EE 244: Fundamental Algorithms for System Modeling, Analysis, and Optimization
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A Logic Primer

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Logic

The α and ω in science.

- Basis of mathematics.
- Also of engineering.
  - Particularly useful for verification (model-checking = checking a model against a logical formula).
  - But also used in other domains, e.g.: Prolog, Datalog, UML OCL (Object Constraint Language), ...

A myriad of logics:

- Propositional logic
- First-order logic
- Temporal logic
- ...

What is logic?

Logic = Syntax + Semantics + Proofs

Proofs

- Manual, or
- Automated: Proofs = Computations

Example:

- Syntax: boolean formulas
- Semantics: boolean functions
- Proofs: is a formula satisfiable? valid (a tautology)?
  - E.g., for boolean logic: an NP-complete problem (a representative for many combinatorial problems).
  - Software tools (SAT solvers) routinely solve such problems today, even with tens of thousands of variables or more.

BOOLEAN LOGIC

(a.k.a. Propositional Logic or Propositional Calculus)

Syntax

Symbols:

- Constants: "false" and "true", or $0, 1$, or $\bot, \top$
- Variable symbols (*atomic propositions*): $p, q, ..., x, y, ...$
- Boolean connectives: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\rightarrow$ (implies), $\equiv$ or $\leftrightarrow$ (is equivalent to)
- Parentheses ( ): used to make syntax unambiguous

Expressions (formulas):

$$
\begin{aligned}
\phi \quad ::= \quad & 0 \mid 1 \mid p \mid q \mid ... \mid x \mid y \mid ... \\
& \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \\
& \mid \neg \phi' \\
& \mid \phi_1 \rightarrow \phi_2 \mid \phi_1 \equiv \phi_2
\end{aligned}
$$

Syntax

Examples:

$$x \vee \neg x$$

$$x \rightarrow y \rightarrow z \text{ (ambiguous)}$$

$$x \rightarrow (y \rightarrow z)$$

$$(x \rightarrow y) \rightarrow z$$

$$(p \rightarrow q) \leftrightarrow (0 \vee \neg p \vee q)$$

$\neg$ usually bings stronger, so $\neg p \vee q$ means $(\neg p) \vee q$.

Similarly, $p \wedge q \vee r$ usually means $(p \wedge q) \vee r$,
$p \wedge q \rightarrow a \vee b$ usually means $(q \wedge q) \rightarrow (a \vee b)$,
etc.

To be sure, better use parentheses!

Alternative syntax

- $\Rightarrow$ instead of $\rightarrow$, but in modern logic notation, $\Rightarrow$ is used for semantical entailment, as in "formula ϕ entails formula ϕ', or $\phi \Rightarrow \phi'$, meaning that ϕ' is true when ϕ is true"
- $\Leftrightarrow$ instead of $\leftrightarrow$
- $+$ instead of $\vee$
- $\cdot$ instead of $\wedge$ (often omitted altogether)
- $\overline{x}$ instead of $\neg x$

E.g.,

$$xy + \overline{z}$$

instead of

$$(x \wedge y) \vee (\neg z)$$

Semantics

The **meaning** of logical formulas.

E.g., what is the semantics of a boolean formula such as $p \rightarrow q$?

"If p, then q", of course.

So, why do we even need to talk about semantics?

Semantics

What is the meaning of a boolean formula?

Different views (all equivalent):

- A "truth table".
- A boolean function.
- A set containing the "solutions" ("models") of the formula.

Why not consider the syntax itself to be the semantics?

Semantics

Formula:

$$x \wedge (y \vee z)$$

Truth table:

x	y	z	result
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

An equivalent formula (different syntax, same semantics):

$$(x \wedge y) \vee (x \wedge z)$$

Semantics

Boolean function: a function $f : \mathbb{B}^n \to \mathbb{B}^m$, where $\mathbb{B} = \{0, 1\}$.

Formula:

$$x \wedge (y \vee z)$$

defines[1] the boolean function: $f : \mathbb{B}^3 \to \mathbb{B}$ such that:

$$f(0,0,0) = 0$$
$$f(0,0,1) = 0$$
$$\ldots$$

[1]assuming an order on the variables: (1) x, (2) y, (3) z.

Semantics

A formula $\phi : x \wedge (y \vee z)$ defines[2] a subset $[\![\phi]\!] \subseteq \mathbb{B}^3$:

$$[\![\phi]\!] = \{(1,0,1), (1,1,0), (1,1,1)\}$$

This is the set of "solutions": all assignments to x, y, z which make the formula true.

To be independent from an implicit order on variables, we can also view $[\![\phi]\!]$ as a set of *minterms*:

$$[\![\phi]\!] = \{x\overline{y}z, xy\overline{z}, xyz\}$$

We can also view $[\![\phi]\!]$ as a set of *sets of atomic propositions*:

$$[\![\phi]\!] = \{\{x,z\}, \{x,y\}, \{x,y,z\}\}$$

What is the type of $[\![\phi]\!]$ in this last case?
$[\![\phi]\!] \subseteq \mathbb{B}^P = 2^P$ where P is the set of atomic propositions ($=$ formula variables).

[2]assuming an order on the variables: (1) x, (2) y, (3) z.

Semantics: satisfaction relation

Satisfaction relation:

$$a \models \phi$$

means a is a "solution" (or model) of ϕ (or "a *satisfies* ϕ").

So

$$a \models \phi \qquad \text{iff} \qquad a \in [\![\phi]\!]$$

Semantics: satisfiability, validity

A formula ϕ is *satisfiable* if $[\![\phi]\!]$ is non-empty, i.e., if there exists $a \models \phi$.

A formula ϕ is *valid* (a *tautology*) if for all a, $a \models \phi$, i.e., if $[\![\phi]\!] = 2^P$.

PREDICATE LOGIC

Limitations of propositional logic

All humans are mortal.

How to write it in propositional logic?

We can associate one proposition p_i for every human i, with the meaning *"human i is mortal"*, and then state:

$$p_1 \wedge p_2 \wedge \cdots \wedge p_{7000000000}$$

But even this is not enough, since we also want to talk about future generations.

Expressing this in (first-order) predicate logic

$$\forall x : H(x) \rightarrow M(x)$$

x: variable

H, M: predicates (functions that return "true" or "false")

$H(x)$: "x is human".

$M(x)$: "x is mortal".

$\forall$: "for all" quantifier.

First-Order Predicate Logic (FOL) – Syntax

Terms:

$$t \quad ::= \quad x \mid c \mid f(t_1, ..., t_n)$$

where x is any variable symbol, c is any constant symbol,[3] and f is any function symbol of some arity n.

Formulas:

$$\begin{array}{rcl} \phi & ::= & P(t_1, ..., t_n) \\ & & \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\neg \phi) \mid \cdots \\ & & \mid (\forall x : \phi) \mid (\exists x : \phi) \end{array}$$

where P is any predicate symbol of some arity n, and t_i are terms.

[3] constants can also be seen as functions of arity 0

FOL – Syntax

Example:

$$\forall x : x > 0 \to x + 1 > 0$$

or, more pedantically:

$$\forall x : \; >(x, 0) \to \; >(+(x, 1), 0)$$

- $0, 1$: constants
- x: variable symbol
- $+$: function symbol of arity 2
- $>$: predicate symbol of arity 2

FOL – Syntax

Note:

- This is also a syntactically well-formed formula:

$$x > 0 \to x + 1 > 0$$

- so is this:

$$\forall x : x > y$$

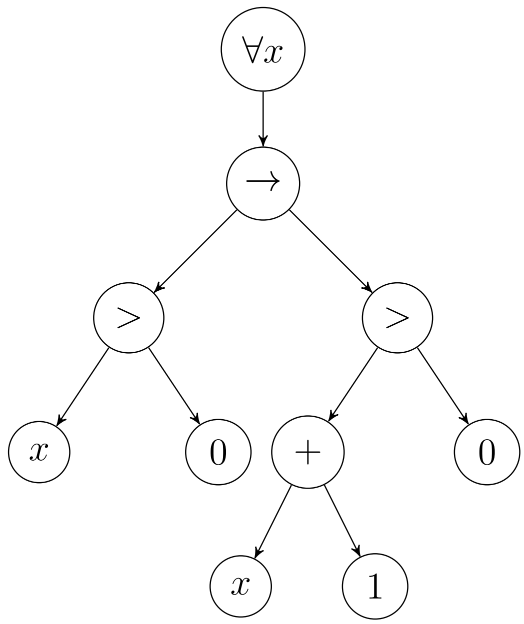
- or this:

$$\forall x : 2z > f(y)$$

Parse Tree of Formula

Formula: $\forall x : x > 0 \to x + 1 > 0$

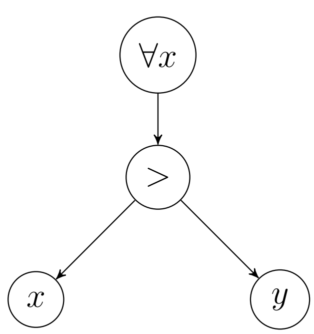
Parse tree:



Free and Bound Variables

Formula: $\forall x : x > y$

Parse tree:



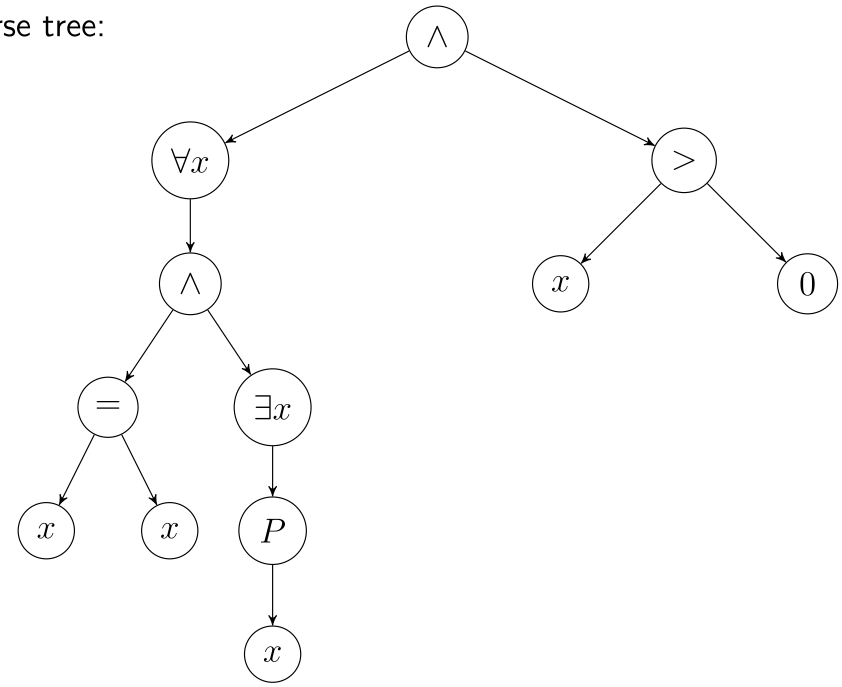
y is *free* in the formula: no ancestor of the leaf node y is a node of the form $\forall y$ or $\exists y$.

x is *bound* in the formula: has ancestor $\forall x$.

Scope of Variables

Formula: $(\forall x : x = x \wedge \exists x : P(x)) \wedge x > 0$

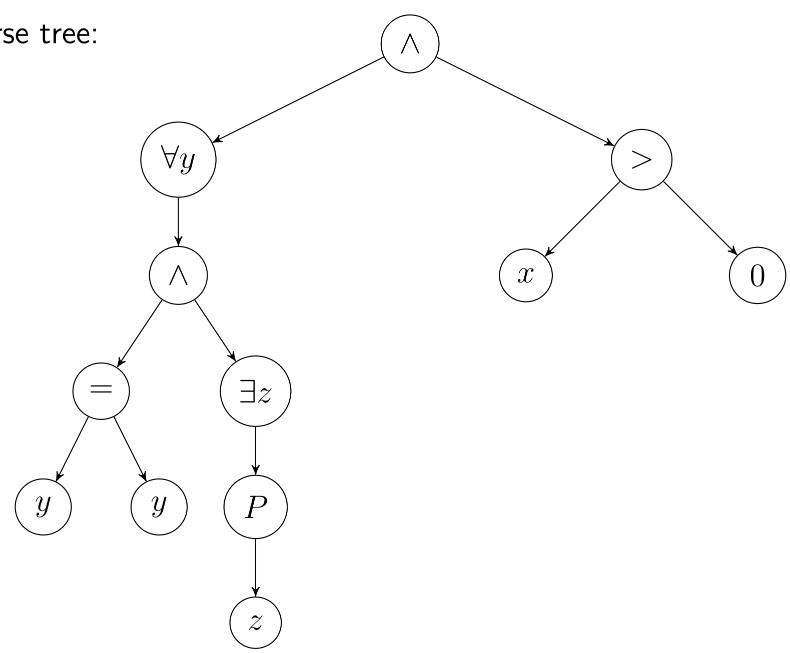
Parse tree:



Renaming

Formula: $(\forall x : x = x \wedge \exists x : P(x)) \wedge x > 0 \rightsquigarrow (\forall y : y = y \wedge \exists z : P(z)) \wedge x > 0$

Parse tree:



FOL – Semantics

In propositional logic, a "solution" (model) of a formula was simply an assignment of truth values to the propositional variables. E.g.,

$$\underbrace{(p := 1, q := 0)}_{\textit{model}} \models \underbrace{p \lor q}_{\textit{formula}}$$

What are the "solutions" (models) of predicate logic formulas?

$$\underbrace{???}_{\textit{model}} \models \underbrace{\forall x : P(x) \to \exists y : Q(x, y)}_{\textit{formula}}$$

Cannot give meaning to the formula without first giving meaning to P, Q.

FOL – Semantics

Let $\mathcal{P}$ and $\mathcal{F}$ be the sets of predicate and function symbols (for simplicity $\mathcal{F}$ also includes the constants).

A *model* $\mathcal{M}$ for the pair $(\mathcal{P}, \mathcal{F})$ consists of the following:

- A non-empty set $\mathcal{U}$, the *universe* of concrete values.
- For each 0-arity symbol $c \in \mathcal{F}$, a concrete value $c_{\mathcal{M}} \in \mathcal{U}$.
- For each $f \in \mathcal{F}$ with arity n, a function $f_{\mathcal{M}} : \mathcal{U}^n \to \mathcal{U}$.
- For each $P \in \mathcal{P}$ with arity n, a set $P_{\mathcal{M}} \subseteq \mathcal{U}^n$.

Note:

- c, f, P are just symbols (syntactic objects).
- $c_{\mathcal{M}}, f_{\mathcal{M}}, P_{\mathcal{M}}$ are semantical objects (values, functions, sets).

FOL – Semantics

Example:

$$\forall x : P(x) \rightarrow \exists y : Q(x, y)$$

Let $\mathcal{M}$ be such that

- $\mathcal{U} = \mathbb{N}$: the set of naturals.
- $P_{\mathcal{M}} = \{0, 2, ...\}$: the set of even naturals.
- $Q_{\mathcal{M}} = \{(0, 1), (1, 2), (2, 3), ...\}$: the set of pairs $(n, n + 1)$, for $n \in \mathbb{N}$.

Then the statement above is true.

Of course, it could have been written "more clearly" (for a human):

$$\forall x : \textit{Even}(x) \rightarrow \exists y : y = x + 1$$

... but a computer (or a person who does not speak English) is equally clueless as to what P or *Even* means ...

FOL – Semantics

Example:

$$\forall x : P(x) \rightarrow \exists y : Q(x, y)$$

Let $\mathcal{M}'$ be another model such that

- $\mathcal{U} = \mathbb{N}$: the set of naturals.
- $P_{\mathcal{M}'} = \{0, 2, ...\}$: the set of even naturals.
- $Q_{\mathcal{M}'} = \{(1, 0), (3, 1), (5, 2), ...\}$: the set of pairs $(2n + 1, n)$, for $n \in \mathbb{N}$.

Then the statement above is false.

FOL – Semantics

What is the meaning of $\forall x : x > y$?

Undefined if we know nothing about the value of y.

We need one more thing: *environments* (or "look-up tables" for variables).

Environment:

$$l : \text{VariableSymbols} \to \mathcal{U}$$

assigns a concrete value to every variable symbol.

Notation:

$$l[x \rightsquigarrow a]$$

is a new environment l' such that $l'(x) = a$ and $l'(y) = l(y)$ for any other variable y.

FOL – Semantics: Giving concrete values to terms

Once we have $\mathcal{M}$ and l, every term evaluates to a concrete value in $\mathcal{U}$.

Example:

$\mathcal{M}$: $\mathcal{U} = \mathbb{N}$, "0" $= 0$, "1" $= 1$, ..., $+ =$ addition function,
...

l: $x \rightsquigarrow 2$, $y \rightsquigarrow 1$

term t	value $\mathcal{M}_l(t)$
$x + 1$	3
$x \cdot y$	2
...	

For a term t, we denote this value by $\mathcal{M}_l(t)$.

FOL – Semantics

Finally we can define the satisfaction relation for first-order predicate logic ($\mathcal{M}$: model, l: environment, ϕ: formula):

$$\mathcal{M}, l \models \phi$$

$\mathcal{M}, l \models P(t_1, ..., t_n)$	iff	$\big(\mathcal{M}_l(t_1), ..., \mathcal{M}_l(t_n)\big) \in P_{\mathcal{M}}$
$\mathcal{M}, l \models \phi_1 \wedge \phi_2$	iff	$\mathcal{M}, l \models \phi_1$ and $\mathcal{M}, l \models \phi_2$
$\mathcal{M}, l \models \neg\phi$	iff	$\mathcal{M}, l \not\models \phi$
$\mathcal{M}, l \models \forall x : \phi$	iff	for all $a \in \mathcal{U} : \mathcal{M}, l[x \rightsquigarrow a] \models \phi$ holds
$\mathcal{M}, l \models \exists x : \phi$	iff	for some $a \in \mathcal{U} : \mathcal{M}, l[x \rightsquigarrow a] \models \phi$ holds

FOL – Semantics: Satisfiability, Validity

A FOL formula ϕ is *satisfiable* if there exist $\mathcal{M}, l$ such that $\mathcal{M}, l \models \phi$ holds.

A formula ϕ is *valid* (a *tautology*) if for all $\mathcal{M}, l$, it holds $\mathcal{M}, l \models \phi$.

FOL – Semantics: Satisfiability, Validity

Examples:

1. $$\forall x : P(x) \to P(x)$$

   Valid.

2. $$x \geq 0 \wedge f(x) \geq 0 \wedge y \geq 0 \wedge f(y) \geq 0 \wedge x \neq y$$

   Satisfiable.

   Example model: $\mathcal{U} = \mathbb{N}$, $x \mapsto 0$, $y \mapsto 1$, $f(\_) \mapsto 0$, $\neq$ is the "not equal to" relation on $\mathbb{N}$: $\neq \mapsto \{(0,1),(0,2),...,(1,0),(1,2),...\}$.

3. $$x + 2 = y \wedge f(\texttt{read}(\texttt{write}(A, x, 3), y - 2)) \neq f(y - x + 1)$$

   Satisfiable with a non-standard interpretation of $+, -$ or $\texttt{read}, \texttt{write}$.

   Unsatisfiable with the standard interpretation of those symbols (theories of arithmetic and arrays). Why?

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