HYBRID OPTIMIZATION PROBLEMS IN AUTOMOTIVE APPLICATIONS

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Abstract: The design of a control for an automotive engine in the cut–off region is considered. The system is modeled as a hybrid control system, and a minimum time problem is formulated. Two relaxed problems obtained by reducing the hybrid model to the continuous and discrete time domains, respectively, are first solved, providing lower bounds on the attainable cost. Then, sub–optimal solutions to the original problem are devised by mapping the solutions of the relaxed problems back to the hybrid domain. The performance of the hybrid laws thus obtained are discussed.

Keywords: Automotive control, Hybrid systems, Optimal control.

1. INTRODUCTION

The general goal of control in automotive applications is to achieve the demanded driving performance while ensuring safety and minimizing fuel consumption and emissions. Vehicle specifications are given by assigning prescribed behaviors in different regions of operation, cf. Heywood (1989). The region of operation considered here is characterized by the driver releasing the gas pedal, requesting no torque to the engine. In this case, an obvious control policy to minimize consumption and emissions is to shut fuel injection off, an operation called *cut–off*. However, cutting off fuel injection produces a torque step which may excite the elasticity of the powertrain resulting in the driver's discomfort. A more considerate injection control policy would tend to minimize discomfort by minimizing the peaks of oscillatory acceleration (cf. Balluchi et al. (1998)) or the time during which perceptible oscillations occur. These optimal control problems are made nontrivial by the fact that the system can only be realistically modeled as a hybrid system consisting of tightly linked discrete event and continuous time dynamics. In Grossman et al. (1993) and Morse (1997), some foundations of hybrid control theory are laid out and some hybrid control problems are described. Optimal control synthesis in the hybrid domain is typically a very difficult problem; for some interesting results see Branicky and Mitter (1995), Ge et al. (1996). Our approach is to introduce relaxed formulations of the problem, in the continuous and discrete time domains respectively, which allow a complete solution by classical methods. The utility of such relaxed solutions is twofold: they provide lower bounds for the minimum attainable cost for *any* hybrid control policy, and suggest suboptimal control laws for the original problem. The results of this approach when applied to the cut–off problem, show that

[1] Research sponsored in part by a CNR grant.

suboptimal strategies are easily obtained and their performance can be quantitatively assessed.

2. RELAXATION APPROACH TO HYBRID OPTIMAL CONTROL

In this paper, we consider deterministic hybrid systems defined as the composition of a Finite State Machine (FSM), a Discrete Event system (DE) and a Continuous Time (CT) dynamics. To each state of the FSM is associated a CT dynamics which evolves until the FSM changes state. The DE system is synchronized with FSM transitions. The three dynamics interact by exchanging signals with each other.

We denote by t the current time and by t_k the instant of time at which the k-th FSM transition occurs. Let S be the set of FSM states s; $z \in \mathbb{R}^n$ denotes the DE state and $x \in \mathbb{R}^q$ the CT state. The triple (s, z, x) represents the hybrid state. Let the CT state and input trajectories be denoted by $x(t)$,$u(t)$, and the sequence of FSM states at times t_k by s_k. Further, since the DE system is synchronous with FSM transitions, we denote by $z(k)$,$u(k)$ the DE states and input sequences. The couple (v, u) is the hybrid system input with $v \in V \subseteq \mathbb{R}^m$ the DE input, and $u \in U \subseteq \mathbb{R}^p$ the CT input. Let $\mathcal{V}$, $\mathcal{U}$ stand for the class of feasible input signals $v : \mathbb{N} \to V$ and $u : \mathbb{R}_0^+ \to U$, respectively.

For all FSM states $s, r \in$ S, if an arc from s to r exists, then a transition from s to r may take place. The $(k+1)$–th such state transition occurs as soon as, at some time $t^* > t_k$, the corresponding guard condition

$$\lambda_{sr}(\nu(k), \mu(t^*)) = 0 \tag{1}$$

becomes true. Upon this condition, $t_{k+1} = t^*$. Signals $\nu(k)$,$\mu(t_k)$ are generated by the DE system and the CT dynamics, respectively, as described in the sequel. Since we do not consider nondeterminism, two guard conditions cannot be simultaneously true.

Associated with each $s \in$ S are the CT dynamics

$$\dot{x}(t) = f_s(x(t), u(t), \nu(k)) \ , \tag{2}$$

$$\mu(t) = h_s(x(t), u(t), z(k)) \ , \tag{3}$$

which evolves from time t_k at which the FSM enters s. Output $\mu(t)$, sampled at time t_{k+1}, is read by the DE system. Notice that CT state resets at FSM transitions are not allowed. The model without resets has been demonstrated to be rich enough for our application purposes. We will investigate state resets in future work.

The DE system

$$z(k+1) = g_{sr}(z(k), v(k), \mu(t_k)) \ , \tag{4}$$

$$\nu(k) = l_{sr}(z(k), v(k), \mu(t_k)) \ . \tag{5}$$

is associated to the FSM transition $s \to r$. Signal $\nu(k)$ is output at the FSM transition and read by the FSM and CT dynamics.

Let $\mathcal{H}$ be a hybrid system which can be modeled as above. Consider a target set $\Psi \subset \text{S} \times \mathbb{R}^n \times \mathbb{R}^q$ and cost functional $\mathcal{F}(s_k, z(k), x(t), v(k), u(t))$.

Problem 1. Given $[s_o, z_o, x_o] \notin \Psi$,

$$\min_{v(k) \in \mathcal{V}, u(t) \in \mathcal{U}} \mathcal{F}(s_k, z(k), x(t), v(k), u(t)) \tag{6}$$

$$\begin{cases} \textit{Dynamics of Hybrid Model } \mathcal{H}, \\ [s_0, z(0), x(0)] = [s_o, z_o, x_o] \, , \\ [s_K, z(K), x(T)] \in \partial\Psi. \end{cases} \tag{7}$$

where T and K denote the (unspecified) time and event horizon, respectively.

2.1 *Relaxations of Hybrid Optimal Control Prob.s*

Problem 1 for the hybrid system $\mathcal{H}$ is very difficult to solve in general, due to the lack of practical solution techniques. We attack the problem by means of a relaxation technique, consisting of defining a new model $\mathcal{M}$ with input ω in class Ω and state ξ, target set Θ, cost functional $\mathcal{G}$, and establishing relations between the original and relaxed system states, inputs, and trajectories such that

- to each state (s, z, x) (input (v, u)) of $\mathcal{H}$ is associated a (possibly non–unique) state ξ (input ω, resp.) of $\mathcal{M}$;
- to each trajectory of $\mathcal{H}$ is associated a (possibly non unique) trajectory of $\mathcal{M}$;
- all relaxed state ξ of $\mathcal{M}$ associated with any state (s, z, x) of $\mathcal{H}$ in Ψ, the are included in Θ;
- all trajectories of $\mathcal{H}$ have cost greater than or equal to the associated relaxed trajectories.

For relaxed problems as defined above, we have:

Fact 2. Given an initial condition $[s_o, z_o, x_o] \notin \Psi$ for $\mathcal{H}$, let $(\hat{v}, \hat{u}) \in \mathcal{V} \times \mathcal{U}$ be an optimal solution to Problem 1. Further, let $\widehat{\omega} \in \Omega$ be an optimal solution to the relaxed problem from an initial state ξ_o associated to $[s_o, z_o, x_o]$, and let $(\hat{s}, \hat{z}, \hat{x})$, $\widehat{\xi}$ be the corresponding state optimal trajectories according to $\mathcal{H}$ and $\mathcal{M}$, respectively. Then,

$$\mathcal{G}(\widehat{\xi}, \widehat{\omega}) \leq \mathcal{F}(\hat{s}, \hat{z}, \hat{x}, \hat{v}, \hat{u}). \tag{8}$$

In particular, we will study two relaxations of Problem 1 in the continuous time (CT) and discrete time (DT) domains, where classical methods are available to solve optimal control problems.

3. PROBLEM FORMULATION

The behavior of the longitudinal motion of a vehicle with a 4-stroke gasoline engine is described by a hybrid model. Consider the single cylinder hybrid model $\mathcal{H}_{1cyl}$ in Figure 1. The FSM with states I,C,E and H describes the evolution through the intake, compression, expansion and exhaust strokes of the internal combustion engine. The DE system, in dashed boxes, models the *torque generation* mechanism. The CT dynamics, in the upper–right corner, describes the *powertrain*. FSM transitions occur when the piston reaches the bottom or top dead center. Then, the guard condition is expressed in terms of the displacement angle $\tilde{\phi}$ of the crank, obtained from the crankshaft angle ϕ_c as described in the upper–left block, where ϕ_{co} represents the angle the crank is mounted on the shaft.
The powertrain is modeled by the linear model

$$\dot{\zeta} = A^p \zeta + b^p u \tag{9}$$
$$\dot{\phi}_c = [0, 1, 0]\,\zeta \tag{10}$$
$$a = c^p \zeta \tag{11}$$

State $\zeta = [\alpha_e, \omega_c, \omega_p]^T$ represents the axle torsion angle, the crankshaft revolution speed, and the wheel revolution speed. The dynamics of ζ contains a dominant real pole λ_1 and a couple of complex poles $\lambda \pm j\mu$ responsible for the powertrain oscillating behavior. The input u is the torque acting on the crank. We assume u is piece-wise constant. The output a is the vehicle acceleration, which can be expressed as $\dot{\omega}_p$ times the wheel radius. Let $x = [\alpha_e, \omega_c, \omega_p, \phi_c]^T$.
A DE system, synchronized with the FSM models the torque generation. Its inputs are: the mass of air-fuel mix $q \in \mathbb{R}^+$ loaded in the intake phase; a binary signal $j \in \{0, 1\}$ which indicates whether or not the fuel is present in the mix; the modulation factor $r \in [r_{\min}, 1]$ due to not optimal spark timing. The DE system output is the torque value $u(k)$ generated by the cylinder. The Z.O.H. transforms $u(k)$ to $u(t)$ which is fed into the CT dynamics. At the $E \to H$ transition the DE systems reads q and j, and stores in $z \in \mathbb{R}$ the amount of torque obtainable, using the mix-to-torque gain G. This value is updated at the $I \to C$ transition by the modulation factor r due to the applied spark advance. The DE output $u(k)$ is always zero except at the $C \to E$ transition when is set to the value stored in z. If $j = 1$ then $u = M = Gq_a^o$.
The model $\mathcal{H}_{4cyl}$ of a 4–cylinder engine is the composition of four cylinders' FSMs and torque generation DE systems, with a common powertrain CT dynamics. Input q is shared among the DE systems. Inputs $j = [j_1, j_2, j_3, j_4]^T$ and $r = [r_1, r_2, r_3, r_4]^T$ are synchronized with the corresponding DE models. Denote by $\mathcal{J}_d$ and $\mathcal{R}_d$

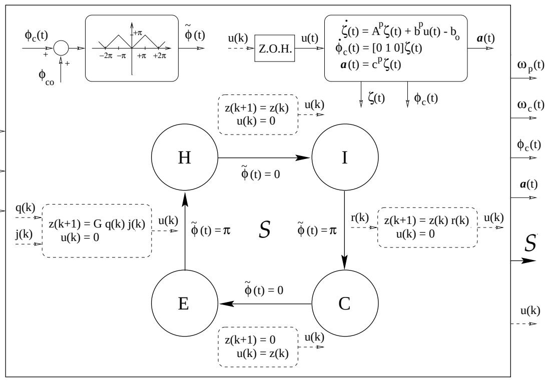

Fig. 1. Hybrid model for a single cylinder engine.

the classes of functions $\mathbb{N} \to \{0, 1\}^4$ and $\mathbb{N} \to [r_{\min}, 1]^4$, feasible for j and r. Crankshaft offsets ϕ_{co_i} are either 0 or π. The FSM state s and the DE state z are four dimensional. Throttle closure produces a decreasing evolution of the manifold pressure towards the idle regime value. The cut–off horizon of control is set to the time at which q_a equals the air quantity with pedal released, q_a^o. The main concern in entering a cut–off regime is limiting powertrain oscillations. Applying modal decomposition, consider the transformed state $[x', x'']^T = T\zeta$ and rewrite (9) as

$$\begin{bmatrix} \dot{x}' \\ \dot{x}'' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} + \begin{bmatrix} b' \\ b \end{bmatrix} u \tag{12}$$

with $A = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$. The oscillating component of the acceleration can be expressed as

$$\tilde{a}(t) = c\,x''(t) \tag{13}$$

where $c \in \mathbb{R}^{1\times 2}$ the product between c^p and the last two columns of T^{-1}. The cut–off control horizon ends when $\tilde{a}(t)$ is lower than a threshold of perception $\tilde{a}_{th} > 0$. Let $\rho = \tilde{a}_{th}\|c\|^{-1}$ and

$$B_\rho = \left\{x'' \in \mathbb{R}^2 \| \|x''\| \le \rho\right\}. \tag{14}$$

Since, under $u = 0$, $\|x''(t)\|$ decreases over time if at some time $\bar{t}$, $x''(\bar{t}) \in B_\rho$, $u = 0$ keeps $x''(t) \in B_\rho$ and $\tilde{a}(t)$ below $\tilde{a}_{th}$ for any $t \ge \bar{t}$. Let C_ρ^ζ denote the image of $\mathbb{R} \times B_\rho$ in the ζ space, and let $\mathrm{S}_\Psi = \{(\mathrm{H,I,C,E}),(\mathrm{I,C,E,H}),(\mathrm{C,E,H,I}),(\mathrm{E,H,I,C})\}$. The cut–off target set is $\Psi = \mathrm{S}_\Psi \times \mathbb{R}^4 \times (C_\rho^\zeta \times \mathbb{R})$. The goal is to perform cut–off entering with the maximum passengers' comfort. In Balluchi et al. (1998), an optimal control which minimizes the peak of $\tilde{a}(t)$ had been presented. Here, we focus on the problem of reaching in minimum time the cut–off state using only on the injection input. Let $\mathcal{H}_I$ be obtained from $\mathcal{H}_{4cyl}$ by setting $v = j$, $\mathcal{V} = \mathcal{J}_{4cyl}$, and $r(k) = [1, 1, 1, 1]^T$, $q(k) = q_a^o$, $\forall k \ge 0$.

Problem 3. Given the hybrid model $\mathcal{H}_I$, for any $\zeta_0 \notin C_\rho^\zeta$,

$$\min_{v(k)\in\mathcal{V}} \int_0^T dt \quad (15)$$

$$\begin{cases} \textit{Dynamics of Hybrid Model } \mathcal{H}_I \\ s_0 = \{\text{H,I,C,E}\}, \\ z(0) = [1,1,1,1]^T G q_a^o, \\ x(0) = [\zeta_0, 0]^T, \\ [s_K, z(K), x(T)] \in \partial\Psi. \end{cases} \quad (16)$$

4. RELAXED OPTIMAL PROBLEMS

4.1 *Continuous–Time Domain Relaxed Problem*

A continuous–time relaxation of $\mathcal{H}_I$ can be obtained by disregarding the exponentially decaying mode of (12), and assuming that torque inputs from the engine to the crankshaft can be varied continuously (instead of only once per cycle).
Let the CT relaxed model state ξ be state x'' associated with the oscillating behavior of the powertrain (12), and assume as input the torque, i.e. $\omega = u$. Define the feasible inputs class

$$\Omega = \left\{\omega | \mathbb{R}_0^+ \to \mathbb{R} \mid 0 \le \omega(t) \le M, \forall t \ge 0\right\} \quad (17)$$

where $M = Gq_a^o$. Notice that actual feasible torque only take values set $\{0, M\}$, and hence the relaxed problem contains a convexification of the input set. Set $\Theta = B_\rho$, and introduce

Problem 4. Given $\xi_o \notin \Theta$,

$$\min_{\omega(t)\in\Omega} \int_0^T dt \quad (18)$$

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + b\omega(t), \\ \xi(0) = \xi_0, \\ \xi(T) \in \partial\Theta. \end{cases} \quad (19)$$

Applying Pontryagin's Maximum Principle a set of necessary conditions for the optimal solutions to Problem 4 is derived. Introduce the adjoint variables $p \in \mathbb{R}^2$ and write the Hamiltonian of the system as $H(\xi(t), p, \omega(t)) = p^T(A\xi + b\omega) - 1$. The adjoint variables p, yield the following unstable oscillating and autonomous dynamics

$$\dot{p} = -\left(\frac{\partial H}{\partial \xi}\right)^T = -A^T p\,. \quad (20)$$

If $\hat{\omega}$ is an optimal solution to Problem 4, and $\hat{\xi}(t)$ is the corresponding trajectory, the maximum condition $\max_{\omega\in\Omega} H(\xi, p, \omega) = H(\xi, p, \hat{\omega}) = 0$, holds. Hence, $\hat{\omega}$ switches according to

$$\hat{\omega} = \begin{cases} M, & \text{if } b^T p > 0 \\ 0, & \text{if } b^T p < 0 \end{cases} . \quad (21)$$

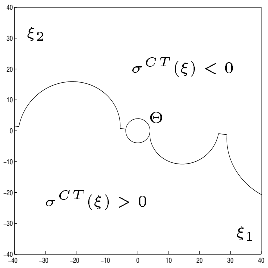

Fig. 2. CT relaxed problem optimal control $\hat{\omega}^{CT}$.

Initial condition ξ_0 and the constraint $\xi(T) \in \partial\Theta$ are imposed on $\xi(t)$. A further boundary condition of the extended dynamics (19), (20) is derived from the transversality condition, i.e. $p(T)$ must be perpendicular to $\partial\Theta$. Then, $p(T)$ is collinear to $\xi(T)$. For $H(\xi(T), p(T), \omega(T)) = 0$, $p(T)$ is opposite to $\xi(T)$ and, without loss of generality, one can choose $p(T) = -\xi(T)$. The optimal solutions can then be obtained by backwards integration of (19) and (20). For all points $\xi(T) = [\rho\cos(\theta),\ \rho\sin(\theta)]^T \in \partial\Theta$ with $\theta \in (-\pi, \pi]$, an optimal trajectory is obtained by backwards integration with either $\omega = 0$ (if $b^T p(T) < 0$) or $\omega = M$ (if $b^T p(T) > 0$) until time t_1 at which $b^T p(t_1) = 0$, where a control switching occurs. The length of the time interval $[t_1, T]$ corresponds to the time vector p takes to reach the switching line $b^T p = 0$ from the final condition $p(T) = -\xi(T)$. Finally, from (21) and (20), one can easily check that the time between two successive switches of the control is constant and equal to $\frac{\pi}{\mu}$. These arguments allow us to map the minimum time switching curve to $\partial\Theta$, from the adjoint space back to the system variable space (see Fig. 2). The optimal control for Problem 4, can then be expressed as

$$\hat{\omega}^{CT} = \begin{cases} 0, & \text{if } \sigma^{CT}(\xi) < 0 \\ M, & \text{if } \sigma^{CT}(\xi) \ge 0 \end{cases} , \quad (22)$$

where $\sigma^{CT}(\xi) = 0$ is the switching line in the ξ state space.

4.2 *Discrete–Time Domain Relaxed Problem*

A discrete–time relaxation of $\mathcal{H}_I$ can be obtained by writing a sampled version of the continuous oscillating dynamics. In the following, a DT model is introduced, whose time scale is adjusted to events in the original system, and is correspondingly warped w.r.t. to real time. The time–warp factor is considered as a control variable. The original hybrid system is thus transformed is a nonlinear DT two–input system.
Assume that the crankshaft velocity ω_c in the evolution of model $\mathcal{H}_I$ remains bounded within an interval $[\omega_c^{\min}, \omega_c^{\max}]$. The event–time sequence t_k can be expressed as $t_k = t_{k-1} + \tau_k$ with

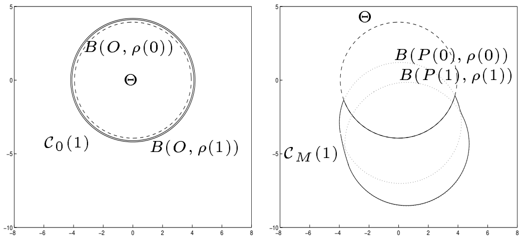

Fig. 3. Points controllable to Θ (dotted) in one step with $\omega_1 \in [0,1]$, $\omega_2 = 0$ (left) and $\omega_2 = M$ (right).

$$\tau_k \;=\; \tau^{\min} + (\tau^{\max} - \tau^{\min})\;\alpha(k) \qquad (23)$$

where $\tau^{\min} = \pi/\omega_c^{\max}$, $\tau^{\max} = \pi/\omega_c^{\min}$ and $\alpha(k)$ is a sequence of warp factors taking values in $[0,1]$. Hence, by relaxing the interaction between DE and CT dynamics in model $\mathcal{H}_I$, the evolution in the x'' sub–space associated with the oscillating behavior of the powertrain is contained in the evolution of the nonlinear two–input DT system

$$x''(k+1) = A_d(\alpha(k))x''(k) + b_d(\alpha(k))u(k) \qquad (24)$$

with $\alpha(k) \in [0,1]$, and where functions $A_d(\alpha(k))$, $b_d(\alpha(k))$ are the discretization of dynamics (19) with sampling period τ_k given by (23), $A_d(\alpha(k)) = e^{A\tau_k}$, $b_d(\alpha(k)) = -(I - A_d(\alpha(k)))\,A^{-1}\,b$. Set $\Theta = B_\rho$, and consider the relaxed model with state $\xi(k) = x''(t_k)$ and input $\omega = [\alpha, u]$ in the class

$$\Omega = \{[\omega_1, \omega_2] : \mathbb{N} \to \in [0,1] \times \{0, M\}\}\ . \qquad (25)$$

Problem 5. Given $\xi_0 \notin \Theta$,

$$\min_{\omega(k)\in\Omega} \sum_{k=1,K} \tau_k(\omega(k)) - \beta_K \qquad (26)$$

$$\begin{cases} \xi(k+1) = A_d(\omega_1(k))\xi(k) + b_d(\omega_1(k))\omega_2(k) \\ \xi(0) = \xi_0, \\ \xi(K) \in \Theta, \end{cases} \qquad (27)$$

where β_K is the amount of time in the final interval $[t_{K-1}, t_K]$ during which the continuous arc of trajectory is inside Θ.

Problem 5 in the DT domain can be solved by the following algorithm based on the analysis of controllable sets. Let $\mathcal{C}(1) = \mathcal{C}_0(1) \cup \mathcal{C}_M(1)$ denote the set of points ξ controllable to Θ in one step, where $\mathcal{C}_0(1)$, $\mathcal{C}_M(1)$ are the sets of points controllable to Θ with $\omega_2 = 0$ and $\omega_2 = M$, respectively. Let $B(\bar{P}, \bar{\rho})$ denote the ball of radius $\bar{\rho}$ and center $\bar{P}$. It is easy to check that

$$\mathcal{C}_0(1) = B(O, \rho(1))/\Theta, \qquad (28)$$

$$\mathcal{C}_M(1) = \cup_{\alpha\in[0,1]} B(P(\alpha), \rho(\alpha))/\Theta, \qquad (29)$$

where $O = [0,0]^T$, $P(\alpha) = -A_d^{-1}(\alpha)b_d(\alpha)$ and $\rho(\alpha) = e^{-\lambda(\tau^{\min} + (\tau^{\max} - \tau^{\min})\alpha)}\rho$. See Fig. 3. The

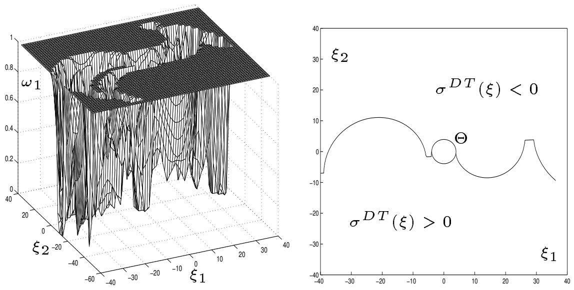

Fig. 4. DE relaxed problem optimal control $\hat{\omega}_1^{DT}$ (left) and $\hat{\omega}_2^{DT}$ (right) with respect to ξ.

boundary of $\mathcal{C}_M(1)$ is given by an arc of $\partial\Theta$, two arcs of $\partial B(P(1), \rho(1))$, an arc of $\partial B(P(0), \rho(0))$ and two of arcs of curves linking them. Due to space limitations, analytical expressions of such arcs are not reported. Given $\xi \in \mathcal{C}(1) \setminus (\mathcal{C}_0(1) \cap \mathcal{C}_M(1))$ a candidate optimal ω is

$$[\omega_1, \omega_2]^T = \begin{cases} [\bar{\alpha}, 0]^T & \text{if } \xi \in \mathcal{C}_0(1)/\mathcal{C}_M(1) \\ [\tilde{\alpha}, M]^T & \text{if } \xi \in \mathcal{C}_M(1)/\mathcal{C}_0(1) \end{cases}$$

with $\bar{\alpha}, \tilde{\alpha} \in [0,1]$ as follows

$$\bar{\alpha} = \begin{cases} 0 & \text{if } \xi \in B(O, \rho(0)) \\ \text{s.t. } \xi \in \partial B(O, \rho(\bar{\alpha})) & \text{if } \xi \notin B(O, \rho(0)) \end{cases}$$

$$\tilde{\alpha} = \begin{cases} 0 & \text{if } \xi \in B(P(0), \rho(0)) \\ \text{s.t. } \xi \in \partial B(P(\tilde{\alpha}), \rho(\tilde{\alpha})) & \text{if } \xi \notin B(P(0), \rho(0)) \end{cases}$$

Points $\xi \in \mathcal{C}_0(1) \cap \mathcal{C}_M(1)$ can be steered to Θ either with $\omega_2 = 0$ or $\omega_2 = M$. To come up with a candidate optimal control, $\mathcal{C}_0(1) \cap \mathcal{C}_M(1)$ is partitioned into sets Q_0, Q_M delimited by the set of points which reach Θ with the same cost under $\omega_2 = 0$ and $\omega_2 = M$, that is $\|e^{A\tau}x\| = \|e^{A\tau}(x + A^{-1}b) - A^{-1}b\| = \rho$. A candidate optimal ω for $\xi \in \mathcal{C}_0(1) \cap \mathcal{C}_M(1)$ is

$$[\omega_1, \omega_2]^T = \begin{cases} [\bar{\alpha}, 0]^T & \text{if } \xi \in Q_0 \\ [\tilde{\alpha}, M]^T & \text{if } \xi \in Q_M \end{cases}$$

with $\bar{\alpha}, \tilde{\alpha}$ defined as above. The algorithm proceeds iteratively computing at each step the controllable set $\mathcal{C}(k)$ in k steps to Θ, using the previous computation of $\mathcal{C}(k-1)$, and dividing $\mathcal{C}(k)$ in two regions where either $\omega_2 = 0$ or $\omega_2 = M$ is optimal. Note that, for all points in $\mathcal{C}(k) \cap \cup_{i=1,k-1}\mathcal{C}(i)$ a check on the optimality of the previously set candidate control has to be done. The algorithm is run until a region of interest around Θ is covered. The optimal control is

$$\begin{bmatrix} \hat{\omega}_1^{DT} \\ \hat{\omega}_2^{DT} \end{bmatrix} = \begin{cases} [1, 0]^T & \text{if } \sigma^{DT}(\xi) < 0 \\ [\alpha, M]^T & \text{if } \sigma^{DT}(\xi) \geq 0 \end{cases} \qquad (30)$$

where $\bar{\alpha}, \tilde{\alpha}$ are chosen as above, and $\sigma^{DT}(\xi) = 0$ is the switching line for control ω_2 in the ξ state space. The optimal controls $\hat{\omega}_1^{DT}$ and $\hat{\omega}_2^{DT}$ for Problem 5 are reported in Fig. 4.

5. HYBRID CONTROLS

The solutions (22) and (30) to the CT and the DT relaxed problems obtained in Section 4.1 and Section 4.2, respectively, cannot be directly applied to the hybrid model $\mathcal{H}_I$ introduced in Section 3, due to the fact that feasible inputs to the CT dynamics in $\mathcal{H}_I$ are piece–wise constant and synchronized with the DE system evolution.

Available controls in $\mathcal{H}_I$ are the injection signals $j_1(k)$,$j_2(k)$,$j_3(k)$,$j_4(k)$. According to the DE model, control signal $j_i(k)$, at current time t_k corresponding to a $E \to H$ FSM transition of the $i-$th cylinder, will produce at the next expansion run the torque $u(k+3)$. This signal will feed the CT dynamics during the interval $[t_{k+3}, t_{k+4})$, steering the CT state from $x''(t_{k+3})$ to $x''(t_{k+4})$. Assuming negligible variations in the crankshaft velocity for $t \in [0, T]$ and $x''(t_k)$ available, an estimate $\tilde{x}_3(k)$ of point $x''(t_{k+3})$ is obtained from the powertrain discrete model (24), setting $\alpha = .5$ and storing the three latest applied injections. Control laws (22) and (30) are bang-bang signals defined in the $\xi \equiv x''$ state space. Injection signal feedbacks for $\mathcal{H}_I$ are derived as follows:

$$j_{i_{CT}}(k) = \begin{cases} 0, & \text{if } \sigma^{CT}(\tilde{x}_3(k)) < 0 \\ 1, & \text{if } \sigma^{CT}(\tilde{x}_3(k)) \geq 0 \end{cases} \quad (31)$$

$$j_{i_{DT}}(k) = \begin{cases} 0, & \text{if } \sigma^{DT}(\tilde{x}_3(k)) < 0 \\ 1, & \text{if } \sigma^{DT}(\tilde{x}_3(k)) \geq 0 \end{cases} \quad (32)$$

where $\sigma^{CT}(\cdot)$,$\sigma^{DT}(\cdot)$ are as in (22),(30), resp..

5.1 *Sub–optimality of hybrid control.*

Given a point $\zeta_0 \notin C^{\zeta}_{\rho}$, let $T^{\mathcal{H}_I}_{CT}$ and $T^{\mathcal{H}_I}_{DT}$ respectively stand for the value of the functional $\int_0^T dt$ along the trajectories of $\mathcal{H}_I$ under feedbacks $j_{i_{CT}}(\cdot)$, as in (31), and $j_{i_{DT}}(\cdot)$, as in (32). Further, let $T^{\mathcal{H}_I}$, T^{CT} and T^{DT} respectively be the optimal solutions to Problem 3, Problem 4 and Problem 5. By (8) and by sub-optimality of (31) and (32),

$$\max(T^{CT}, T^{DT}) \leq T^{\mathcal{H}_I} \leq \max\left(T^{\mathcal{H}_I}_{CT}, T^{\mathcal{H}_I}_{DT}\right).$$

An evaluation of the optimal solution to Problem 3 is obtained from the estimation of quantities T^{CT},T^{DT} and $T^{\mathcal{H}_I}_{CT}$,$T^{\mathcal{H}_I}_{DT}$. Such comparison is shown in Fig. 5, for $\zeta_0 = A^{-1}b\beta$ with $\beta \in [30, 55]$. The behaviors of the hybrid model $\mathcal{H}_I$ under feedbacks (31) and (32) are compared to establish which one of the two solutions exhibits the best performance in terms of cost and stability. Note that, switching curves $\sigma^{DT}(\xi) = 0$ and $\sigma^{CT}(\xi) = 0$ have similar shape and the former is rotated counterclockwise w.r.t. the latter (cf. Fig. 4, Fig. 2). Further, x'' trajectories encircle the origin clockwise. We argue that if $\omega_c^{\max}/\omega_c^{\min}$

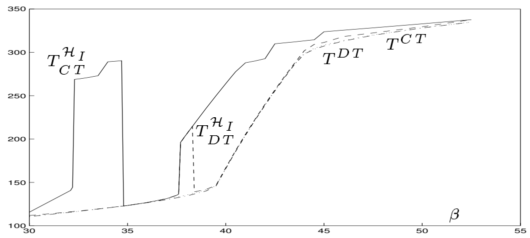

Fig. 5. $T^{\mathcal{H}_I}$ is lower bounded by T^{CT} (dash-dotted) and T^{DT} (dotted) and upper bounded by $T^{\mathcal{H}_I}_{CT}$ (solid) $T^{\mathcal{H}_I}_{DT}$ (dashed), reported in mseconds, for $\zeta_0 = A^{-1}b\beta$.

is small then the DT relaxed model is closer to the behavior of $\mathcal{H}_I$ than the CT one, indeed it anticipates switchings to take into account of torque synchronization. In this case $T^{\mathcal{H}_I}_{CT} > T^{\mathcal{H}_I}_{DT}$. The CT relaxed model exhibits a behavior very close to $\mathcal{H}_I$ if $\omega_c^{\max}/\mu$ is large.

CONCLUSIONS

Optimal control problems have been formalized for a class of hybrid systems. A relaxation method which allows to derive related optimal control problems in the continuous and discrete time domains has been introduced. The relaxed problems, solvable with classical techniques, provide bounds for the attainable cost, and suggest suboptimal control laws for the original problem. The method has been applied to the cut–off control problem with good results.

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