Towards a Denotational Semantics for Discrete-Event Systems

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Abstract

This work focuses on establishing a semantic interpretation for discrete-event systems. In this paper we describe some of the basic mathematical structures, building on the tagged-signal model of [4, 3], and the term-based formalism of [5].

1. Prerequisites

In this section we introduce some terminology and review some fundamental concepts and results from set theory and order theory [2, 1].

1.1. Isomorphisms

Let $(P, \leq_P)$, $(Q, \leq_Q)$, $(R, \leq_R)$ be ordered sets.

A map $\varphi : P \to Q$ is an *order-embedding,* and we write $\varphi : P \hookrightarrow Q$, iff for all $p_1, p_2 \in P$:

$$p_1 \leq_P p_2 \quad \Leftrightarrow \quad \varphi(p_1) \leq_Q \varphi(p_2)$$

An order-embedding $\varphi : P \hookrightarrow Q$ is an *order-isomorphism* iff it maps P onto Q. The ordered sets P and Q are *order-isomorphic*, and we write $P \cong Q$, iff there exists an order-isomorphism from P to Q.

The concept of isomorphism obeys laws of reflexivity, symmetry, and transitivity. In particular, $P \cong P$, $P \cong Q$ implies $Q \cong P$, and $P \cong Q \cong R$ implies $P \cong R$.

1.2. Well Orderings and Ordinals

Let $(P, \leq)$ be an ordered set.

The ordered set $(P, \leq)$ is *well ordered* iff it is totally ordered and every non-empty subset of P has a least element.

If $p \in P$, then the set $\{p' \mid p' < p\}$ is called the *initial segment up to* p and is denoted by $\operatorname{seg} p$.

For any two well ordered sets, either they are order-isomorphic or one is order-isomorphic to an initial segment of the other.

A set A is called *transitive* iff $a \in b \in A$ implies $a \in A$ for all sets a, b.

An *ordinal* is a transitive set that is well ordered by set containment $\in$. We denote the class of all ordinals by Ω.

Every well ordered set is order-isomorphic to a unique ordinal, called the *ordinal number* of the set. Two well ordered sets are order-isomorphic iff they have the same ordinal number.

For any set A, the set $A \cup \{A\}$ is called the *successor* of A and is denoted by A^+.

For all ordinals α and β, either $\alpha \in \beta$, or $\alpha = \beta$, or $\beta \in \alpha$.

Any transitive set of ordinals is an ordinal. The empty set $\emptyset$ is an ordinal. If α is an ordinal, then α^+ is an ordinal. If A is a set of ordinals, then $\bigcup A$ is an ordinal.

For all ordinals α and β, $\alpha \in \beta$ if and only if $\alpha^+ \in \beta$ or $\alpha^+ = \beta$.

An ordinal α is a *successor ordinal* iff $\alpha = \beta^+$ for some ordinal β. Otherwise $\alpha = \bigcup \alpha$, and either $\alpha = \emptyset$ or α is a *limit ordinal*.

The class of all ordinals Ω is not a set (Burtali-Forti paradox).

A set A is *dominated* by a set B iff there exists a one-to-one function from A into B.

For any set A, there exists an ordinal α not dominated by A (Hartogs' theorem).

Note that we adopt von Neumann's approach to the construction of the natural numbers. Under this approach, each natural number is the set of all smaller natural numbers. Hence, every natural number is a finite ordinal and the set of all natural numbers $\omega = \{0, 1, 2, \ldots\}$ coincides with the least limit ordinal.

1.3. Complete Partial Orders

Let $(P, \leq)$ be a partially ordered set.

An element $\bot \in P$ such that $\bot \leq p$ for any $p \in P$ is called a *bottom* or *zero* element. A partially ordered set is *pointed* iff it has a bottom element.

A subset D of P is *directed* iff it is non-empty and every pair of elements in D has an upper bound in D.

A pointed partially ordered set in which every directed subset has a least upper bound is called a *complete partial order* or simply *cpo*.

A subset C of P is *consistent* iff it is non-empty and every finite subset of C has an upper bound in P.

A cpo $(P, \leq)$ is *consistently complete* iff every consistent set $C \subseteq P$ has a least upper bound in P.

An element p of a cpo $(P, \leq)$ is *finite* iff whenever $p \leq \bigvee D$ for a directed set $D \subseteq P$, $p \leq d$ for some $d \in D$.

A cpo $(P, \leq)$ is *algebraic* iff for any $p \in P$, there exists a directed set $D \subseteq P$ of finite elements such that $p = \bigvee D$. An algebraic cpo is called ω-algebraic iff the set of all finite elements is denumerable.

1.4. Least Fixed Points

Let $(P, \leq_P)$, $(Q, \leq_Q)$ be complete partial orders.

A function $f : P \to Q$ is *order-preserving* iff for all $p_1, p_2 \in P$ such that $p_1 \leq_P p_2$, $f(p_1) \leq_Q f(p_2)$.

A function $f : P \to Q$ is *continuous* iff for every directed set $D \subseteq P$, the set $\{f(d) \mid d \in D\}$ is a directed subset of Q, and $f(\bigvee_P D) = \bigvee_Q \{f(d) \mid d \in D\}$. Every continuous function is order-preserving.

An element $p \in P$ is a *fixed point* of the function $f : P \to P$ iff $f(p) = p$. A fixed point p of f is the *least fixed point* of f, and we use the expression $\mu x.f(x)$ to denote it, iff for any fixed point p' of f, $p \leq p'$.

If $f : P \to P$ is a continuous function, then f has a least fixed point and $\mu x.f(x) = \bigvee_{n \in \omega} f^n(\bot)$.

If $f : P \to P$ is an order-preserving function, then f has a least fixed point and $\mu x.f(x) = f^{\alpha}(\bot)$ for some ordinal α, where $f^{\beta^+}(\bot) = f(f^{\beta}(\bot))$ for any ordinal β, and $f^{\gamma}(\bot) = \bigvee \{f^{\beta}(\bot) \mid \beta \in \gamma\}$ if γ is a limit ordinal.

2. Signals and Tuples of Signals

2.1. Signals

Let $\mathcal{V}$ be a non-empty set of possible *values*, and $\mathcal{T}$ a non-empty set of *tags*. While we impose no structure on the set of values $\mathcal{V}$, we require that the set of tags be a totally ordered set $(\mathcal{T}, \leq)$.

Definition 2.1 (Event)**.** *An event e is a tuple (τ, v) with $\tau \in \mathcal{T}$ and $v \in \mathcal{V}$.*

We denote the set of all events by $\mathcal{E}$, that is $\mathcal{E} = \mathcal{T} \times \mathcal{V}$.

Definition 2.2 (Signal)**.** *A signal s is a partial function from the set of tags $\mathcal{T}$ to the set of values $\mathcal{V}$, that is $s \in (\mathcal{T} \rightharpoonup \mathcal{V})$.*

Alternatively, a signal s can be defined as a subset of $\mathcal{E}$, such that for all events $(\tau_1, v_1), (\tau_2, v_2) \in s$, $v_1 \neq v_2$ implies $\tau_1 \neq \tau_2$. The two definitions are equivalent and will be used interchangeably according to context. We denote the set of all signals by $\mathcal{S}$, that is $\mathcal{S} = (\mathcal{T} \rightharpoonup \mathcal{V}) \subset \mathscr{P}\mathcal{E}$.

The *tag-set* $\mathcal{T}(s)$ of a signal $s \in \mathcal{S}$ is defined to be the domain of s, that is $\mathcal{T}(s) = \operatorname{dom} s$.

For notational convenience, we will write $s_1(\tau) \simeq s_2(\tau)$ iff the signals s_1 and s_2 are either both defined, or both undefined at tag τ, and if defined $s_1(\tau) = s_2(\tau)$.

Definition 2.3 (Natural Signal)**.** *A signal $s \in \mathcal{S}$ is natural iff there exists an order-embedding from its tag-set $\mathcal{T}(s)$ to the set of all natural numbers ω.*

We denote the set of all natural signals by $\mathcal{S}_\omega$.

Definition 2.4 (Ordinal signal)**.** *A signal $s \in \mathcal{S}$ is ordinal iff its tag-set $\mathcal{T}(s)$ is well ordered.*

Equivalently, a signal is ordinal iff there exists an order-isomorphism from its tag-set to some ordinal. A natural signal $s \in \mathcal{S}_\omega$ is clearly ordinal, since the existence of an order-embedding from its tag-set $\mathcal{T}(s)$ to the set of all natural numbers ω implies the existence of an order-isomorphism from $\mathcal{T}(s)$ to some ordinal α, where in particular $\alpha \in \omega$ or $\alpha = \omega$. We denote the set of all ordinal signals by $\mathcal{S}_\Omega$. Note that there is no ambiguity in terming the class of all ordinal signals a set, since $\mathcal{S}_\Omega \subseteq \mathcal{S}$.

2.2. Tuples of Signals

Let Var be an infinite set of *variables*. Let $x, y, z, x_i, \ldots$ range over Var and $I, J, K, \ldots$ over $\mathscr{P}Var$.

Given sets I and A, the set of all functions from I to A is denoted by $(I \to A)$ or A^I. An element of A^I can be thought of as an I-*tuple* of elements of A, and we usually write a_i instead of $a(i)$ for $a \in A^I$, $i \in I$. If $a \in A^I$ and $b \in A^J$ with $I \cap J = \emptyset$, then $a \oplus b \in A^{I \cup J}$ denotes the $(I \cup J)$-tuple satisfying:

$$(a \oplus b)(i) = \begin{cases} a_i & \text{if } i \in I, \\ b_i & \text{if } i \in J. \end{cases}$$

A *tuple of signals* or *signal tuple* $\boldsymbol{s}$ is an I-tuple of signals in $\mathcal{S}$ for some set of variables $I \subseteq Var$. Notice that the set of all tuples of signals is essentially the set of all partial functions from the set of all variables to the set of all signals, that is $(Var \rightharpoonup \mathcal{S})$.

The *tag-set* $\mathcal{T}(\boldsymbol{s})$ of a tuple $\boldsymbol{s} \in \mathcal{S}^I$ is defined as the union of the tag-sets of the tupled signals, that is $\mathcal{T}(\boldsymbol{s}) = \bigcup_{i \in I} \mathcal{T}(\boldsymbol{s}_i)$.

Occasionally, it will be convenient to consider an alternative definition for tuples of signals, according to which a signal tuple $\boldsymbol{s}$ is an element of the set $(\mathcal{T} \rightharpoonup Var \rightharpoonup \mathcal{V})$. In fact, the two definitions are equivalent in the sense that for any $\boldsymbol{s} \in (Var \rightharpoonup \mathcal{S})$, there exists a unique $\boldsymbol{s}' \in (\mathcal{T} \rightharpoonup Var \rightharpoonup \mathcal{V})$, such that for all $i \in Var$ and $\tau \in \mathcal{T}$, either $\boldsymbol{s}(i)(\tau)$ and $\boldsymbol{s}'(\tau)(i)$ are both undefined, or they are both defined and $\boldsymbol{s}(i)(\tau) = \boldsymbol{s}'(\tau)(i)$, and vice versa.

With this definitional equivalence in mind, we will write $\boldsymbol{s}_1(\tau) \simeq \boldsymbol{s}_2(\tau)$ for all tuples $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}^I$ and any tag τ, iff $\boldsymbol{s}_1(i)(\tau) \simeq \boldsymbol{s}_2(i)(\tau)$ for all $i \in I$. Similarly, for any signal tuple $\boldsymbol{s}$ and any set of tags $T \subseteq \mathcal{T}$, we will let $\boldsymbol{s} \restriction T$ denote the signal tuple $\{(i, \boldsymbol{s}_i \restriction T) \mid i \in \operatorname{dom} \boldsymbol{s}\}$.

Definition 2.5 (Natural signal tuple). *A signal tuple $\boldsymbol{s} \in \mathcal{S}^I$ is natural iff there exists an order-embedding from its tag-set $\mathcal{T}(\boldsymbol{s})$ to the set of all natural numbers ω.*

We denote the set of all natural I-tuples of signals by $[\mathcal{S}^I]_\omega$. Note that this is different from the set of all I-tuples of natural signals $[\mathcal{S}_\omega]^I$. In fact, it is easy to verify that $[\mathcal{S}^I]_\omega \subseteq [\mathcal{S}_\omega]^I$. We denote the set of all natural tuples of signals by $(Var \rightharpoonup \mathcal{S})_\omega$.

Definition 2.6 (Ordinal signal tuple). *A signal tuple $\boldsymbol{s} \in \mathcal{S}^I$ is ordinal iff its tag-set $\mathcal{T}(\boldsymbol{s})$ is well ordered.*

Equivalently, a tuple of signals is ordinal iff there exists an order-isomorphism from its tag-set to some ordinal. As in the case of natural and ordinal signals, any natural tuple of signals is ordinal. We denote the set of all ordinal I-tuples of signals by $[\mathcal{S}^I]_\Omega$. Again, this is different from the set of all I-tuples of ordinal signals $[\mathcal{S}_\Omega]^I$, and it is easy to verify that $[\mathcal{S}^I]_\Omega \subseteq [\mathcal{S}_\Omega]^I$. We denote the set of all ordinal tuples of signals by $(Var \rightharpoonup \mathcal{S})_\Omega$.

Proposition 2.1. *If $I \subseteq Var$ is a finite set of variables, then $[\mathcal{S}^I]_\Omega = [\mathcal{S}_\Omega]^I$.*

Proof. Consider an arbitrary tuple of signals $\boldsymbol{s} \in \mathcal{S}^I$. If $\mathcal{T}(\boldsymbol{s})$ is well ordered, then $\mathcal{T}(\boldsymbol{s}_i)$ will be well ordered for any $i \in I$, since $\mathcal{T}(\boldsymbol{s}_i) \subseteq \mathcal{T}(\boldsymbol{s})$. Hence, $[\mathcal{S}^I]_\Omega \subseteq [\mathcal{S}_\Omega]^I$.

For the other inclusion, notice that any non-empty subset of tags $T \subseteq \mathcal{T}(\boldsymbol{s})$ can be expressed as $\bigcup_{i \in I} \big(T \cap \mathcal{T}(\boldsymbol{s}_i)\big)$. If $\mathcal{T}(\boldsymbol{s}_i)$ is well ordered for any $i \in I$, then $T \cap \mathcal{T}(\boldsymbol{s}_i)$ will have a least tag for any $i \in I$. The set of these least tags will be totally ordered and finite by assumption. Therefore it will have a least element that is also least in $\bigcup_{i \in I} \big(T \cap \mathcal{T}(\boldsymbol{s}_i)\big)$. Hence $\mathcal{T}(\boldsymbol{s})$ will be well ordered, and $[\mathcal{S}^I]_\Omega \supseteq [\mathcal{S}_\Omega]^I$. $\square$

2.3. Ordering Signals and Tuples of Signals

Definition 2.7 (Signal prefix). *A signal $s_1 \in \mathcal{S}$ is a prefix of a signal $s_2 \in \mathcal{S}$, and we write $s_1 \sqsubseteq s_2$, iff $s_1 \subseteq s_2$ and for all tags $\tau_1 \in \mathcal{T}(s_1)$ and $\tau_2 \in \mathcal{T}(s_2) \setminus \mathcal{T}(s_1)$, $\tau_1 < \tau_2$.*

The signal-prefix relation $\sqsubseteq \subset \mathcal{S} \times \mathcal{S}$ is a partial order, and for any signal $s \in \mathcal{S}$, $\emptyset \sqsubseteq s$.

Proposition 2.2. *Two signals $s_1, s_2 \in \mathcal{S}$ have an upper bound if and only if they are comparable.*

Proof. If s_1 and s_2 are comparable, then they trivially share an upper bound, moreover $s_1 \sqcup s_2 \in \{s_1, s_2\}$.

In the other direction, if $s_1 = \emptyset$ or $s_2 = \emptyset$, then s_1 and s_2 are trivially comparable. Otherwise, assume that s_1 and s_2 are incomparable. Then, and without loss of generality, there exist tags $\tau_1 \in \mathcal{T}(s_1)$ and $\tau_2 \in \mathcal{T}(s_2)$, such that $s_1(\tau_1) \not\simeq s_2(\tau_1)$ and $\tau_1 < \tau_2$. For any signal $s \sqsupseteq s_1$, $s \supseteq s_1$, and we deduce that $s(\tau_1) \not\simeq s_2(\tau_1)$. Hence s_2 is incomparable to any upper bound of s_1. $\square$

Proposition 2.3. *If $C \subseteq \mathcal{S}$ is a non-empty set of signals, then the following are equivalent:*

(i) C is totally ordered,

(ii) C is directed,

(iii) C is consistent.

Proof. If C is totally ordered, then for any pair of signals $s_1, s_2 \in C$, either $s_1 \sqsubseteq s_2$ or $s_2 \sqsubset s_1$ implying $s_1 \sqcup s_2 \in \{s_1, s_2\} \subseteq C$. Hence C is directed.

If C is directed, then every pair of signals in C has an upper bound in C, and by straightforward induction, any finite subset of C will have an upper bound in C. Hence C is consistent.

If C is consistent, then any pair of signals $s_1, s_2 \in C$ has an upper bound, and s_1 and s_2 need to be comparable by Proposition 2.2. Hence C is totally ordered. $\square$

Proposition 2.4. *The least upper bound $\bigsqcup C$ of a set of signals $C \subseteq \mathcal{S}$ exists if and only if C is totally ordered, in which case $\bigsqcup C = \bigcup C$.*

Proof. Let C be totally ordered and consider the set $\bigcup C \subseteq \mathcal{E}$. We first need to show that $\bigcup C$ is a signal. Assume to the contrary that $\bigcup C \notin \mathcal{S}$. Then there exist events $(\tau_1, v_1), (\tau_2, v_2) \in \bigcup C$ with $v_1 \neq v_2$ and $\tau_1 = \tau_2$. Consequently, there need to be signals $s_1, s_2 \in C$ such that $(\tau_1, v_1) \in s_1$ and $(\tau_2, v_2) \in s_2$, which cannot be comparable, contradicting the fact that C is totally ordered. Now for any signal $s \in C$, $s \subseteq \bigcup C$ and $(\bigcup C) \setminus s = \bigcup_{s' \in C}(s' \setminus s)$. Since C is totally ordered, $s' \setminus s \neq \emptyset$ if and only if $s \sqsubset s'$. Hence, by Definition 2.7, for all tags $\tau \in \mathcal{T}(s)$ and $\tau' \in \mathcal{T}(s' \setminus s)$, $\tau < \tau'$. And since $\mathcal{T}((\bigcup C) \setminus s) = \mathcal{T}(\bigcup_{s' \in C}(s' \setminus s)) = \bigcup_{s' \in C} \mathcal{T}(s' \setminus s)$, we conclude that $s \sqsubseteq \bigcup C$. Therefore $\bigcup C$ is an upper bound of C. If $u \in \mathcal{S}$ is another upper bound of C, then for any signal $s \in C$, $s \subseteq u$ implying that $\bigcup C \subseteq u$. Suppose there are tags $\tau \in \bigcup C$ and $\tau' \in \mathcal{T}(u \setminus \bigcup C)$ such that $\tau' < \tau$. Then there needs to be a signal $s \in C$ with $\tau \in \mathcal{T}(s)$ and

$\tau' \notin \mathcal{T}(s)$, which cannot be comparable with u, in contradiction to the fact that u is an upper bound of C. Hence $\bigcup C \sqsubseteq u$, establishing that $\bigcup C$ is a least upper bound of C.

For the reverse implication, notice that if C is not totally ordered, then by Proposition 2.3 it is not consistent and therefore cannot have a least upper bound. □

It is evident that the set of all signals $\mathcal{S}$ is a consistently complete cpo under the signal-prefix relation. The situation becomes less clear in the case of natural and ordinal signals.

Proposition 2.5. *The ordered set $(\mathcal{S}_\omega, \sqsubseteq)$ is a consistently complete cpo.*

Proof. The set $(\mathcal{S}_\omega, \sqsubseteq)$ is a pointed partially ordered set with bottom element the empty signal $\emptyset$. Let $D \subseteq \mathcal{S}_\omega$ be a directed set. Then by Proposition 2.3, D will be totally ordered, and by Proposition 2.4, $\bigsqcup D = \bigcup D$. We need to show that $\bigcup D$ is a natural signal. Observe that $\mathcal{T}(\bigcup D) = \bigcup_{s \in D} \mathcal{T}(s)$. For any signal $s \in D$ we can find an order-embedding $\varphi_s : \mathcal{T}(s) \hookrightarrow \omega$ from its tag-set to the set of all natural numbers. Consider the map $\varphi : \mathcal{T}(\bigcup D) \to \omega$, such that for any $\tau \in \mathcal{T}(\bigcup D)$:

$$\varphi(\tau) = \min \{\varphi_s(\tau) \mid s \in D \text{ and } \tau \in \mathcal{T}(s)\}$$

It is easy to verify that φ is an order-embedding. Hence $\bigcup D \in \mathcal{S}_\omega$ and $(\mathcal{S}_\omega, \sqsubseteq)$ is a cpo.

Let $C \subseteq \mathcal{S}_\omega$ be a consistent set. Then by Proposition 2.3, C will be directed. Hence $\bigsqcup C \in \mathcal{S}_\omega$ and $(\mathcal{S}_\omega, \sqsubseteq)$ is consistently complete. □

Proposition 2.6. *The ordered set $(\mathcal{S}_\Omega, \sqsubseteq)$ is a consistently complete cpo.*

Proof. Similar to Proposition 2.5. (under construction...) □

It is a simple exercise to show that the cpo of natural signals, and similarly the cpo of ordinal signals, is algebraic, though we do not pursuit this here.

Let $I \subseteq Var$ be a set of variables. Perhaps surprisingly, we do not use the induced pointwise order to order the set of signal tuples $\mathcal{S}^I$.

Definition 2.8 (Signal-tuple prefix)**.** *A signal tuple $\boldsymbol{s}_1 \in \mathcal{S}^I$ is a prefix of a signal tuple $\boldsymbol{s}_2 \in \mathcal{S}^I$, and we write $\boldsymbol{s}_1 \sqsubseteq \boldsymbol{s}_2$, iff $\boldsymbol{s}_1(i) \subseteq \boldsymbol{s}_2(i)$ for all $i \in I$, and for all tags $\tau_1 \in \mathcal{T}(\boldsymbol{s}_1)$ and $\tau_2 \in \mathcal{T}(\boldsymbol{s}_2) \setminus \mathcal{T}(\boldsymbol{s}_1)$, $\tau_1 < \tau_2$.*

The signal-tuple prefix relation $\sqsubseteq \subset \mathcal{S}^I \times \mathcal{S}^I$ is a partial order, and for any signal tuple $\boldsymbol{s} \in \mathcal{S}^I$, $\emptyset^I \sqsubseteq \boldsymbol{s}$. Notice that for all tuples $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}^I$, $\boldsymbol{s}_1 \sqsubseteq \boldsymbol{s}_2$ implies $\boldsymbol{s}_1(i) \sqsubseteq \boldsymbol{s}_2(i)$ for all $i \in I$.

We can immediately deduce that $(\mathcal{S}^I, \sqsubseteq)$ is a consistently complete cpo. This is less obvious in the case of natural and ordinal tuples of signals.

Proposition 2.7. *The ordered set $([\mathcal{S}^I]_\omega, \sqsubseteq)$ is a consistently complete cpo.*

Proof. Similar to Proposition 2.5. □

Proposition 2.8. *The ordered set $([\mathcal{S}^I]_\Omega, \sqsubseteq)$ is a consistently complete cpo.*

Proof. Similar to Proposition 2.6. □

3. Functions over Tuples of Signals

3.1. Composition

For sets $I, J, K \subseteq Var$ and functions $f : \mathcal{S}^I \to \mathcal{S}^J$ and $g : \mathcal{S}^J \to \mathcal{S}^K$, the *sequential composition* of f and g is the function $g \circ f : \mathcal{S}^I \to \mathcal{S}^K$ such that for any $\boldsymbol{s} \in \mathcal{S}^I$, $(g \circ f)(\boldsymbol{s}) = g(f(\boldsymbol{s}))$. The *n-fold* composition of a function $f : \mathcal{S}^I \to \mathcal{S}^I$ with itself is denoted by f^n and defined recursively as $f \circ f^{n-1}$, with f^0 being the identity function on $\mathcal{S}^I$. For sets $I, J, K, L \subseteq Var$, with $J \cap L = \emptyset$, and functions $f : \mathcal{S}^I \to \mathcal{S}^J$ and $g : \mathcal{S}^K \to \mathcal{S}^L$, the *parallel composition* of f and g is the function $f \oplus g : \mathcal{S}^{I \cup K} \to \mathcal{S}^{J \cup L}$ such that for any $\boldsymbol{s} \in \mathcal{S}^{I \cup K}$, $(f \oplus g)(\boldsymbol{s}) = f(\boldsymbol{s} \restriction I) \oplus g(\boldsymbol{s} \restriction K)$.

3.2. Causality

Let A be a non-empty set and $(\Gamma, \leq)$ a pointed totally ordered set with a zero element $\mathbf{0} \in \Gamma$, such that for any $\gamma \in \Gamma$, $\mathbf{0} \leq \gamma$. A function $d : A \times A \to \Gamma$ is an *ultrametric distance* function iff for all $a_1, a_2, a_3 \in A$:

(i) $d(a_1, a_2) = \mathbf{0} \Leftrightarrow a_1 = a_2$,

(ii) $d(a_1, a_2) = d(a_2, a_1)$,

(iii) $d(a_1, a_3) \leq \max \{d(a_1, a_2), d(a_2, a_3)\}$.

Let $(\Gamma, \leq)$ be a complete totally ordered set with a zero element $\mathbf{0} \in \Gamma$, and $\varphi : \mathcal{T} \hookrightarrow \Gamma$ an order-embedding from the set $(\mathcal{T}, \leq)$ to the set $(\Gamma \setminus \{\mathbf{0}\}, \geq)$. We define the *generalized Cantor metric* $d_{\mathcal{C}} : \mathcal{S} \times \mathcal{S} \to \Gamma$ such that for all signals $s_1, s_2 \in \mathcal{S}$:

$$d_{\mathcal{C}}(s_1, s_2) = \bigvee\nolimits_{\Gamma} \{\varphi(\tau) \mid \tau \in \mathcal{T} \text{ and } s_1(\tau) \not\simeq s_2(\tau)\}$$

It is easy to verify that $d_{\mathcal{C}}$ is an ultrametric.

We can extend the generalized Cantor metric directly to tuples of signals with consideration to their alternative definition. Hence, for any set of variables I and all tuples $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}^I$:

$$d_{\mathcal{C}}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \bigvee\nolimits_{\Gamma} \{\varphi(\tau) \mid \tau \in \mathcal{T} \text{ and } \boldsymbol{s}_1(\tau) \not\simeq \boldsymbol{s}_2(\tau)\}$$

Definition 3.1 (Causal function)**.** *A function $f : \mathcal{S}^I \to \mathcal{S}^J$ is causal iff for all tuples $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}^I$:*

$$d_{\mathcal{C}}(f(\boldsymbol{s}_1), f(\boldsymbol{s}_2)) \leq d_{\mathcal{C}}(\boldsymbol{s}_1, \boldsymbol{s}_2)$$

Definition 3.2 (Strictly causal function)**.** *A function $f : \mathcal{S}^I \to \mathcal{S}^J$ is strictly causal iff for all tuples $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}^I$, $\boldsymbol{s}_1 \neq \boldsymbol{s}_2$ implies:*

$$d_{\mathcal{C}}(f(\boldsymbol{s}_1), f(\boldsymbol{s}_2)) < d_{\mathcal{C}}(\boldsymbol{s}_1, \boldsymbol{s}_2)$$

Clearly, any strictly causal function is causal. Causal functions are closed under both sequential and parallel composition. A sequential composition of causal functions is strictly causal if at least one of the composed functions is strictly causal. A parallel composition, however, is strictly causal iff all of the composed functions are strictly causal.

3.3. Naturality and Ordinality

Definition 3.3 (Natural function)**.** *A function $f : \mathcal{S}^I \to \mathcal{S}^J$ is natural iff for any tuples $\boldsymbol{s} \in [\mathcal{S}^I]_\omega$, $f(\boldsymbol{s}) \in [\mathcal{S}^J]_\omega$.*

Natural functions, while closed under sequential composition, are not closed under parallel composition.

Definition 3.4 (Ordinal function)**.** *A function $f : \mathcal{S}^I \to \mathcal{S}^J$ is ordinal iff for any tuples $\boldsymbol{s} \in [\mathcal{S}^I]_\Omega$, $f(\boldsymbol{s}) \in [\mathcal{S}^I]_\Omega$.*

Ordinal functions are closed under sequential composition and finite parallel composition. That is, if A is a finite set, $f_a : \mathcal{S}^{I_a} \to \mathcal{S}^{J_a}$ is an ordinal function for any $a \in A$, and $a \neq a'$ implies $J_a \cap J_{a'} = \emptyset$ for all $a, a' \in A$, then $\oplus_{a \in A} f_a$ is an ordinal function. Establishing this is easy.

3.4. Fixed-point Properties

We define an *ordinal retraction* ψ_α for each ordinal α, such that for any ordinal tuple $\boldsymbol{s} \in (Var \rightharpoonup \mathcal{S})_\Omega$:

$$\psi_\alpha(\boldsymbol{s}) = \begin{cases} \boldsymbol{s} \restriction \operatorname{seg}\tau & \text{if } \alpha \cong \{\tau' \in \mathcal{T}(\boldsymbol{s}) \mid \tau' < \tau\} \\ \boldsymbol{s} & \text{otherwise.} \end{cases}$$

for some $\tau \in \mathcal{T}(\boldsymbol{s})$. Notice that for any ordinal α, ψ_α is a well defined function. For any two well ordered sets, either they are order-isomorphic, or one is order-isomorphic to a unique initial segment of the other. Hence, there is no ambiguity in the choice of τ when the ordinal α is order-isomorphic to an initial segment of $\mathcal{T}(\boldsymbol{s})$.

Ordinal retractions are easily seen to be continuous functions. For all ordinals α and β, $\psi_\alpha \circ \psi_\beta = \psi_\gamma$, with $\gamma = \alpha$ if $\alpha \in \beta$ and $\gamma = \beta$ otherwise. For any ordinal α and any ordinal tuple of signals $\boldsymbol{s}$, $\psi_\alpha(\boldsymbol{s}) \sqsubseteq \boldsymbol{s}$. For all ordinal tuples of signals $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ such that $\boldsymbol{s}_1 \sqsubset \boldsymbol{s}_2$, if α_1 is the ordinal number of $\mathcal{T}(\boldsymbol{s}_1)$, then $\psi_{\alpha_1}(\boldsymbol{s}_2) = \boldsymbol{s}_1$, and for any ordinal $\beta \ni \alpha_1$, $\psi_\beta(\boldsymbol{s}_2) \sqsupset \boldsymbol{s}_1$.

It is interesting to note that the class of all ordinal retractions is a set, since for any ordinal α, $\psi_\alpha \in \big((Var \rightharpoonup \mathcal{S})_\Omega \to (Var \rightharpoonup \mathcal{S})_\Omega\big)$.

Proposition 3.1. *There exists an ordinal β such that for all ordinals $\alpha, \alpha' \in \beta^+$, $\alpha \neq \alpha'$ implies $\psi_\alpha \neq \psi'_\alpha$, and for any ordinal $\gamma \ni \beta$, $\psi_\gamma = \psi_\beta = id$.*

Proof. Omitted. □

We define the *ordinal approximation* Φ_Ω as a mapping from the ordinal functions to the restrictions of ordinal functions to the set of all ordinal tuples of signals. In particular, for any ordinal function $f : \mathcal{S}^I \to \mathcal{S}^I$ and any ordinal tuple $\boldsymbol{s} \in [\mathcal{S}^I]_\Omega$:

$$\Phi_\Omega(f)(\boldsymbol{s}) = \psi_{\alpha^+}(f(\psi_\alpha(\boldsymbol{s})))$$

where α is the ordinal number of $\mathcal{T}(\boldsymbol{s} \sqcap f(\boldsymbol{s}))$. Notice that the greatest lower bound of $\boldsymbol{s}$ and $f(\boldsymbol{s})$ exists and is ordinal since the cpo $([\mathcal{S}^I]_\Omega, \sqsubseteq)$ is consistently complete. The set $\mathcal{T}(\boldsymbol{s} \sqcap f(\boldsymbol{s}))$ coincides with $\mathcal{T}(\boldsymbol{s})$ iff $\boldsymbol{s} \sqsubseteq f(\boldsymbol{s})$, in which case $\psi_\alpha(\boldsymbol{s}) = \boldsymbol{s}$. Similarly, $\mathcal{T}(\boldsymbol{s} \sqcap f(\boldsymbol{s}))$ coincides with $\mathcal{T}(f(\boldsymbol{s}))$ iff $\boldsymbol{s} \sqsupseteq f(\boldsymbol{s})$, in which case $\psi_\alpha(f(\boldsymbol{s})) = f(\boldsymbol{s})$. In any case, $\psi_\alpha(\boldsymbol{s}) = \psi_\alpha(f(\boldsymbol{s}))$ and both $\mathcal{T}(\psi_\alpha(\boldsymbol{s}))$ and $\mathcal{T}(\psi_\alpha(f(\boldsymbol{s})))$ have ordinal number α.

It is important to note that if $f : \mathcal{S}^I \to \mathcal{S}^I$ is a strictly causal ordinal function, then for any ordinal tuple $\boldsymbol{s} \in [\mathcal{S}^I]_\Omega$, $f(\psi_\alpha(\boldsymbol{s})) \sqsupseteq \psi_\alpha(\boldsymbol{s})$, with α being the ordinal number of $\mathcal{T}(\boldsymbol{s} \sqcap f(\boldsymbol{s}))$. This follows directly by the definition of strict causality.

Lemma 3.1. *If $f : \mathcal{S}^I \to \mathcal{S}^I$ is a strictly causal ordinal function, then the function $\Phi_\Omega(f) : [\mathcal{S}^I]_\Omega \to [\mathcal{S}^I]_\Omega$ is order-preserving.*

Proof. Consider two ordinal tuples $\boldsymbol{s}_1, \boldsymbol{s}_2 \in [\mathcal{S}^I]_\Omega$ such that $\boldsymbol{s}_1 \sqsubseteq \boldsymbol{s}_2$. Let α_1 be the ordinal number of $\mathcal{T}(\boldsymbol{s}_1 \sqcap f(\boldsymbol{s}_1))$, and α_2 be the ordinal number of $\mathcal{T}(\boldsymbol{s}_2 \sqcap f(\boldsymbol{s}_2))$.

If $\boldsymbol{s}_1 = \boldsymbol{s}_2$, then trivially $\Phi_\Omega(f)(\boldsymbol{s}_1) = \Phi_\Omega(f)(\boldsymbol{s}_2)$.

If $\boldsymbol{s}_1 \sqsubset \boldsymbol{s}_2$, then $d_{\mathcal{C}}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \varphi(\tau)$, where τ is the least tag of $\mathcal{T}(\boldsymbol{s}_2) \setminus \mathcal{T}(\boldsymbol{s}_1)$. Since f is strictly causal, $d_{\mathcal{C}}(f(\boldsymbol{s}_1), f(\boldsymbol{s}_2)) < d_{\mathcal{C}}(\boldsymbol{s}_1, \boldsymbol{s}_2)$, which implies $f(\boldsymbol{s}_1)(\tau') \simeq f(\boldsymbol{s}_2)(\tau')$ for any $\tau' \leq \tau$. Moreover, $\psi_{\alpha_1}(f(\boldsymbol{s}_1)) = f(\boldsymbol{s}_1) \restriction \operatorname{seg}\tau''$ for some tag $\tau'' \leq \tau$, with $\mathcal{T}(f(\boldsymbol{s}_1) \restriction \operatorname{seg}\tau'')$ being order-isomorphic to α_1. However, $f(\boldsymbol{s}_1) \restriction \operatorname{seg}\tau'' = f(\boldsymbol{s}_2) \restriction \operatorname{seg}\tau''$ and therefore $\psi_{\alpha_1}(f(\boldsymbol{s}_2)) = \psi_{\alpha_1}(f(\boldsymbol{s}_1))$. Clearly, $\psi_{\alpha_1}(\boldsymbol{s}_2) = \psi_{\alpha_1}(\boldsymbol{s}_1)$ and $\psi_{\alpha_1}(f(\boldsymbol{s}_1)) = \psi_{\alpha_1}(\boldsymbol{s}_1)$, and thus $\psi_{\alpha_1}(\boldsymbol{s}_2) = \psi_{\alpha_1}(f(\boldsymbol{s}_2)) = \psi_{\alpha_1}(\boldsymbol{s}_1)$. Hence, $\psi_{\alpha_1}(\boldsymbol{s}_1) \sqsubseteq \psi_{\alpha_2}(\boldsymbol{s}_2)$. If $\psi_{\alpha_1}(\boldsymbol{s}_1) = \psi_{\alpha_2}(\boldsymbol{s}_2)$, then trivially $\Phi_\Omega(f)(\boldsymbol{s}_1) = \Phi_\Omega(f)(\boldsymbol{s}_2)$. Otherwise, $\psi_{\alpha_1}(\boldsymbol{s}_1) \sqsubset \psi_{\alpha_2}(\boldsymbol{s}_2)$ and $d_{\mathcal{C}}(\psi_{\alpha_1}(\boldsymbol{s}_1), \psi_{\alpha_2}(\boldsymbol{s}_2)) = \varphi(\tau''')$, where τ''' is the least tag of $\mathcal{T}(\psi_{\alpha_2}(\boldsymbol{s}_2)) \setminus \mathcal{T}(\psi_{\alpha_1}(\boldsymbol{s}_1))$. Strict causality of f implies $f(\psi_{\alpha_1}(\boldsymbol{s}_1))(\tau') \simeq f(\psi_{\alpha_2}(\boldsymbol{s}_2))(\tau')$ for any $\tau' \leq \tau'''$. However, $\psi_{\alpha_2}(\boldsymbol{s}_2) \sqsubseteq f(\psi_{\alpha_2}(\boldsymbol{s}_2))$ and since

$\tau''' \in \mathcal{T}(\psi_{\alpha_2}(\boldsymbol{s}_2))$, $f(\psi_{\alpha_1}(\boldsymbol{s}_1))(\tau') \simeq \psi_{\alpha_2}(\boldsymbol{s}_2)(\tau')$ for any $\tau' \leq \tau'''$. Now τ''' is easily seen to be the greatest tag of $\mathcal{T}(\psi_{\alpha_1^+}(\psi_{\alpha_2}(\boldsymbol{s}_2))$, which is order-isomorphic to α_1^+. Consequently, $\psi_{\alpha_1^+}(f(\psi_{\alpha_1}(\boldsymbol{s}_1))) = \psi_{\alpha_1^+}(\psi_{\alpha_2}(\boldsymbol{s}_2))$. Since $\alpha_1 \in \alpha_2$, $\psi_{\alpha_1^+}(\psi_{\alpha_2}(\boldsymbol{s}_2)) \sqsubseteq \psi_{\alpha_2}(\boldsymbol{s}_2)$ and therefore $\psi_{\alpha_1^+}(f(\psi_{\alpha_1}(\boldsymbol{s}_1))) \sqsubseteq \psi_{\alpha_2}(\boldsymbol{s}_2) \sqsubseteq \psi_{\alpha_2^+}(f(\psi_{\alpha_2}(\boldsymbol{s}_2)))$, yielding $\Phi_\Omega(f)(\boldsymbol{s}_1) \sqsubseteq \Phi_\Omega(f)(\boldsymbol{s}_2)$. □

Lemma 3.2. *If $f : \mathcal{S}^I \to \mathcal{S}^I$ is a strictly causal ordinal function, then any fixed point of $\Phi_\Omega(f)$ is a fixed point of f.*

Proof. Consider an ordinal tuple $\boldsymbol{s} \in [\mathcal{S}^I]_\Omega$ such that $\boldsymbol{s} \neq f(\boldsymbol{s})$. Let α be the ordinal number of $\mathcal{T}(\boldsymbol{s} \sqcap f(\boldsymbol{s}))$, and β be the ordinal number of $\mathcal{T}(\boldsymbol{s})$.

If $\alpha = \beta$, then $\psi_\alpha(\boldsymbol{s}) = \boldsymbol{s}$ and $\boldsymbol{s} \sqsubset f(\boldsymbol{s})$. Hence, $\Phi_\Omega(f)(\boldsymbol{s}) = \psi_{\alpha^+}(f(\boldsymbol{s})) \sqsupset \boldsymbol{s}$.

If $\alpha^+ = \beta$, then $\psi_\alpha(\boldsymbol{s}) \sqsubset \boldsymbol{s}$ and $d_\mathcal{C}(\psi_\alpha(\boldsymbol{s}), \boldsymbol{s}) = \varphi(\tau)$, where τ is the greatest tag of $\mathcal{T}(\boldsymbol{s})$. Notice that $\mathcal{T}(\boldsymbol{s})$ has a greatest element in this case since β is a successor ordinal. Since f is strictly causal, $d_\mathcal{C}(f(\psi_\alpha(\boldsymbol{s})), f(\boldsymbol{s})) < d_\mathcal{C}(\psi_\alpha(\boldsymbol{s}), \boldsymbol{s})$, which implies $f(\psi_\alpha(\boldsymbol{s}))(\tau') \simeq f(\boldsymbol{s})(\tau')$ for any $\tau' \leq \tau$. However, $f(\boldsymbol{s})(\tau') \not\simeq \boldsymbol{s}(\tau')$ for some $\tau' \leq \tau$ and therefore $f(\psi_\alpha(\boldsymbol{s}))(\tau') \not\simeq \boldsymbol{s}(\tau')$. Hence, $\Phi_\Omega(f)(\boldsymbol{s}) = \psi_{\alpha^+}(f(\psi_\alpha(\boldsymbol{s}))) \neq \boldsymbol{s}$.

If $\alpha^+ \in \beta$, then trivially $\Phi_\Omega(f)(\boldsymbol{s}) = \psi_{\alpha^+}(f(\psi_\alpha(\boldsymbol{s}))) \neq \boldsymbol{s}$. □

Theorem 3.1. *Let $f : \mathcal{S}^I \to \mathcal{S}^I$ be a strictly causal ordinal function. Then $\mu x.\Phi_\Omega(f)(x)$ is the unique fixed point of f.*

Proof. The function $\Phi_\Omega(f)$ will be order-preserving, by Lemma 3.1, and will therefore have a least fixed point $\mu x.\Phi_\Omega(f)(x)$. By Lemma 3.2, $\mu x.\Phi_\Omega(f)(x)$ will be a fixed point of f. To establish its uniqueness, consider an ordinal tuple $\boldsymbol{s} \in [\mathcal{S}^I]_\Omega$ such that $\boldsymbol{s} \neq \mu x.\Phi_\Omega(f)(x)$, and suppose to the contrary that $\boldsymbol{s} = f(\boldsymbol{s})$. Since f is strictly causal, $d_\mathcal{C}(f(\mu x.\Phi_\Omega(f)(x)), f(\boldsymbol{s})) < d_\mathcal{C}(\mu x.\Phi_\Omega(f)(x), \boldsymbol{s})$, in contradiction to $d_\mathcal{C}(f(\mu x.\Phi_\Omega(f)(x)), f(\boldsymbol{s})) = d_\mathcal{C}(\mu x.\Phi_\Omega(f)(x), \boldsymbol{s})$ as derived by direct substitution. □

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