EECS 144/244: Fundamental Algorithms for System Modeling, Analysis, and Optimization

Discrete Systems

Lecture: State-Space Exploration

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ANALYSIS OF DISCRETE SYSTEMS: STATE-SPACE EXPLORATION

State-Space Exploration

Goal: explore all reachable states of the system (modeled as a transition system).

Main application: exhaustive verification by *reachability analysis*.

- Check that system is never in an "incorrect" state (*safety*)
  - deadlock state
  - state which does not satisfy a given (state) property
  - e.g., "train is at intersection but gate is not lowered"
  - "autopilot is off but pilot thinks it is on"
  - ...

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  - e.g., “train is at intersection but gate is not lowered”
  - “autopilot is off but pilot thinks it is on”
  - ...
- Also the basis for checking *liveness* properties: every so often system does something useful.

State-Space Exploration

- For finite-state systems, it can be done fully automatically! (in principle)
- Forms the basis for *model-checking*
  - Turing award 2007: Clarke, Emerson, Sifakis.
  - Established practice in the industry (mainly hardware, but increasingly also software).

Model-Checking and State-Space Exploration

Model-checking: check a model M (transition system) against a specification ϕ:

$$M \models \phi?$$

- Often ϕ is a temporal logic formula:
  - If ϕ is an LTL formula: $M \models \phi$ means $\sigma \models \phi$ for *every* trace of M.
  - If ϕ is a CTL formula: $M \models \phi$ means $s_0 \models \phi$ for *every* initial state s_0 of M.

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State-space exploration: the basis for model-checking. E.g.,

- Can build monitor M_ϕ and reduce the LTL model-checking question to a state-space exploration question about the composition of M and M_ϕ (we will revisit this when we talk about composition).
- Can extend symbolic reachability analysis (later in this lecture) to CTL model-checking.

State-Space Exploration Algorithms

- Enumerative (also called "explicit state").
- Symbolic
  - Bounded model-checking using SAT/SMT solvers.
  - Symbolic reachability.

An Enumerative Algorithm: Depth-First Search

Assume given: Kripke structure (P, S, S_0, L, R).

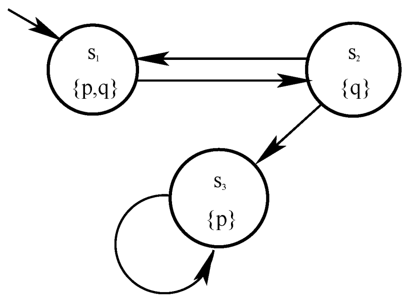
main:

$V := \emptyset$; /* V: set of visited states */
for all $s \in S_0$ **do**
  DFS(s);
end for

DFS(s):

check s; /* is s a deadlock? is given $p \in L(s)$? ... */
$V := V \cup \{s\}$;
for all s' such that $(s, s') \in R$ **do**
  if $s' \notin V$ **then**
    DFS(s'); /* recursive call */
  end if
end for

An Enumerative Algorithm: Depth-First Search



Let's simulate the algorithm on this graph.

An Enumerative Algorithm: Depth-First Search

Quiz:

- Does the algorithm terminate?
- Does it visit all reachable states?
- Does it visit any unreachable states?
- What is the complexity of the algorithm?

Enumerative Methods

Many algorithms: DFS, BFS, A*, ...

Many approaches to combat state-space explosion: partial-order reduction, symmetry reduction, bit-state hashing, ...

In-depth discussion: Computer-Aided Verification course (see also bibliography).

SYMBOLIC METHODS

Symbolic Methods: Why?

The plague of exhaustive verification: *state explosion*.

- A chip with 100 flip-flops: 2^{100} (potentially reachable) states.
- That is $1267650600228229401496703205376$ states.
- Even if each state costs 1 bit to store, this still makes $2^{100-60-8} = 2^{32} = 4,294,967,296$ exabytes ...

Symbolic methods aim to improve this.

A seminal paper was subtitled "*Symbolic model checking:* 10^{20} *states and beyond.*" [Burch et al., 1990].

10^{20} is still less than 2^{67}, but a great leap forward at that time.

Symbolic Representation of State Spaces

Key idea:

Instead of reasoning about individual states, reason about **sets** *of states.*

How do we represent a set of states?

Symbolic representation:

Set $=$ *predicate.*

Set of states $=$ *predicate on state variables.*

Symbolic Representation of Sets of States

Examples:

1. Assume 3 state variables, p, q, r, of type boolean.

$$S_1 : \qquad p \vee q$$

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2. Assume 3 state variables, x, i, b, of types real, integer, boolean.

$$S_2 : \qquad x > 0 \wedge (b \rightarrow i \geq 0)$$

   How many states are in S_2?

Symbolic Representation of Transition Relations

Key idea:

> *Use a predicate on* **two copies** *of the state variables: unprimed (current state) + primed (next state).*

If $\vec{x}$ is the vector of state variables, then the transition relation R is a predicate on $\vec{x}$ and $\vec{x}'$:

$$R(\vec{x}, \vec{x}')$$

e.g.,

$$R(x, i, b, x', i', b')$$

Symbolic Representation of Transition Relations

Examples:

1. Assume one state variable, p, of type boolean.

$$R_1 : \qquad (p \rightarrow \neg p') \wedge (\neg p \rightarrow p')$$

Which transition relation does this represent? Is it a relation or a function (deterministic)?

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   Which transition relation does this represent? Is it a relation or a function (deterministic)?

2. Assume one state variable, n, of type integer.

$$R_2 : \qquad n' = n + 1 \vee n' = n$$

   Which transition relation does this represent? Is it a relation or a function (deterministic)?

Symbolic Representation of Kripke Structures

Kripke structure:

$$(P, S, S_0, L, R)$$

Symbolic representation:

$$(P, \mathit{Init}, \mathit{Trans})$$

where

- $P = \{x_1, x_2, ..., x_n\}$: set of (boolean) state variables, also also taken to be the atomic propositions.[1]
- Predicate $\mathit{Init}(\vec{x})$ on vector $\vec{x} = (x_1, ..., x_n)$ represents the set S_0 of initial states.
- Predicate $\mathit{Trans}(\vec{x}, \vec{x}')$ represents the transition relation R.

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Basis of the language of NuSMV.

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Example: NuSMV model

```
MODULE inverter(input)
VAR
  output : boolean;
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  output = FALSE
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  next(output) = !input | next(output) = output
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What is the Kripke structure defined by this NuSMV program?

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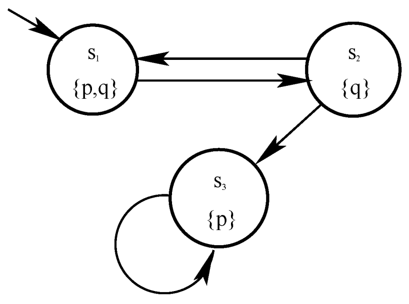
What about P and L?

Example: Kripke Structure



Represent this symbolically.

FINITE-HORIZON REACHABILITY
(a.k.a. BOUNDED MODEL-CHECKING)

Bounded Model-Checking

Question:

Can a "bad" state be reached in up to n steps (transitions)?

i.e., does there exist a path

$$s_0, s_1, ..., s_k$$

where $k \leq n$, $s_0 \in S_0$ and $s_k \in Bad$, for some given set Bad.

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Key idea:

Reduce the above question to a SAT (satisfiability) problem.

- SAT problem NP-complete for propositional logic, generally undecidable for predicate logic.
- In practice, today's SAT solvers can handle formulas with thousands of variables.
- SMT (SAT Modulo Theory) solvers also making good progress.
- BMC exploits this.

Bounded Model-Checking (BMC)

Suppose I have predicates $Init(\vec{x})$, $Trans(\vec{x}, \vec{x}')$, and $Bad(\vec{x})$.

How to use them for BMC?

- Bad state reachable in 0 steps iff

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- Bad state reachable in 1 step iff

$$\mathsf{SAT}\big(\mathit{Init}(\vec{x}_0) \wedge \mathit{Trans}(\vec{x}_0, \vec{x}_1) \wedge \mathit{Bad}(\vec{x}_1)\big)$$

- ...
- Bad state reachable in n steps iff

$$\mathsf{SAT}\big(\mathit{Init}(\vec{x}_0) \wedge \mathit{Trans}(\vec{x}_0, \vec{x}_1) \wedge \cdots \wedge \mathit{Trans}(\vec{x}_{n-1}, \vec{x}_n) \wedge \mathit{Bad}(\vec{x}_n)\big)$$

for all $k = 0, 1, ..., n$ **do**
  $\phi := \mathit{Init}(\vec{x}_0) \wedge \mathit{Trans}(\vec{x}_0, \vec{x}_1) \wedge \cdots \wedge \mathit{Trans}(\vec{x}_{k-1}, \vec{x}_k) \wedge \mathit{Bad}(\vec{x}_k)$;
  if SAT(ϕ) **then**
    print "Bad state reachable in k steps";
    output solution as counter-example;
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end for
print "Bad state unreachable up to n steps";

SAT Solving

SAT: given propositional logic formula, check whether it is satisfiable, and return a solution if it is.

Usually formula given in CNF: *Conjunctive Normal Form*.

CNF and DNF

Literal: a variable x or its negation $\overline{x}$.

Clause: a disjunction of literals. E.g.:

$$\begin{aligned} \text{clause 1} &: \quad x + y \\ \text{clause 2} &: \quad \overline{x} + z + w \end{aligned}$$

CNF: conjunction of clauses, i.e., conjunction of disjunctions of literals (also called POS - "product of sums"). E.g.:

$$(x + y) \cdot (\overline{x} + z + w) \cdots$$

DNF: disjunction of conjunctions of literals (also called SOP - "sum of products"). E.g.:

$$(xy) + (\overline{x}zw) + \cdots$$

Translations to/from CNF and DNF

Every formula can be trivially transformed into DNF. How?

Are there more efficient ways to transform into DNF? (Hint: how easy is it to check whether a DNF formula is SAT? how hard is SAT?)

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$(x \wedge y) \rightarrow (z \wedge w) \quad \rightsquigarrow$

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Homework: write a procedure to convert any boolean expression to NNF.

Given a formula in NNF, how to transform it into CNF?

Translations to/from CNF and DNF

CNF(ϕ):
if ϕ is a literal **then**
    return ϕ;
else if ϕ is $\phi_1 \wedge \phi_2$ **then**
    return CNF(ϕ_1) $\wedge$ CNF(ϕ_2);
else if ϕ is $\phi_1 \vee \phi_2$ **then**
    return DistributeOr$\big($CNF(ϕ_1), CNF(ϕ_2)$\big)$;
else
    error: ϕ not in NNF;
end if

DistributeOr(ϕ_1, ϕ_2):
if ϕ_1 is $\phi_{11} \wedge \phi_{12}$ **then**
    return DistributeOr(ϕ_{11}, ϕ_2) $\wedge$ DistributeOr(ϕ_{12}, ϕ_2);
else if ϕ_2 is $\phi_{21} \wedge \phi_{22}$ **then**
    return DistributeOr(ϕ_1, ϕ_{21}) $\wedge$ DistributeOr(ϕ_1, ϕ_{22});
else
    return $\phi_1 \vee \phi_2$;   /* both must be literals at this point */
end if

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How large can CNF(ϕ) be in the worst-case?

Translations to/from CNF and DNF

More efficient translations to CNF exist: they preserve satisfiability of the original formula, by adding extra variables.

Example:

$$ab + cd \quad \rightsquigarrow \quad \underbrace{(a+c)(a+d)(b+c)(b+d)}_{\text{normal translation}}$$

$$ab + cd \quad \rightsquigarrow \quad \underbrace{(z_1+z_2)(\overline{z_1}+a)(\overline{z_1}+b)(\overline{z_2}+c)(\overline{z_2}+d)}_{\text{translation adding variables } z_1, z_2}$$

Idea: z_1 represents ab and z_2 represents cd.

SAT Solving

SAT: given propositional logic formula, check whether it is satisfiable, and return a solution if it is.

Usually formula given in CNF: *Conjunctive Normal Form*.

Solvers exploit this form, e.g.:

- Easy to detect *conflicts* (unsatisfiability).

SAT Solving

Let ϕ be a formula in CNF.
Let $x_1, x_2, ..., x_n$ be the variables appearing in ϕ.

Brute-force algorithm:

- Explore the tree of all assignments to $x_1, x_2, ..., x_n$.
- As soon as a partial assignment results in a conflict, prune the entire subtree and backtrack.
- Either a complete assignment that works is found, or formula is UNSAT.

Worst-case performance: $O(2^n)$.

SAT Solving: Resolution

From two clauses $x + A$ and $\overline{x} + B$, derive new clause $A + B$.

Example:

$$(x + y)(\overline{x} + z) \equiv (x + y)(\overline{x} + z)(y + z)$$

Why?

SAT Solving: Resolution

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Example:

$$(x + y)(\overline{x} + z) \equiv (x + y)(\overline{x} + z)(y + z)$$

Why?

A form of *learning* (new clauses).

SAT Solving: Resolution

We can assume that every variable x appears both in "positive" form x and in "negative" form $\overline{x}$ in ϕ, that is, ϕ contains at least two clauses $(x + \cdots)$ and $(\overline{x} + \cdots)$.

What if it's not the case?

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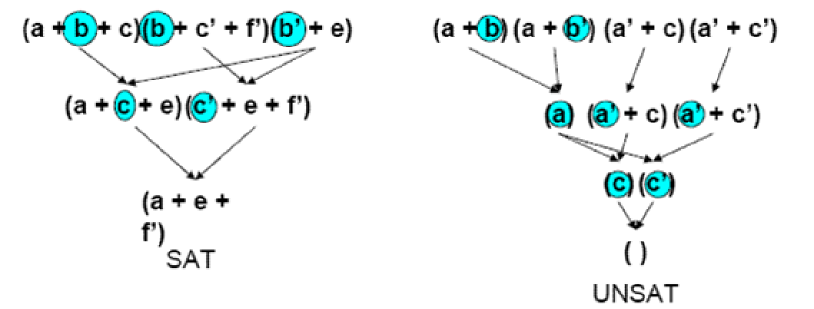
What if it's not the case?

Davis-Putnam algorithm:

for every variable x in ϕ **do**
    for every pair of clauses $A : (x + \cdots)$ and $B : (\overline{x} + \cdots)$ in ϕ **do**
        resolve A and B and add resolvent to ϕ;
        check UNSAT (empty clause);
    end for
    remove from ϕ original clauses containing x or $\overline{x}$;
    remove variables that don't appear both in positive and negative form;
    check SAT;
end for

SAT Solving: Davis-Putnam Algorithm

Examples (a' means negation, i.e., $\overline{a}$):



From Sanjit Seshia.

SAT Solving

Modern SAT solvers use much more involved (and efficient) techniques.

In-depth discussion: Computer-Aided Verification course.

Back to BMC: Soundness vs. Completeness

for all $k = 0, 1, ..., n$ **do**
  $\phi := \mathit{Init}(\vec{x}_0) \wedge \mathit{Trans}(\vec{x}_0, \vec{x}_1) \wedge \cdots \wedge \mathit{Trans}(\vec{x}_{k-1}, \vec{x}_k) \wedge \mathit{Bad}(\vec{x}_k)$;
  if SAT(ϕ) **then**
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end for
print "Bad state unreachable up to n steps";

BMC algorithm is *sound* in the following sense:

- if algorithm reports "reachable" then indeed a bad state is reachable
- if algorithm reports "unreachable" then a bad state is unreachable **but only up to n steps**.

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Can we make BMC *complete*?

- Reports unreachable iff bad states are unreachable (w.r.t. any bound).

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- Is this even possible in general?

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- Is this even possible in general? For finite-state systems?

Complete BMC: the trivial threshold

for all $k = 0, 1, ..., n$ **do**
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Can we turn BMC into a complete method for finite-state structures? How?

Complete BMC: the trivial threshold

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    if SAT($\phi$) then
        print "Bad state reachable in $k$ steps";
        output solution as counter-example;
    end if
end for
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A finite-state Kripke structure $K = (P, S, S_0, L, R)$ is essentially a finite graph.

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If we know $|S|$ (the number of all possible states) then we can set $n := |S|$.

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A finite-state Kripke structure $K = (P, S, S_0, L, R)$ is essentially a finite graph.

Can we turn BMC into a complete method for finite-state structures? How?

If we know $|S|$ (the number of all possible states) then we can set $n := |S|$.

With 100 boolean variables, $|S| = 2^{100}$, so this doesn't work.

Complete BMC: the Completeness Threshold

Reachability diameter : maximum number of steps that it takes to reach all states.

$d := \min\{i \mid \forall s \in \text{Reach} : \exists \text{ path } s_0, s_1, ..., s_j \text{ in } K : j \leq i \wedge s_0 \in S_0 \wedge s_j = s\}$

where Reach is the set of reachable states of K.

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$d \leq |\mathsf{Reach}| \leq |S|$. It could in practice be much smaller.

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Problem: we don't know Reach, therefore how to compute d?

Complete BMC: the Completeness Threshold

Recurrence diameter : length of the longest loop-free path.

$r := \max\{i \mid \exists$ path $s_0, s_1, ..., s_i$ in $K : s_0 \in S_0 \wedge \forall 0 \leq j < k \leq i : s_j \neq s_k\}$

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Then: $d \leq r$. Why?

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where Reach is the set of reachable states of K.

Then: $d \leq r$. Why?

How to compute r?

Express it in propositional logic!

$$r := \max\{i \mid \mathsf{SAT}\Big(\mathit{Init}(\vec{x}_0) \wedge \mathit{Trans}(\vec{x}_0, \vec{x}_1) \wedge \cdots \wedge \mathit{Trans}(\vec{x}_{i-1}, \vec{x}_i) \wedge \bigwedge_{j=0}^{i-1} \bigwedge_{k=j+1}^{i} \vec{x}_j \neq \vec{x}_k \Big)\}$$

EXHAUSTIVE SYMBOLIC REACHABILITY (a.k.a. SYMBOLIC MODEL-CHECKING)

Symbolic Reachability

Recall:

- Set of states = predicate $\phi(\vec{x})$ on vector of state variables $\vec{x}$. E.g.:
  - $Init(x, y, z) \,:\, x \wedge \neg y$
  - $Bad(x_1, x_2) \,:\, x_1 = crit \wedge x_2 = crit$
- Transition relation = predicate $Trans(\vec{x}, \vec{x}')$ on state variables and next-state variables. E.g.:
  - $Trans(x, y, x', y') : x' = x + 1 \wedge (y' = 0 \vee y' = 1)$

Symbolic Reachability: Predicate Transformer

Then:

- Successors can be computed by a **predicate transformer** :

$$\mathbf{succ}\big(\phi(\vec{x})\big) := \big(\exists \vec{x} : \phi(\vec{x}) \wedge Trans(\vec{x}, \vec{x}')\big)[\vec{x}' \rightsquigarrow \vec{x}]$$

  - $\exists \vec{x} : \phi(\vec{x}) \wedge Trans(\vec{x}, \vec{x}')$: successors of states in ϕ
  - $[\vec{x}' \rightsquigarrow \vec{x}]$: renames variables so that resulting predicate is over current state variables

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Example:

$$\begin{array}{rcl} \phi & = & 0 \leq x \leq 5 \\ \mathit{Trans} & = & x \leq x' \leq x+1 \\ \mathbf{succ}(\phi) & = & (\exists x : 0 \leq x \leq 5 \wedge x \leq x' \leq x+1)[x' \rightsquigarrow x] \end{array}$$

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 & = & (0 \le x' \le 6)[x' \leadsto x] \\
 & = & 0 \le x \le 6
\end{array}$$

Symbolic Reachability: Predicate Transformer

$$\mathbf{succ}\big(\phi(\vec{x})\big) := \big(\exists \vec{x} : \phi(\vec{x}) \land \mathit{Trans}(\vec{x}, \vec{x}')\big)[\vec{x}' \leadsto \vec{x}]$$

How do we implement $\mathbf{succ}$ in practice?

In particular, how to do quantifier elimination automatically?

In the case of propositional logic, quantifier elimination becomes a simple procedure:

$$\exists x : \phi(x) \quad \equiv$$

Symbolic Reachability: Predicate Transformer

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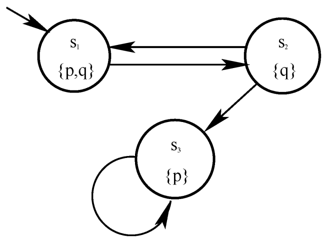
In particular, how to do quantifier elimination automatically?

In the case of propositional logic, quantifier elimination becomes a simple procedure:

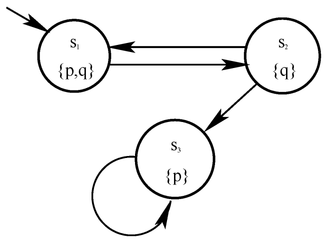
$$\exists x : \phi(x) \quad \equiv \quad \phi(x)[x \leadsto 0] \lor \phi(x)[x \leadsto 1]$$

Predicate Transformer: Another Example



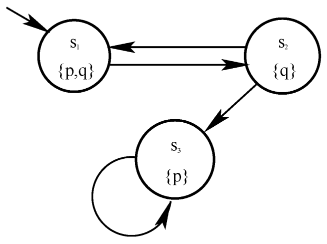
$$\mathbf{succ}(p \wedge q) \quad = \quad (\exists p, q : p \wedge q \wedge \mathit{Trans})[p' \rightsquigarrow p, q' \rightsquigarrow q]$$

Predicate Transformer: Another Example



$$
\begin{aligned}
\mathbf{succ}(p \wedge q) &= (\exists p, q : p \wedge q \wedge \mathit{Trans})[p' \rightsquigarrow p, q' \rightsquigarrow q] \\
&= (\exists p, q : p \wedge q \wedge \overline{p}' \wedge q')[p' \rightsquigarrow p, q' \rightsquigarrow q]
\end{aligned}
$$

Predicate Transformer: Another Example



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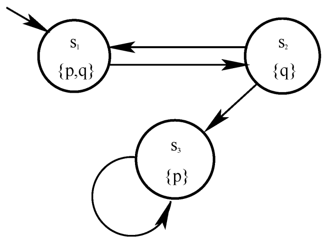

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&= (\overline{p}' \wedge q')[p' \rightsquigarrow p, q' \rightsquigarrow q] \\
&= \overline{p} \wedge q
\end{aligned}
$$

Symbolic Reachability

$Reachable := Init$;
terminate := false;
repeat
    $tmp := Reachable \vee \mathbf{succ}(Reachable)$;
    if $tmp \equiv Reachable$ **then**
        terminate := true;
    else
        $Reachable := tmp$;
    end if
until terminate
return $Reachable$;

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repeat
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    if $tmp \equiv Reachable$ **then**
        terminate := true;
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    end if
until terminate
return $Reachable$;

Does the algorithm terminate? Why?

$Reachable := Init$;
terminate := false;
repeat
    $tmp := Reachable \vee \textbf{succ}(Reachable)$;
    if $tmp \equiv Reachable$ **then**
        terminate := true;
    else
        $Reachable := tmp$;
    end if
until terminate
return $Reachable$;

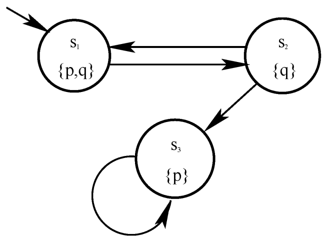
Does the algorithm terminate? Why?

Quiz: modify the algorithm to make it check reachability of a set of bad states characterized by predicate Bad.

Symbolic Reachability: checking for Bad states

```
Reachable := Init;
terminate := false;
error := false;
repeat
    tmp := Reachable ∨ succ(Reachable);
    if tmp ≡ Reachable then
        terminate := true;
    else
        Reachable := tmp;
    end if
    if SAT(Reachable ∧ Bad) then
        error := true;
    end if
until terminate or error
return (Reachable,error);
```

Symbolic Model-Checking: Example



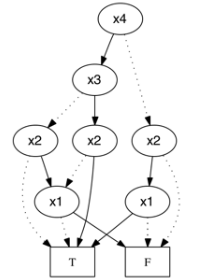
Let's model-check this symbolically!
We want to check that all reachable states satisfy $p \vee q$.
In temporal logic parlance:

$$\begin{array}{rl} \text{CTL:} & AG(p \vee q) \\ \text{LTL:} & \Box(p \vee q) \end{array}$$

Symbolic Model-Checking: Implementation

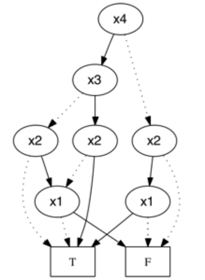
- For finite-state systems, boolean variables can be used to encode state.
- All predicates then become boolean expressions.
- Efficient data structures for boolean expressions:
  - BDDs (Binary Decision Diagrams)
- Efficient algorithms for implementing logical operations (conjunction, disjunction, satisfiability check) on BDDs.
- In-depth discussion: Computer-Aided Verification course.

Example: BDD



Can you guess which boolean expression this BDD represents?

Example: BDD



Can you guess which boolean expression this BDD represents?

$$\overline{x_2}\ \overline{x_3}x_4 + \overline{x_1}(x_2\overline{x_3}x_4 + \overline{x_2}x_3x_4) + x_2x_3x_4 + x_1x_2\overline{x_4}$$

Symbolic Model-Checking

Homework: Assuming that the system only has boolean variables, express deadlock as a predicate on state (or next state) variables. Note: the predicate must have no quantifiers.

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