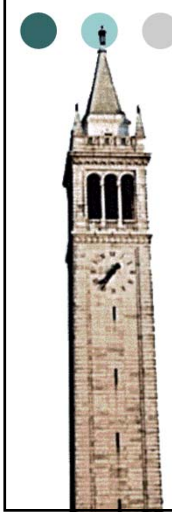


Fundamental Algorithms for System Modeling, Analysis, and Optimization

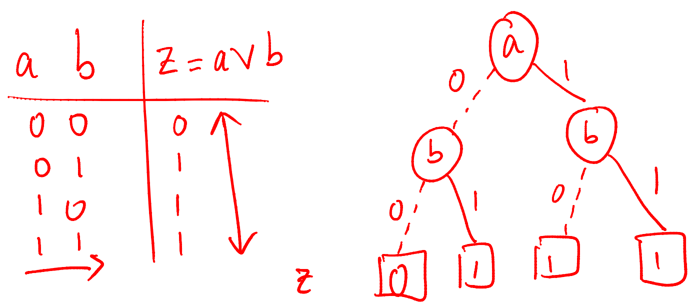
Edward A. Lee, Jaijeet Roychowdhury, Sanjit A. Seshia
UC Berkeley
EECS 144/244

Lec 12: Binary Decision Diagrams (BDDs)

Thanks to R. Rutenbar for some slides

Boolean Function Representations

- Syntactic: e.g.: CNF, DNF (SOP), Circuit
- Semantic: e.g.: Truth table, Binary Decision Tree, BDD



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Reduced Ordered BDDs

- Introduced by Randal E. Bryant in mid-80s
  - IEEE Transactions on Computers 1986 paper is one of the most highly cited papers in EECS
- Useful data structure to represent Boolean functions
  - Applications in logic synthesis, verification, program analysis, AI planning, …
- Commonly known simply as BDDs
  - Lee [1959] and Akers [1978] also presented BDDs, but not **RO**BDDs
- Many variants of BDDs have also proved useful
- Links to coding theory (trellises), etc.

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RoadMap for this Lecture

- Cofactor of a Boolean function
- From truth table to BDD
- Properties of BDDs
- Operating on BDDs
- Variants

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Cofactors

- A Boolean function F of n variables $x_1, x_2, \ldots, x_n$

  $$F : \{0,1\}^n \rightarrow \{0,1\}$$

- Suppose we define new Boolean functions of n-1 variables as follows:

  $F_{x_1}(x_2, \ldots, x_n) = F(1, x_2, x_3, \ldots, x_n)$

  $F_{x_1'}(x_2, \ldots, x_n) = F(0, x_2, x_3, \ldots, x_n)$

- F_{x_i} and $F_{x_i'}$ are called cofactors of F. F_{x_i} is the positive cofactor, and $F_{x_i'}$ is the negative cofactor

Shannon Expansion

- $F(x_1, \ldots, x_n) = x_i \,.\, F_{x_i} + x_i' \,.\, F_{x_i'}$

- Proof?

Case-splitting

$x_i = 1$: LHS = $F(x_1, \ldots \underset{i}{1}, \ldots x_n)$

RHS = F_{x_i}

$x_i = 0$: ...

Shannon expansion with many variables

- $F(x, y, z, w) =$

  $xy\, F_{xy} + x'y\, F_{x'y} + xy'\, F_{xy'} + x'y'\, F_{x'y'}$

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Properties of Cofactors

- Suppose you construct a new function H from two existing functions F and G: e.g.,
  - H = F'
  - H = F.G
  - H = F + G
  - Etc.

- What is the relation between cofactors of H and those of F and G?

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Very Useful Property

- Cofactor of NOT is NOT of cofactors
- Cofactor of AND is AND of cofactors
- …

- Works for any binary operator

k-ary

BDDs from Truth Tables

Truth Table

↓

Binary Decision Tree

↓

Binary Decision Diagram (BDD)

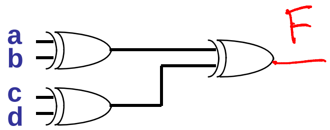
↓

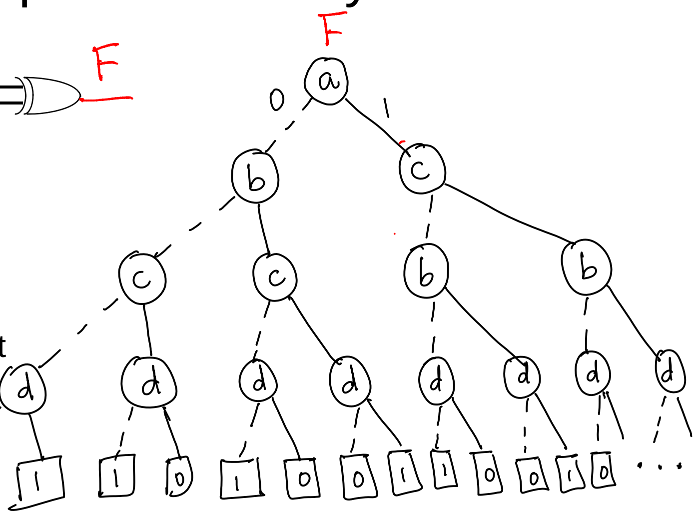
Ordered Binary Decision Diagram (OBDD)

↓

Reduced Ordered Binary Decision Diagram
(ROBDD, simply called BDD)

Example: Odd Parity Function





Edge labels along a root-leaf path form an assignment to a,b,c,d

Binary Decision Tree

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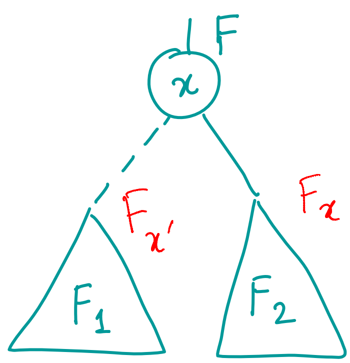
Nodes & Edges



- How is F related to x, F_1, F_2?

$F = xF_2 + x'F_1$

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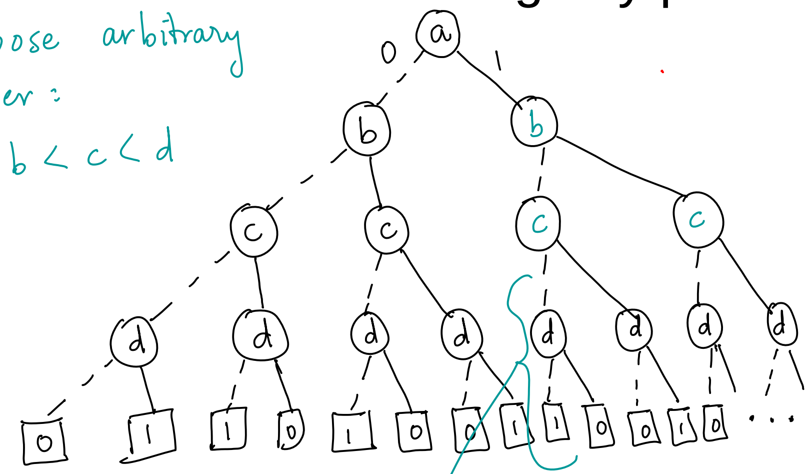
Ordering: variables appear in same order from root to leaf along any path

Impose arbitrary order:

$a < b < c < d$



Each node is some cofactor of the function

Why didn't this part change?

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Reduction

- Identify Redundancies
- 3 Rules:

1. Merge equivalent leaves
2. Merge isomorphic nodes
3. Eliminate redundant tests

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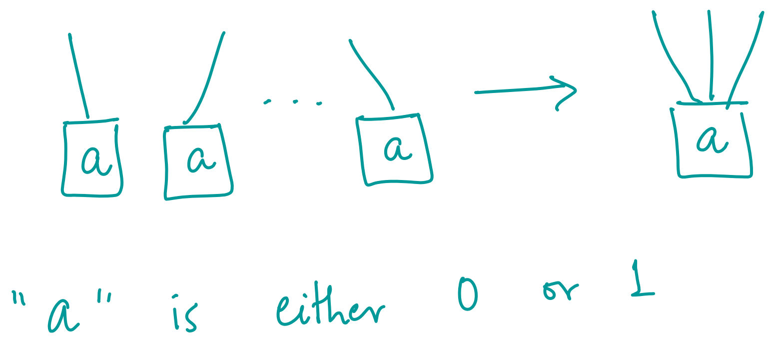
Merge Equivalent Leaves



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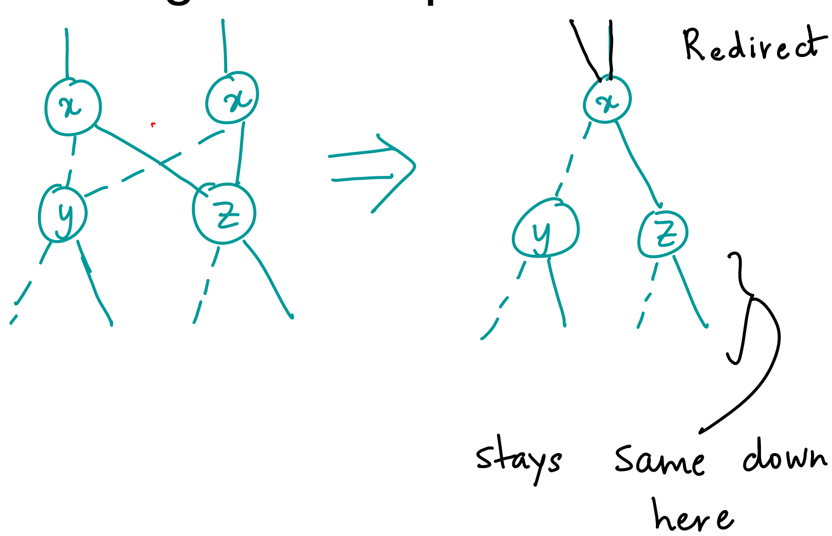
Merge Isomorphic Nodes



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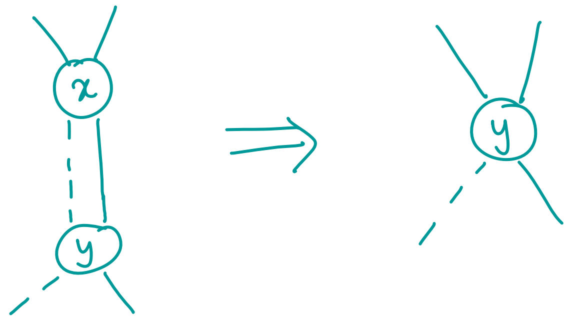
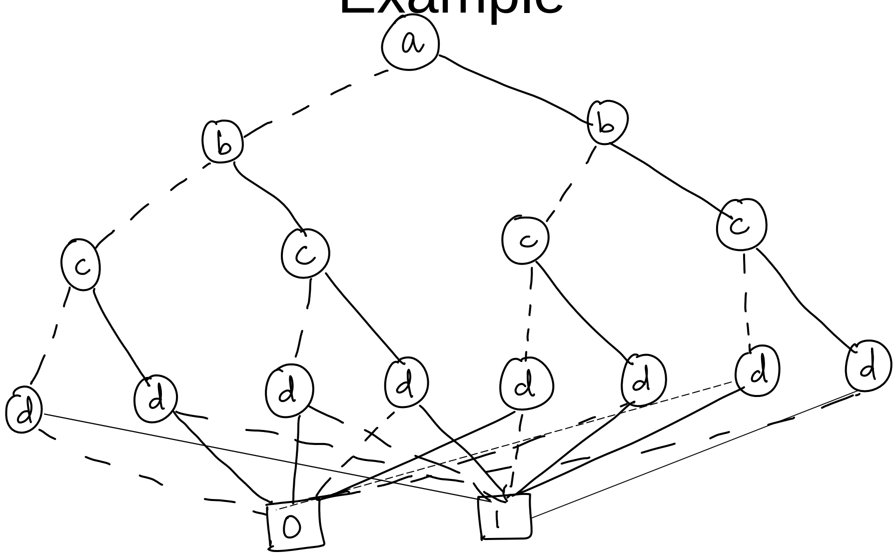
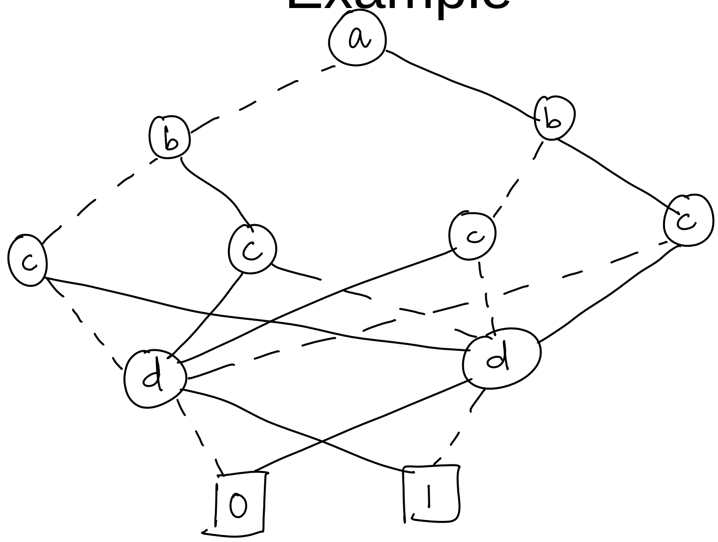
Eliminate Redundant Tests



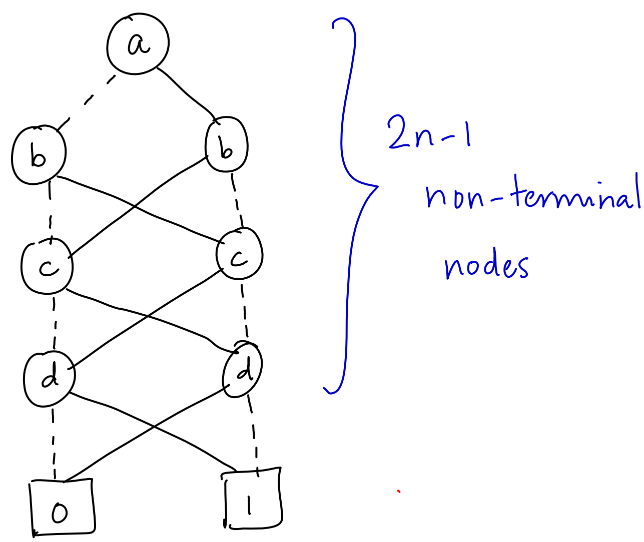
Example



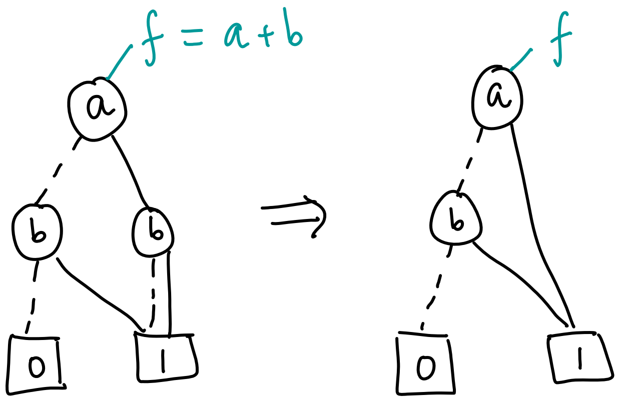
Example



Final ROBDD for Odd Parity Function



Example of Rule 3



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What can BDDs be used for?

- Uniquely representing a Boolean function
  - And a Boolean function can represent sets
- Symbolic simulation of a combinational (or sequential) circuit
- Equivalence checking and verification
  - Satisfiability (SAT) solving
- Finding / counting *all* solutions to a SAT (combinatorial) problem
- Operations on “quantified” Boolean formulas

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Set $S = \{e_1, e_2, e_3, \ldots, e_N\}$

$e_i = \langle 0001011\ldots\rangle = \langle x_1 x_2 \ldots x_k\rangle$

$k = \lceil \log N \rceil$

$S = \text{ON-SET}(f)$

$$f(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } \langle x_1 x_2 \ldots x_k\rangle \in S \\ 0 & \text{o.w.} \end{cases}$$

characteristic Boolean f^n of S

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(RO)BDDs are canonical

- Theorem (R. Bryant): If G, G' are ROBDD's of a Boolean function f with k inputs, using same variable ordering, then G and G' are identical.

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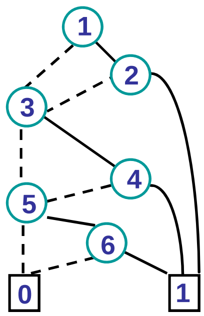
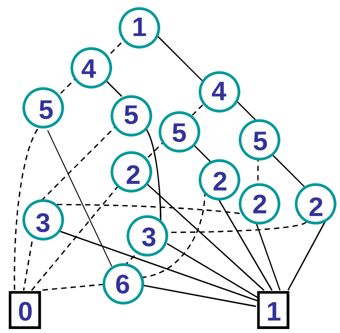
Sensitivity to Ordering

- Given a function with n inputs, one input ordering may require exponential # vertices in ROBDD, while other may be linear in size.
- Example: $f = x_1 x_2 + x_3 x_4 + x_5 x_6$

$x_1 < x_2 < x_3 < x_4 < x_5 < x_6$



$x_1 < x_4 < x_5 < x_2 < x_3 < x_6$



Constructing BDDs in Practice

- Strategy: Define how to perform basic Boolean operations
- Build a few core operators and define everything else in terms of those

Advantage:
- **Less programming work**
- **Easier to add new operators later by writing “wrappers”**

Core Operators

- Just two of them!

1. Restrict(Function F, variable v, constant k)
   - Shannon cofactor of F w.r.t. v=k

2. ITE(Function I, Function T, Function E)
   - “if-then-else” operator

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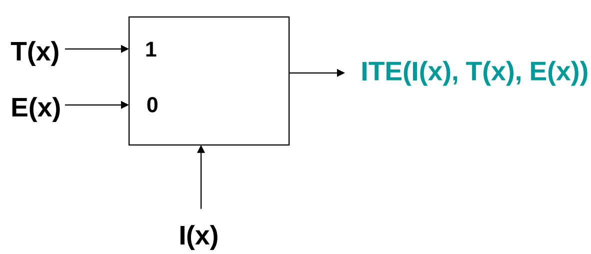
ITE

- Just like:
  – “if then else” in a programming language
  – A mux in hardware
- ITE(I(x), T(x), E(x))
  – If I(x) then T(x) else E(x)



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The ITE Function

- ITE(I(x), T(x), E(x))
- =
- I(x) . T(x)   + I'(x). E(x)

What good is the ITE?

- How do we express
- NOT?
- OR?
- AND?

How do we implement ITE?

- Divide and conquer!
- Use Shannon cofactoring…
- Recall: Operator of cofactors is Cofactor of operators…

ITE Algorithm

```
ITE (bdd I, bdd T, bdd E) {
  if (terminal case) { return computed result; }
  else { // general case
    Let x be the topmost variable of I, T, E;
    PosFactor = ITE(I_x , T_x , E_x) ;
    NegFactor = ITE(I_x' , T_x' , E_x');
    R = new node labeled by x;
    R.low = NegFactor; // R.low is 0-child of R
    R.high = PosFactor; // R.high is 1-child of R
    Reduce(R);
    return R;
}
```

Terminal Cases (complete these)

- ITE(1, T, E) =
- ITE(0, T, E) =
- ITE(I, T, T) =
- ITE(I, 1, 0) =
- …

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General Case

- Still need to do cofactor (Restrict)
- How hard is that?
  - Which variable are we cofactoring out? (2 cases)

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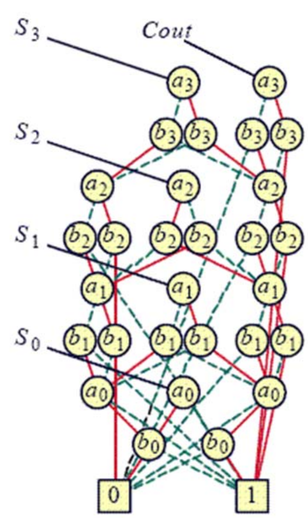
Practical Issues

- Previous calls to ITE are cached
  - "memoization"

- Every BDD node created goes into a "unique table"
  - Before creating a new node R, look up this table
  - Avoids need for reduction

Sharing: Multi-Rooted DAG

- BDD for 4-bit adder: 5 output bits → 5 Boolean functions
- Each output bit (of the sum & carry) is a distinct rooted BDD
- But they share sub-DAGs



More on BDDs

- Circuit width and bounds on BDD size (reading exercise – slide summary posted)
- Dynamically changing variable ordering
- Some BDD variants

Sifting

- Dynamic variable re-ordering, proposed by R. Rudell
- Based on a primitive "swap" operation that interchanges x_i and x_{i+1} in the variable order
  - Key point: the swap is a local operation involving only levels i and i+1
- Overall idea: pick a variable x_i and move it up and down the order using swaps until the process no longer improves the size
  - A "hill climbing" strategy

Some BDD Variants

- Free BDDs (FBDDs)
  - Relax the restriction that variables have to appear in the same order along all paths
  - How can this help? → smaller BDD
  - Is it canonical? → NO

Some BDD Variants

- MTBDD (Multi-Terminal BDD)
  - Terminal (leaf) values are not just 0 or 1, but some finite set of numerical values
  - Represents function of Boolean variables with non-Boolean value (integer, rational)
    - E.g., input-dependent delay in a circuit, transition probabilities in a Markov chain
  - Similar reduction / construction rules to BDDs

Some BDD packages

- CUDD – from Colorado University, Fabio Somenzi's group
  - We will use the PerlDD front-end to CUDD in HW3
- BuDDy – from IT Univ. of Copenhagen

Reading

- Bryant's 1992 survey paper is required reading (posted on the website)
- Optional reading: you can check out Don Knuth's chapter on BDDs (available off his website)