CPO of Continuous Functions

EECS 290N, Spring 2009

Edward Lee, EECS Department, UC Berkeley

February 25, 2009

Theorem: For CPOs $(A, \leq)$ and $(C, \leq)$, let $X \subset [A \to C]$ be the set of all continuous total functions from A to C. Then $(X, \leq)$ is a CPO under the pointwise order $\leq$.

Proof: First we need to show that X has a bottom element. This is easy. The bottom element is a function $g \in X$ where for all $a \in A$, $g(a) = \bot$. This function is obviously continuous and total, and hence is in X.

Second, we need to show that any chain of functions in X has a LUB and that the LUB is continuous. Consider a chain of functions

$$C = \{f_1, f_2, ...\} \subset X.$$

Since each of these functions is continuous (and hence monotonic), then for any $a \in A$, the following set is also a chain,

$$C'_a = \{f_1(a), f_2(a), ...\} \subset C.$$

Since C is a CPO, this set has a LUB. Define the function $g \colon A \to C$ such that for all $a \in A$,

$$g(a) = \bigvee C'_a.$$

Then in the pointwise order, it must be that

$$g = \bigvee C = \bigvee \{f_1, f_2, ...\}.$$

It remains to show that g is in X. To show this, we must show that it is continuous. We must show that for all chains $D \subset A$,

$$g(\bigvee D) = \bigvee \hat{g}(D).$$

Writing the elements of $D = \{d_1, d_2, ...\}$, observe that

$$\begin{aligned}
\bigvee \hat{g}(D) &= \bigvee \{g(d_1), g(d_2), ...\} \\
&= \bigvee \{\bigvee \{f_1(d_1), f_2(d_1), ...\}, \bigvee \{f_1(d_2), f_2(d_2), ...\} ...\} \\
&= \bigvee \{\bigvee \{f_1(d_1), f_1(d_2), ...\}, \bigvee \{f_2(d_1), f_2(d_2), ...\} ...\} \\
&= \bigvee \{\bigvee \hat{f}_1(D), \bigvee \hat{f}_2(D), ...\} \\
&= \bigvee \{f_1(\bigvee D), f_2(\bigvee D), ...\} \\
&= g(\bigvee D).
\end{aligned}$$

Note that the above use the axiom of choice, which states that given a set of sets, one can construct a new set by collecting one element from each of the sets in the set of sets.