EE 144/244: Fundamental Algorithms for System Modeling, Analysis, and Optimization
Fall 2016

Timed Automata

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Timed Automata

- A formal model for dense-time systems [Alur and Dill(1994)]
- Developed mainly with verification in mind:
  - in the basic TA variant, model-checking is decidable
- But also an elegant theoretical extension of the standard theory of regular and ω-regular languages.
- Many different TA variants, some undecidable.
- We will look at a basic variant.

Timed Automaton

A TA is a tuple

$$(C, Q, q_0, \mathsf{Inv}, \triangleright)$$

- C: finite set of *clocks*
- Q: finite set of *control states*; $q_0 \in Q$: initial control state
- Inv: a function assigning to each $q \in Q$ an *invariant*
- $\triangleright$: a finite set of *actions*, each being a tuple

$$(q, q', g, C')$$

  - $q, q' \in Q$: source and destination control states
  - g: clock *guard*
  - C': set of clocks to *reset* to 0, $C' \subseteq C$
- Invariants and guards are simple constraints on clocks, e.g.,

$$c \leq 1, \quad 0 < c_1 < 2 \wedge c_2 = 4, \quad \text{etc.}$$

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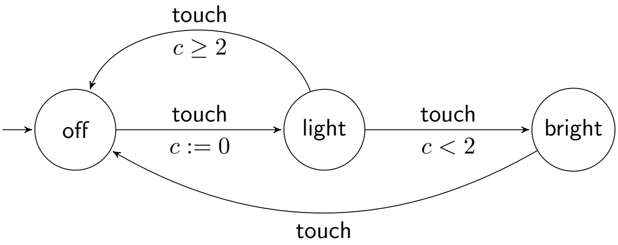
Can also have atomic propositions labeling control states, labels on actions, communication via shared memory or message passing, etc.

Example: Timed Automaton

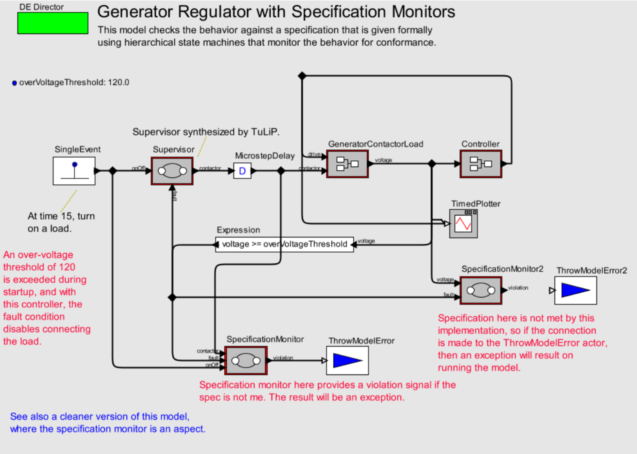
A simple light controller:



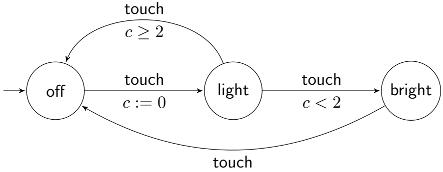
- $C = \{c\}$
- $Q = \{$ off, light, bright $\}$
- $q_0 =$ off
- touch: action label (can be seen as the input symbol)
- Inv$(q) = \mathit{true}$ for all $q \in Q$
- Actions: (off, light, $\mathit{true}, \{c\}$), (light, off, $c \geq 2, \{\}$), ...

Event-based vs. state-based models

High-level:



Low-level:



Timed Automata: Semantics

A TA $(C, Q, q_0, \mathsf{Inv}, \triangleright)$ defines a transition system

$$(S, S_0, R)$$

such that

- Set of states: $S = Q \times \mathbb{R}_+^C$
  - $\mathbb{R}_+^C$: the set of all functions $v : C \to \mathbb{R}_+$
  - each v is called a *valuation*: it assigns a value to every clock
- Set of initial states: $S_0 = \{(q_0, v_0)\}$, where we define $v_0(c) = 0$ for all $c \in C$ (i.e., all clocks are initially set to 0)
- Set of transitions: $R = R_t \cup R_d$
  - R_t: set of transitions modeling passage of time
  - R_d: set of discrete transitions ("jumps" between control states)

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- Set of initial states: $S_0 = \{(q_0, v_0)\}$, where we define $v_0(c) = 0$ for all $c \in C$ (i.e., all clocks are initially set to 0)
  - we could also define $S_0 = \{q_0\} \times \mathbb{R}_+^C$ – what does this say?
- Set of transitions: $R = R_t \cup R_d$
  - R_t: set of transitions modeling passage of time
  - R_d: set of discrete transitions ("jumps" between control states)

Timed Automata: Discrete and Time Transitions

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\begin{aligned}
R_t &= \{\big((q,v),(q,v+t)\big) \mid \forall t' \le t : v + t' \models \mathsf{Inv}(q)\} \\
R_d &= \{\big((q,v),(q',v')\big) \mid \exists a = (q,q',g,C') \in \rhd : \\
&\qquad\qquad v \models g \wedge v' = v[C' := 0]\}
\end{aligned}
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where:

- $v + t$ is a new valuation u such that $u(c) = v(c) + t$ for all c
- if g is a constraint, then $v \models g$ means v satisfies g
- $v[C' := 0]$ is a new valuation u such that $u(c) = 0$ if $c \in C'$ and $u(c) = v(c)$ otherwise

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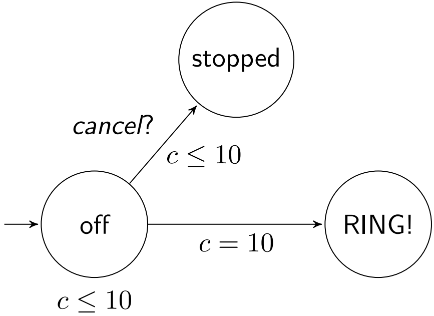
Instead of $\big((q,v),(q,v+t)\big) \in R_t$ we write $(q,v) \xrightarrow{t} (q,v+t)$.
Instead of $\big((q,v),(q',v')\big) \in R_d$ we write $(q,v) \xrightarrow{a} (q',v')$.

Example: Alarm Modeled as a Timed Automaton



Inv(off) $= c \leq 10$: automaton cannot spend more than 10 time units at control state "off".

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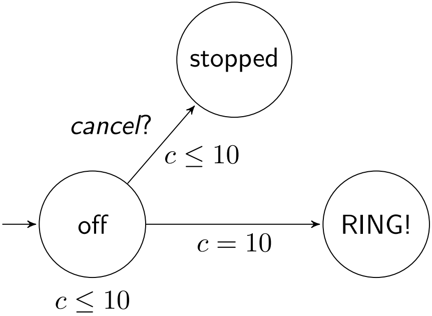

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What if we omit the invariant?

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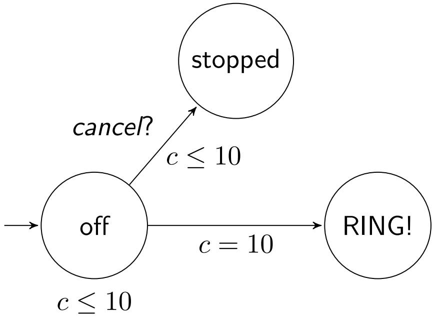

Does it work correctly if *cancel* arrives exactly when $c = 10$?

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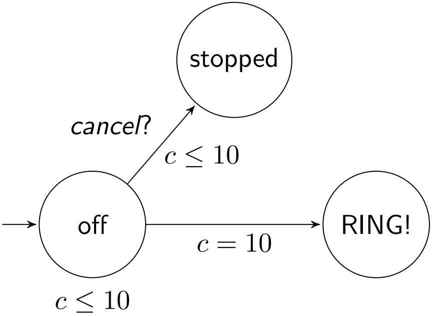

Does it work correctly if *cancel* arrives exactly when $c = 10$?

Depends on the semantics of composition: if it's non-deterministic (as usually done) then alarm may still ring. Otherwise, must give higher priority to the cancel transition.

Timed Automata Model-Checking: Reachability

- Basic question: is a given **control state** q **reachable**?
  - i.e., does there exist some reachable state $s = (q, v)$ in the transition system defined by the timed automaton?
- Many interesting questions about timed automata can be reduced to this question.

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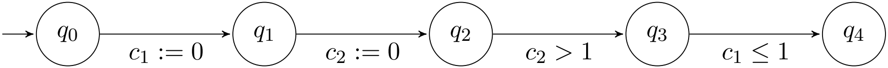
  - i.e., does there exist some reachable state $s = (q, v)$ in the transition system defined by the timed automaton?
- Many interesting questions about timed automata can be reduced to this question.
- Is the basic control-state reachability question decidable?

Timed Automata Reachability

Not the same as discrete-state reachability!



q_4 is reachable if we ignore the timing constraints.
But is it really reachable?

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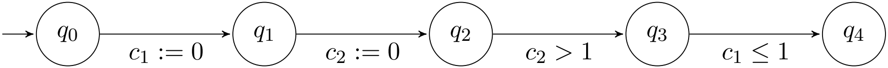
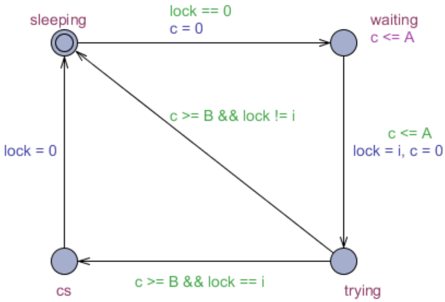


q_4 is reachable if we ignore the timing constraints.
But is it really reachable?

No: at q_3, $c_2 > 1$ and $c_1 \geq c_2$, therefore $c_1 > 1$ also.

Timed Automata Model-Checking: Reachability

A less obvious example: Fischer's mutual exclusion protocol.



Suppose we have many processes, each behaving like the TA above.
Is mutual-exclusion guaranteed?
I.e., at most 1 process is in critical section (control state cs) at any given time.

Timed Automata Model-Checking: Reachability

Brute-force idea: exhaustive state-space exploration of the transition system defined by the timed automaton

- does not work since state-space is infinite (even uncountable)

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- does not work since state-space is infinite (even uncountable)

Yet problem is decidable! [Alur-Dill'94]

Key idea:

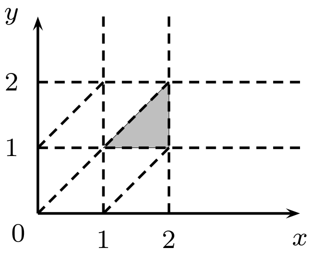
- *Region equivalence*: partitions the state-space into **finite** number of equivalence classes (*regions*)
- Perform reachability on finite (abstract) state-space
- Can prove that q is reachable in the abstract space iff it is reachable in the concrete space

The Region Equivalence

Key idea: two valuations v_1, v_2 are equivalent iff:

1. v_1 satisfies a guard g iff v_2 satisfies g.
2. v_1 can lead to some v_1' satisfying a guard g with a discrete transition iff v_2 can do the same.
3. v_1 can lead to some v_1' satisfying a guard g with a time transition iff v_2 can do the same.

Region = equivalence class w.r.t. region equivalence = set of all equivalent valuations.

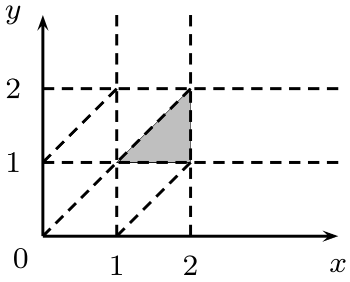


Region in gray:
$1 < x < 2 \wedge 1 < y < 2 \wedge x > y$.
Other regions:
$x = y = 0$,
$0 < x = y < 1$,
$x = 0 \wedge 0 < y < 1$,
etc.

Pictures in this and other slides taken from [Bouyer(2005)].

The Region Equivalence: Finiteness

Finite number of equivalence classes: bounded by constant $c =$ maximal constant appearing in a guard or invariant.



Some regions are unbounded, e.g.:

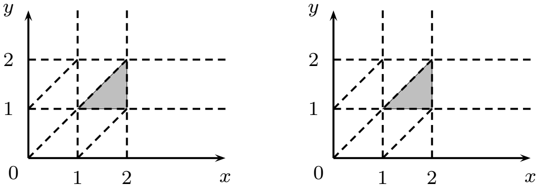
$x > 2 \wedge 0 < y < 1$

$x > 2 \wedge y > 2$

etc.

The Region Graph

A graph of regions: one region space for each control location.



q_1 q_2

Nodes: pairs (q, r) where

- q is a control location of the timed automaton.
- r is a region.

Two types of edges:

- $(q, r) \xrightarrow{a} (q', r')$: discrete transition
- $(q, r) \xrightarrow{time} (q, r')$: time transition

Decidability

Theorem ([Alur and Dill(1994)])

$\exists$ reachable state (q, v) in a timed automaton
iff
$\exists$ reachable node (q, r) in its region graph.

Finite # regions and control states $\Rightarrow$ Region graph is finite $\Rightarrow$ Reachability is decidable.

The Problem with Regions

STATE EXPLOSION!

Worst-case number of regions:

$$O(2^n \cdot n! \cdot c^n)$$

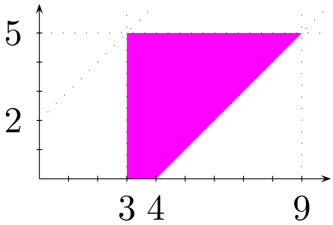
where n is the number of clocks and c is the maximal constant.

This is actually often close to the actual number of regions $\Rightarrow$ no practical tool uses regions.

Model-checkers for TA (Uppaal, Kronos, ...) have improved upon the region-graph idea and use *symbolic* techniques.

From Regions to Zones

Zone: a convex union of regions, e.g., $x_1 \geq 3 \wedge x_2 \leq 5 \wedge x_1 - x_2 \leq 4$.



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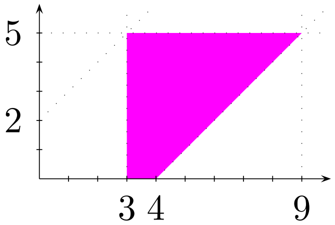

Key property: can be represented efficiently using *difference bound matrices* (DBMs) [Dill(1989)].

$$x_1 \geq 3 \ \wedge x_2 \leq 5 \wedge x_1 \leq x_2 + 4 \quad : \quad \begin{array}{c} \\ x_0 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \\ \left(\begin{array}{ccc} \infty & -3 & \infty \\ \infty & \infty & 4 \\ 5 & \infty & \infty \end{array} \right) \end{array}$$

Symbolic Manipulations of Zones using DBMs

DBMs = the BDDs of the timed automata world.

Time elapse, guard intersection, clock resets, are all easily implementable in DBMs.

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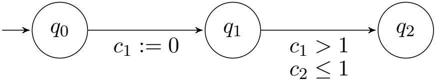
Time elapse, guard intersection, clock resets, are all easily implementable in DBMs.

Is zone union implementable with DBMs?

No! The union of two zones in general is not a zone.

$\Rightarrow$ often state explosion even with zones ...

Timed Automata Reachability: Simple Example



Is q_2 reachable? (initially, $c_1 = c_2 = 0$)

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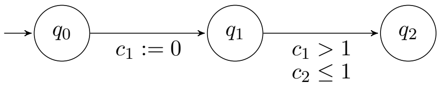

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step	symbolic state
initially:	$(q_0, c_1 = 0 \wedge c_2 = 0)$

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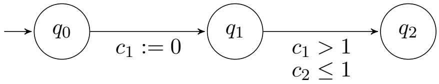

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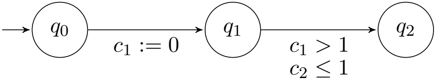

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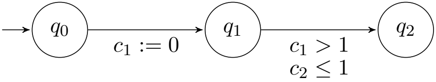

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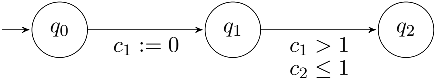

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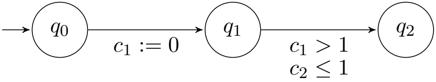

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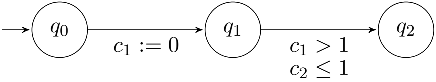

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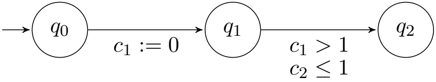

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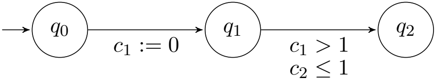

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take discrete transition to q_2:	cannot because
	$c_1 > 1 \wedge c_2 \leq 1 \wedge c_2 \geq c_1$ is UNSAT

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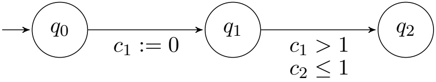

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therefore q_2 not reachable	

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