigma)^\omega$ if $\gamma^{-1}(\pi) \in \phi$. As above, we say that a play $\pi = s_0 \sigma_0 s_1 \dots \sigma_{n-1} s_n \sigma_n \dots \in \mathsf{Plays}(G^{\mathsf{K}})$ *satisfies* an objective ϕ iff the sequence of observations $o_0 o_1 \dots o_n \dots$ such that for all $i \geq 0$, $\ell_i \in \gamma(o_i)$ belongs to ϕ. The following lemma follows from the results of [2].

Lemma 3 ([2]). *If Player* 1 *has a sure-winning strategy in G^{K} for an objective ϕ, then Player 1 has an observation-based sure-winning strategy in G for ϕ. If Player* 1 *does not have a deterministic sure-winning strategy in G^{K} for a Borel objective ϕ, then Player* 1 *does not have an observation-based sure-winning strategy in G for ϕ.*

Together with Theorem 1, Lemma 3 implies the first part of the following theorem, also used in [2]. The second part of the theorem generalizes the result to quantitative Borel objectives.

Theorem 2 (Sure-winning reduction). *Let G be a game structure, and $G^{\mathsf{K}} = \mathsf{Pft}(G)$. The following assertions hold. (1) Player* 1 *has an observation-based sure-winning strategy in G for a qualitative Borel objective ϕ if and only if Player* 1 *has a sure-winning strategy in G^{K} for ϕ* [2]. *(2) $\sup_{\alpha \in \mathcal{A}_G^O} \inf_{\pi \in \mathsf{Outcome}(G,\alpha)} f(\pi) = \sup_{\alpha \in \mathcal{A}_{G^{\mathsf{K}}}} \inf_{\pi \in \mathsf{Outcome}(G,\alpha)} f(\pi)$.*

For the second part, let $v = \sup_{\alpha \in \mathcal{A}_{G^{\mathsf{K}}}} \inf_{\pi \in \mathsf{Outcome}(G,\alpha)} f(\pi)$. Given $\epsilon > 0$, consider the qualitative objective $\phi = \{\pi \mid f(\pi) \geq v - \epsilon\}$. By the first part of the theorem, there is a sure-winning strategy in G^{K} iff there is an observation-based sure-winning strategy in G for the qualitative objective ϕ. Since ϵ is arbitrary, the result follows. It follows from Theorem 2 that to solve a game structure G of imperfect information it suffices to construct the game structure G^{K} of perfect information and solve the corresponding objective on G^{K}.

4 Games with Variables

We now consider game structures whose states are determined by valuations to a set of state variables, and formulate several games of imperfect information by restricting the variables that can be observed.

Games with Variables. A *game with variables* consists of: (1) a finite set $X = \{ x_1, x_2, \ldots, x_n \}$ of n boolean variables; a *valuation* v is a truth value assignment to all the variables, and we write V to denote the set of all valuations; (2) a finite set Γ of input letters; and (3) a non-deterministic transition function $\delta : V \times \Gamma \to 2^V \setminus \emptyset$ given a current valuation and an input letter gives the non-empty possible set of next valuations. We specify the games with variables as a tuple $G = (V, \Gamma, \delta)$. We introduce some notation. Given a natural number n we denote by $[n]$ the set $\{ 1, 2, \ldots, n \}$. For $I \subseteq [n]$ and $v \in V$, we denote by $v \upharpoonright I$ the restriction of the valuation on the set I of variables. Similarly, for $I \subseteq [n]$ we denote by $V \upharpoonright I$ the restriction of the set of valuations on the set I of variables. In games with variables we have two players: the controller and the system. The controller chooses the input letter and the system resolves the non-determinism in the transition function. We will consider several ways to restrict the knowledge of the controller by limiting what variables it can observe.

Games with Fixed-partial-information. To begin with, we consider games with variables where the information of the controller is restricted to a fixed set of size $k \leq n$ of variables. Without loss of generality, we consider the case when the controller can only observe the variables $x_1, x_2, \ldots, x_k$. Such games with variables have *fixed partial-information.*

Reduction. Let $G = (V, \Gamma, \delta)$ be a game with variables. A strategy for the controller in G is $[k]$-restricted if the strategy only observes the variables $\{ x_1, x_2, \ldots, x_k \}$. We present a reduction of games with variables with fixed-partial-information to the class of imperfect information games of Section 2. The reduction to a game with imperfect information $\widehat{G}_{\upharpoonright[k]} = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$ is as follows: (1) the set of states $L = V$, the set of valuations; (2) the input letters $\Sigma = \Gamma$; (3) the set of observations is the set of restrictions of the valuations to $\{x_1, \ldots, x_k\}$: $\mathcal{O} = V \upharpoonright \{1, 2, \ldots, k\}$; (4) $\gamma(o) = \{l \in L \mid l \upharpoonright \{1, 2, \ldots, k\} = o\}$; and (5) $(l, \sigma, l') \in \Delta$ iff $l' \in \delta(l, \sigma)$.

Theorem 3. *Let $G = (V, \Gamma, \delta)$ be a game with variables, and $p : V \to \{ 0, 1, \ldots, d \}$ be a priority function on V. Let $\widehat{p} : 2^V \setminus \emptyset \to \{ 0, 1, \ldots, d \}$ be a priority function derived from p as follows: for a non-empty set $Y \subseteq V$ we have $\widehat{p}(Y) = \max\{ p(v) \mid v \in Y \}$ if $p(v)$ is even for all $v \in Y$; otherwise $\widehat{p}(Y) = \min\{ p(v) \mid v \in Y, p(v) \text{ is odd} \}$. There is a $[k]$-restricted strategy for the controller in G to satisfy the objective $\mathsf{Parity}(p)$ iff there is a strategy in $\widehat{G}^{\mathsf{K}} = \mathsf{Pft}(\widehat{G}_{\upharpoonright[k]})$ to satisfy $\mathsf{Parity}(\widehat{p})$.*

Example 1. Consider a plant with variables $\{ x_1, x_2, \ldots, x_n \}$ such that the set $\{ x_1, x_2, \ldots, x_k \}$, for $k \leq n$, is the set of *public* variables that can be accessed by the controller and all the other variables are *private*, i.e., cannot be accessed by the controller. Games with fixed-partial-information provide an appropriate framework to model the interaction of the controller and the plant.

Games with Budget Constraints. We now consider games with variables where the set of variables that the controller can observe is not fixed, but there is a hard constraint on the amount of information that the controller can observe at any round. We will again present a reduction to games of imperfect information, but the reduction is more involved than the case of fixed partial-information.

Games with hard constraints. Let $G = (V, \Gamma, \delta)$ be a game with variables, and let c be a cost function that assigns a cost $c(i) > 0$ to variable x_i, i.e., there is a cost $c(i)$ for the controller to know the value of the variable x_i. The controller can choose to know the truth values of a subset of variables and then choose the input letter. For a budget $B > 0$, a strategy of the controller is B-restricted if at each round the controller can ask for the truth values of a subset I of variables such that the sum of the costs of the variables does not exceed B, that is, $\sum_{i \in I} c(i) \leq B$. Observe that the choice of the set of variables is not fixed and can vary in each round. For a set $I \subseteq [n]$ we denote by $c(I) = \sum_{i \in I} c(i)$ the sum of the cost of the variables in I. We present a reduction of games with variables and a budget B to imperfect-information games of Section 2. The reduction to an imperfect-information game $\overline{G}_{\restriction B} = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$ is as follows:
(1) *States.* The set of states is $L = V \times \{ I \subseteq [n] \mid c(I) \leq B \} \cup \overline{V}$, where $\overline{V}$ is a copy of the valuations. That is the set of states consists of a pair of valuation and a subset I such that $c(I)$ does not exceed the budget B, and copy $\overline{V}$ of V.
(2) *Input letters.* The set of input letters is $\Sigma = \Gamma \cup \{ I \subseteq [n] \mid c(I) \leq B \}$. The set of input letters is the set of input letters Γ of the game G and also consists of subsets $I \subseteq [n]$ such that $c(I) \leq B$.
(3) *Observations.* The set of observations is $\mathcal{O} = \{ (o, I) \mid I \subseteq [n], c(I) \leq B, o \in V \restriction I \} \cup \{ \overline{o} \}$. The set of observations consists of pairs (o, I) where $I \subseteq [n]$ and o is a valuation restricted to I, and there is a special observation $\overline{o}$.
(4) *Observation map.* The observation map is as follows: $\gamma(o, I) = \{ (l, I) \in L \mid l \restriction I = o \}$ and $\gamma(\overline{o}) = \overline{V}$. Observe that each state in $\overline{V}$ has the observation $\overline{o}$.
(5) *Transition function.* The transition function is as follows: for $\sigma \in \Gamma$, $((l, I), \sigma, \widehat{l'}) \in \Delta$ iff $l' \in \delta(l, \sigma)$ and $(\overline{l}, \sigma, (l, \sigma)) \in \Delta$ for $\sigma = I \subseteq [n]$ such that $c(I) \leq B$. For a state (l, I) if an input letter σ from Γ is chosen, then a next state $\overline{l}'$ is possible iff $l' \in \delta(l, \sigma)$. For a state $\overline{l} \in \overline{V}$ the input letter can be chosen as a subset I such that $c(I) \leq B$, and the next state is (l, I). Observe that we assumed that input letters from Γ can be chosen at states (l, I), and at states from $\overline{V}$ a subset I of $[n]$ can be chosen. However, this can be easily transformed to a game where at every state all input letters are available as follows: we add an auxiliary state that is losing for the controller, and at a state if an input letter is not available, we make it available and add a transition to the losing state. For simplicity, we ignore the details of this reduction.

The set of observation-based strategies of $\overline{G}_{\restriction B}$ represents the set of B-restricted strategies. Let $\overline{G}^{\mathsf{K}}$ be the perfect-information game obtained from the subset construction of $\overline{G}_{\restriction B}$, i.e., $\overline{G}^{\mathsf{K}} = \mathsf{Pft}(\overline{G}_{\restriction B})$.

Theorem 4. *Let $G = (V, \Gamma, \delta)$ be a game with variables with a cost function c on variables and $p : V \to \{ 0, 1, \ldots, d \}$ be a priority function on V. For $B > 0$, consider the perfect-information game structure $\overline{G}^{\mathsf{K}} = \mathsf{Pft}(\overline{G}_{\restriction B})$. Let $\overline{p}$ be a priority function on $\overline{G}^{\mathsf{K}}$ defined as follows: for $s \subseteq \overline{V}$ we have $\overline{p}(s) = d$; and for $s \subseteq L \setminus \overline{V}$ we have $\overline{p}(s) = \max\{p(v) \mid (v, I) \in Y\}$ if $p(v)$ is even for all $(v, I) \in s$; otherwise $\overline{p}(s) = \min\{ p(v) \mid (v, I) \in Y, p(v) \text{ is odd} \}$. There is a B-restricted strategy for the controller in G to satisfy the objective $\mathsf{Parity}(p)$ iff there is a strategy in $\overline{G}^{\mathsf{K}}$ to satisfy $\mathsf{Parity}(\overline{p})$.*

Example 2. Consider the interaction of a controller with a plant with variables $\{ x_1, x_2, \ldots, x_n \}$ where all the variables are public. Assume the variables are accessed through a network with a bandwidth constraint B. Let c be a cost function that associates with a variable x_i the cost $c(i)$ that specifies the bandwidth requirement to access variable x_i. The games with hard-constraints provide the right framework to model such interactions.

Budget Optimization Problems. We now consider games with soft-constraints. These are games with variables with a cost function on variables. In contrast to games with hard-constraints where the budget B is a hard-constraint, in games with soft-constraints the controller can choose to know the value of a subset I of variables and incur a cost $c(I)$, and the goal is to either minimize the long-run average of the cost, or minimize the maximum cost, along with satisfying a given parity objective. A strategy in such games is called soft-constrained if whenever it asks for the valuation of a set I of variables, then it only observes the valuation of the set I of variables.

Reduction. Let $G = (V, \Gamma, \delta)$ be a game with variables, and let c be a cost function that assigns cost $c(i) > 0$ to variable x_i. We present a reduction of games with variables with soft-constraints to an imperfect-information game $\widetilde{G}_{\mathsf{soft}} = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$ as follows:
(1) *States.* The set of states is $L = V \times \{ I \subseteq [n] \} \cup \widetilde{V}$, where $\widetilde{V}$ is a copy of the valuations. That is the set of states consists of a pair of valuation and a subset $I \subseteq [n]$ and copy of the valuations.
(2) *Input letters.* The set of input letters is $\Sigma = \Gamma \cup \{ I \subseteq [n] \}$. The set of input letters is the set of input letters Γ of the game G and consists of subsets $I \subseteq [n]$.
(3) *Observations.* The set of observations is $\mathcal{O} = \{(o, I) \mid I \subseteq [n], o \in V \restriction I\} \cup \{\widetilde{o}\}$. The set of observations consists of pairs (o, I) where $I \subseteq [n]$ and o is a valuation restricted to I, and there is a special observation $\widetilde{o}$.
(4) *Observation map.* The observation map is as follows: $\gamma(o, I) = \{ (l, I) \in L \mid l \restriction I = o \}$ and $\gamma(\widetilde{o}) = \widetilde{V}$. Observe that each state in $\widetilde{V}$ has the observation $\widetilde{o}$.
(5) *Transition function.* The transition function is as follows: for $\sigma \in \Gamma$, $((l, I), \sigma, \widehat{l'}) \in \Delta$ iff $l' \in \delta(l, \sigma)$ and $(\widetilde{l}, \sigma, (l, \sigma)) \in \Delta$ for $\sigma = I \subseteq [n]$. For a state (l, I) if an input letter σ from Γ is chosen, then a next state $\widetilde{l'}$ is possible iff $l' \in \delta(l, \sigma)$. For a state $\widetilde{l} \in \widetilde{V}$ the input letter can be chosen as a subset $I \subseteq [n]$. Observe that we assumed that input letters from Γ can be chosen at states (l, I), and at states from $\widetilde{V}$ a subset I of $[n]$ can be chosen.

(6) *Reward function.* The reward function r on input letters is as follows: for input letters $\sigma \in \Gamma$ we have $r(\sigma) = 0$ and for $I \subseteq [n]$ we have $r(I) = -c(I)$, i.e., the reward collected is the negative of the cost.
The set of observation-based strategies of $\widetilde{G}_{\mathsf{soft}}$ represents the set of soft-constrained strategies. Let $\widetilde{G}^{\mathsf{K}}$ be the perfect-information game obtained from the subset construction of $\widetilde{G}_{\mathsf{soft}}$, i.e., $\widetilde{G}^{\mathsf{K}} = \mathsf{Pft}(\widetilde{G}_{\mathsf{soft}})$.

Theorem 5. *Let $G = (V, \Gamma, \delta)$ be a game with variables with a cost function c on variables and $p : V \to \{0, \ldots, d\}$ be a priority function on V. Consider the perfect-information game structure $\widetilde{G}^{\mathsf{K}} = \mathsf{Pft}(\widetilde{G}_{\mathsf{soft}})$. Let $\widetilde{p}$ be a priority function on $\widetilde{G}^{\mathsf{K}}$ defined as: for $s \subseteq \overline{V}$, let $\widetilde{p}(s) = d$; and for $s \subseteq L \setminus \widetilde{V}$, let $\widetilde{p}(s) = \max\{ p(v) \mid (v, I) \in Y \}$ if $p(v)$ is even for all $(v, I) \in s$; otherwise $\widetilde{p}(s) = \min\{ p(v) \mid (v, I) \in Y, p(v) \text{ is odd} \}$. The following assertions hold: (1) there is a soft-constrained strategy for the controller in G to satisfy $\mathsf{Parity}(p)$ and ensure the long-run average of the costs is at most λ iff $\sup_{\alpha \in \widetilde{G}^{\mathsf{K}}} \inf_{\pi \in \mathsf{Outcome}(\widetilde{G}^{\mathsf{K}}, \pi)} \mathsf{MP}(\widetilde{p}, r)) \geq -\frac{\lambda}{2}$; and (2) there is a soft-constrained strategy for the controller in G to satisfy $\mathsf{Parity}(p)$ and ensure the maximum of the costs is at most λ iff $\sup_{\alpha \in \widetilde{G}^{\mathsf{K}}} \inf_{\pi \in \mathsf{Outcome}(\widetilde{G}^{\mathsf{K}}, \pi)} \mathsf{MinP}(\widetilde{p}, r)) \geq -\lambda$.*

Observe that in item 1 of Theorem 5 the right-hand side is $-\frac{\lambda}{2}$ instead of $-\lambda$. This is because in the modeling of a game with variables with soft-constraints, each step of the original game is simulated in two-steps rather than one, and hence we need a factor of 2 in the result.

Example 3. Consider the interaction of a plant with variables $\{ x_1, x_2, \ldots, x_n \}$ and a controller where all the variables are public. The values of the variables can be obtained through sensors, and the value of variable x_i can be obtained through a sensor by consuming $c(i)$ units of power. Games with soft-constraints provide suitable framework for such games. If the goal is to minimize the average-power consumption, then the long-run average criterion is appropriate, and if the goal is to minimize the peak-power consumption, then the appropriate objective is to minimize the maximum cost.

Solution of perfection-information games. The results of [3] present solutions of perfect-information games with mean-payoff parity objectives. The result of Theorem 5 present a reduction of games with variables with soft-constraints to minimize long run average of the costs along with satisfying a parity objective to perfect-information games with mean-payoff parity objectives. Theorem 5 also presents the reduction of games with variables with soft-constraints to minimize the maximum cost along with satisfying a parity objective to perfect-information games with min-parity objectives. We now briefly describe how to use solutions of perfect-information parity games to obtain solutions of perfect-information min-parity games. The solution of perfect-information games with min-parity objectives can be obtained as follows: (a) sort the rewards on the edges; (b) with a binary search on the range of rewards, keep only edges above a certain reward value and solve the resulting qualitative parity game. The solution of perfect-information games with parity objectives is widely studied in literature, see [8, 18, 7] for algorithmic solution of perfect-information parity games. Hence perfect-information min-parity games with n states and m edges can be solved with $\log(m)$ calls to perfect-information parity games. It may be noted that from the above solution we can find the the minimum budget B that is required to satisfy games with variables with hard-constraints to satisfy a given parity objective.

Computational complexity. It follows from the results of [2, 15] that games with fixed-partial information are EXPTIME-hard even for reachability objectives. The games with fixed-partial information can be obtained as a special case of games with budget constraints as follows: set the budget as $B = k$, and the cost for bits $1, 2, \ldots, k$ as 1, and $k + 1$ for all other bits. Hence it follows that games with budget constraints are EXPTIME-hard; and it also follows that the budget optimization problem is EXPTIME-hard for reachability objectives (and also for the more general parity objectives). From Theorem 4, Theorem 5, and the solution of perfect-information games we obtain an EXPTIME upper bound for the solution of games with budget constraints and the budget optimization problem. Thus we have the following result.

Theorem 6. *Let $G = (V, \Gamma, \delta)$ be a game with variables with a cost function c on variables and $p : V \to \{0, 1, \ldots, d\}$ be a priority function on V. For $B > 0$, it is EXPTIME-complete to decide whether there is a B-restricted strategy for the controller in G to satisfy the objective $\mathsf{Parity}(p)$; and the problem is EXPTIME-hard even for reachability objectives.*

5 Discrete Time Control of Rectangular Automata

We now apply the theory of controller synthesis with budget constraints to the discrete time control problem for *rectangular automata* [6]. We obtain our results using a general decidability result about imperfect-information games on infinite state spaces that have a stable partition with a finite quotient.

R-stable games. In this section we drop the assumption of finite state space of games. Let $G = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$ be a game structure of imperfect-information such that L is infinite. Let $R = \{ r_1, r_2, \ldots, r_l \}$ be a finite partition of L. A set $Q \subseteq L$ is *R-definable* if $Q = \bigcup_{r \in Z} r$, for some $Z \subseteq R$. The game G is *R-stable* if the following conditions hold for all $\sigma \in \Sigma$: (a) the set $\{ l \in L \mid \exists l' \in L.(l, \sigma, l') \in \Delta \}$ is R-definable; (b) for all $r \in R$, the set $\mathsf{Post}^G_\sigma(r)$ is R-definable; (c) for all $r, r' \in R$, if for some $x \in r$ we have $\mathsf{Post}^G_\sigma(\{ x \}) \cap r' \neq \emptyset$, then for all $x' \in r$ we have $\mathsf{Post}^G_\sigma(\{ x' \}) \cap r' \neq \emptyset$; and (d) for all $o \in \mathcal{O}$, the set $\gamma(o)$ is R-definable.

Lemma 4. *The following assertions hold. (1) Let G be a game structure of imperfect information, and let R be a finite partition of the state space of G such that the game G is R-stable. Then the perfect-information game $\mathsf{Pft}(G)$ is 2^R-stable. (2) Let $\overline{G}$ be a perfect-information game structure with a parity objective with d-priorities. If $\overline{G}$ is $\overline{R}$-stable, for a given finite partition $\overline{R}$, then the sure winning sets in $\overline{G}$ can be computed in time $O(|\overline{R}|^d)$.*

We present the definition of rectangular automata with budget constraints and then reduce the problem to a game of imperfect information. Using a result of [6] we establish the game of imperfect information is R-stable for a finite set R.

Rectangular constraints. Let $Y = \{ y_1, y_2, \ldots, y_k \}$ be a set of real-valued variables. A *rectangular inequality* over Y is of the form $x_i \sim d$, where d is an integer constant, and $\sim \in \{ \leq, <, \geq, > \}$. A *rectangular predicate* over Y is a conjunction of rectangular inequalities. We denote the set of rectangular predicates over Y as $\mathit{Rect}(Y)$. The rectangular predicate ϕ defines the set of vectors $[\![\phi]\!] = \{ y \in \mathbb{R}^k \mid \phi[Y := y] \text{ is true} \}$. For $1 \leq i \leq k$, let $[\![\phi]\!]_i$ be the projection on variable y_i of the set $[\![\phi]\!]$. A set of the form $[\![\phi]\!]$, where ϕ is a rectangular predicate, is called a rectangle. Given a non-negative integer $m \in \mathbb{N}$, the rectangular predicate ϕ is m-bounded if $|d| \leq m$, for every conjunct $y_i \sim d$ of ϕ. Let us denote by $\mathit{Rect}_m(Y)$ the set of m-bounded rectangular predicates on Y.

Rectangular automata with budget constraints. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of boolean variables and V the set of all valuations. A *rectangular automaton with budget constraints* H is a tuple $\langle V, \mathit{Lab}, \mathit{Edg}, Y, \mathit{Init}, \mathit{Inv}, \mathit{Flow}, \mathit{Jump}, c \rangle$ where (a) Lab is a finite set of *labels*; (b) $\mathit{Edg} \subseteq V \times \mathit{Lab} \times V$ is a finite set of *edges*; (c) $Y = \{ y_1, y_2, \ldots, y_k \}$ is a finite set of *variables*; (d) $\mathit{Init} : V \to \mathit{Rect}(Y)$ gives the *initial condition* $\mathit{Init}(v)$ of a valuation v; (e) $\mathit{Inv} : V \to \mathit{Rect}(Y)$ gives the *invariant condition* $\mathit{Inv}(v)$ of valuation v (i.e., the automaton can stay in v as long as the values of variables lie in $[\![\mathit{Inv}(v)]\!]$); (f) $\mathit{Flow} : V \to \mathit{Rect}(\dot{Y})$ governs the evolution of the variables in each valuation; (g) c is a cost function that assigns cost $c(i)$ to variable x_i, for $1 \leq i \leq n$; and (h) Jump maps each edge e to a predicate $\mathit{Jump}(e)$ of the form $\phi \wedge \phi' \wedge \bigwedge_{i \notin \mathit{Update}(e)} (y'_i = y_i)$, where $\phi \in \mathit{Rect}(Y)$, $\phi' \in \mathit{Rect}(Y')$, and $\mathit{Update}(e) \subseteq \{ 1, 2, \ldots, k \}$. The variables in Y' refer to the updated values of the variables after the edge has been traversed. Each variable y_i with $i \in \mathit{Update}(e)$ is updated nondeterministically to a new value in $[\![\phi']\!]_i$. A rectangular automaton is m-bounded if all rectangular constraints are m-bounded.

Nondecreasing and bounded variables. Let H be a rectangular automaton, and let $i \in \{ 1, 2, \ldots, k \}$. The variable y_i of H is *nondecreasing* if for all $v \in V$, the invariant interval $[\![\mathit{Inv}(v)]\!]_i$ and the flow interval $[\![\mathit{Flow}(v)]\!]_i$ are subsets of the nonnegative reals. The variable y_i of H is *bounded* if for all $v \in V$, the invariant interval $[\![\mathit{Inv}(v)]\!]_i$ is a bounded set. The automaton H is bounded (resp. nondecreasing) if all the variables are bounded (resp. nondecreasing). In sequel we consider automata that are bounded or nondecreasing.

Game semantics. The rectangular automaton game with a budget constraint B is played as follows: the game starts with a valuation v and values for the continuous variables $y \in [\![\mathit{Init}(v)]\!]$. At each round the controller can choose to observe a subset I of the boolean variables such that $c(I) \leq B$; and then the controller decides to take one of the enabled edges (if one exists). Then the environment nondeterministically updates the continuous variables according to the flow predicates by letting time pass for 1 time unit. Then the new round of the game starts. We now present a reduction to imperfect-information game, and then show that the game is stable with respect to a finite partition.

Reduction. A rectangular automaton H with a budget constraint B reduces to an imperfect-information game $\overline{H}_{\restriction B} = \langle L, l_0, \Sigma, \Delta, \mathcal{O}, \gamma \rangle$ as follows:
(1) *States.* The set of states is $L = V \times \mathbb{R}^k \times \{ I \subseteq [n] \mid c(I) \leq B \} \cup \overline{V} \times \mathbb{R}^k$, where $\overline{V}$ is a copy of the valuations. That is the set of states consists of a tuple of valuation, values of variables and a subset I such that $c(I)$ does not exceed the budget B, and copy of the valuations and the values of variables.
(2) *Input letters.* The set of input letters is $\Sigma = \mathit{Lab} \cup \{1\} \cup \{I \subseteq [n] \mid c(I) \leq B\}$. The set of input letters is the set of labels Lab of H, unit time 1 and subsets $I \subseteq [n]$ such that $c(I) \leq B$.
(3) *Observations.* The set of observations is $\mathcal{O} = \{ (o, I) \mid I \subseteq [n], c(I) \leq B, o$ is a valuation from $V \restriction I \} \cup \{ \overline{o} \}$. The set of observations consists of pairs (o, I) where $I \subseteq [n]$ and o is a valuation restricted to I, and there is a special observation $\overline{o}$.
(4) *Observation map.* The observation map is as follows: $\gamma(o, I) = \{ (l, y, I) \in L \mid l \restriction I = o \}$ and $\gamma(\overline{o}) = \overline{V} \times \mathbb{R}^k$. Observe that each state in $\overline{V} \times \mathbb{R}^k$ has the same observation $\overline{o}$.
(5) *Transition function.* The transition function is as follows: (a) $((\overline{v}, y), \sigma, (v, y, \sigma)) \in \Delta$, for $\sigma = I \subseteq [n]$ such that $c(I) \leq B$; (b) $((v, y, I), \sigma, \overline{v}', y') \in \Delta$, such that there exists $e = (v, \sigma, v') \in \mathit{Edg}$ with $(y, y') \in [\![\mathit{Jump}(e)]\!]$; and (c) $((v, y, I), 1, (v, y', I)) \in \Delta$ such that there exists a continuously differentiable function $f : [0,1] \to \mathit{Inv}(v)$ such that $f(0) = y$, $f(1) = y'$ and for all $t \in (0,1)$ we have $\dot{f}(t) \in [\![\mathit{Flow}(v)]\!]$.

The set of observation-based strategies of $\overline{H}_{\restriction B}$ represents the set of B-restricted strategies.

Equivalence relation. Let H be a m-bounded rectangular automaton with a budget constraint B, and let $\overline{H}_{\restriction B}$ be the game of imperfect information obtained by the reduction. We define the equivalence relation $\equiv_m$ on the state space as follows: $(v, y, I) \equiv_m (v', y', I)$ (resp. $(\overline{v}, y) \equiv_m (\overline{v}', y')$) iff (a) $v = v'$ (resp. $\overline{v} = \overline{v}'$); and (b) for all $1 \leq i \leq k$, either $\lfloor y_i \rfloor = \lfloor y_i' \rfloor$ and $\lceil y_i \rceil = \lceil y_i' \rceil$, or both y_i and y_i' are greater than m. We denote by $R_{\equiv_m}$ the set of equivalence classes of $\equiv_m$. It is easy to observe that $R_{\equiv_m}$ is finite (in fact exponential in the size of H). An extension of the result of [6] gives us the following result.

Lemma 5. *Let H be a m-bounded rectangular automaton game with a budget constraint B. The imperfect-information game $\overline{H}_{\restriction B}$ is $R_{\equiv_m}$-stable.*

Theorem 7. *Let H be a rectangular automaton with a budget constraint B and let $p : V \to \{ 0, 1, \ldots, d \}$ be a priority function on V. Consider the perfect-information game structure $\overline{H}^{\mathsf{K}} = \mathsf{Pft}(\overline{H}_{\restriction B})$. Let $\overline{p}$ be a priority function on $\overline{H}^{\mathsf{K}}$ defined as follows: for $s \subseteq \overline{V}$ we have $\overline{p}(s) = d$; and for $s \subseteq L \setminus \overline{V}$ we have $\overline{p}(s) = \max\{ p(v) \mid (v, I) \in Y \}$ if $p(v)$ is even for all $(v, I) \in s$; otherwise $\overline{p}(s) = \min\{ p(v) \mid (v, I) \in Y, p(v)$ is odd $\}$. There is a B-restricted strategy for the controller in H to satisfy the objective $\mathsf{Parity}(p)$ iff there is a strategy in $\overline{H}^{\mathsf{K}}$ to satisfy $\mathsf{Parity}(\overline{p})$.*

From Lemma 4, Lemma 5, and Theorem 7 we obtain the following corollary.

Corollary 1. *Let H be a rectangular automaton with a budget constraint B and let $p : V \to \{ 0, 1, \ldots, d \}$ be a priority function on V. Whether there is a B-restricted strategy for the controller in H to satisfy the objective* $\mathsf{Parity}(p)$ *can be decided in 2EXPTIME.*

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