EECS 144/244: System Modeling, Analysis, and Optimization

Continuous Systems

Lecture: Linear Systems

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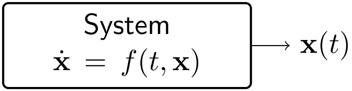
University of California, Berkeley

April 30, 2013

Autonomous Systems

So far we worked mostly on systems of the form



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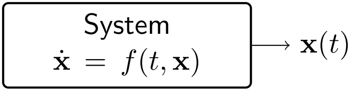


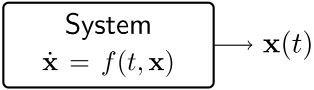
for which we studied existence, unicity, and numerical computation of solutions.

Question: is this system deterministic ?

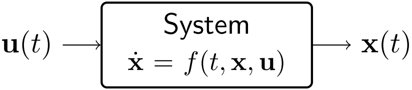
Given $\mathbf{x}_0$, it produces a unique solution, called *flow* $\mathbf{x}(t, \mathbf{x}_0)$

Systems with Inputs

Closed system



Open loop



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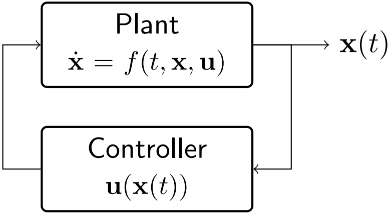

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Short answer

OK if $F(t, \mathbf{x}(t)) = f(t, \mathbf{x}(t), \mathbf{u}(t))$ satisfies the same conditions as before (continuous, (locally) Lipshitz)

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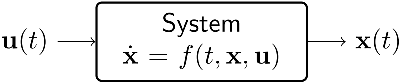
OK if $F(t, \mathbf{x}(t)) = f(t, \mathbf{x}(t), \mathbf{u}(t))$ satisfies the same conditions as before (continuous, (locally) Lipshitz)

Although, the story is more complex

- Typically inputs are discontinuous (step functions)
- Complex feedback can create algebraic loops resulting in DAEs rather than ODEs (consult David)

Some Properties of Interest



Stability

For any "reasonable" input $\mathbf{u}$, $\mathbf{x}$ does not "explode" in some sense.

Different notions:

- Bounded Input Bounded Output: $\forall t, \|\mathbf{u}(t)\| < C_u \Rightarrow \forall t, \|\mathbf{x}(t)\| \leq C_{\mathbf{x}}$
- Globally asymptotic: $\forall \mathbf{x}(0), \lim_{t \to \infty} \mathbf{x}(t) = 0$
- etc

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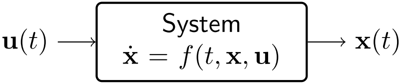

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Controllability

Can we find an input $\mathbf{u}$ that drives the system to some goal state ?

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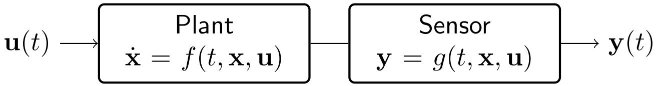

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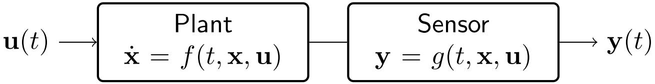

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For linear systems, these questions are well understood

Linear Functions

A *linear* function is a mapping which satisfies :

1. Additivity: $f(x+y) = f(x) + f(y)$
2. Scaling: $f(\lambda x) = \lambda f(x)$

or equivalently,

$$\forall \alpha, \beta, f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

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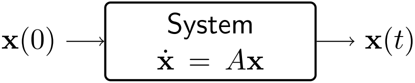
Linear functions from $\mathbb{R}$ to $\mathbb{R}$? $f(x) = a \cdot x$

Linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$? $f(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix.

Linear Systems

Linear dynamical systems



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Allows to characterize infer sets of behaviors from typical simple inputs

- Step signals $u(t) = 1(t)$
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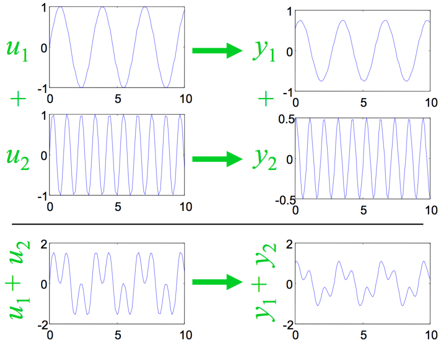

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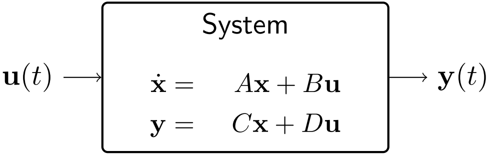

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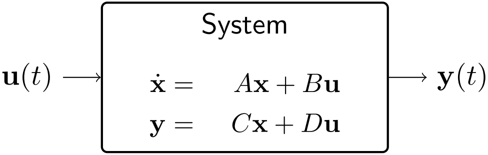

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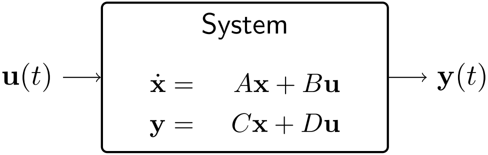

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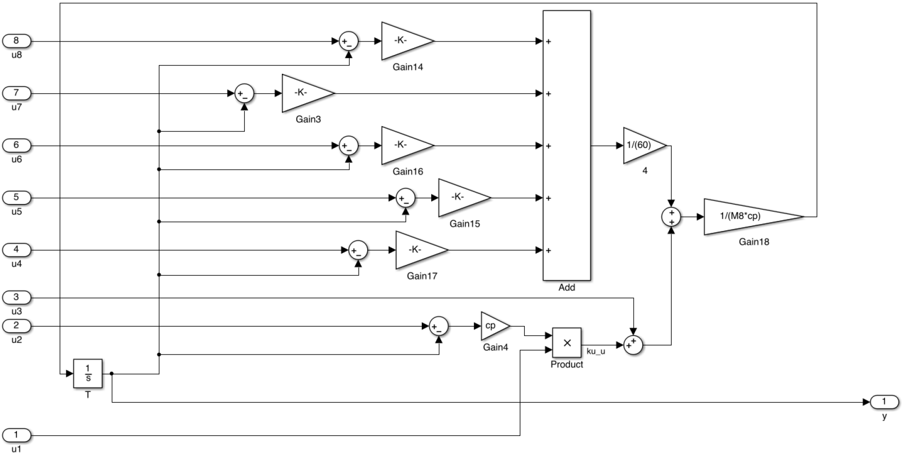
- If $\mathbf{x}_0 \neq 0$, the system is not linear, strictly speaking. However, it is *affine*
- So is $f(\mathbf{x}, \mathbf{u}) = A\mathbf{x} + B\mathbf{u}$, by the way.

Example

Is this system linear ?



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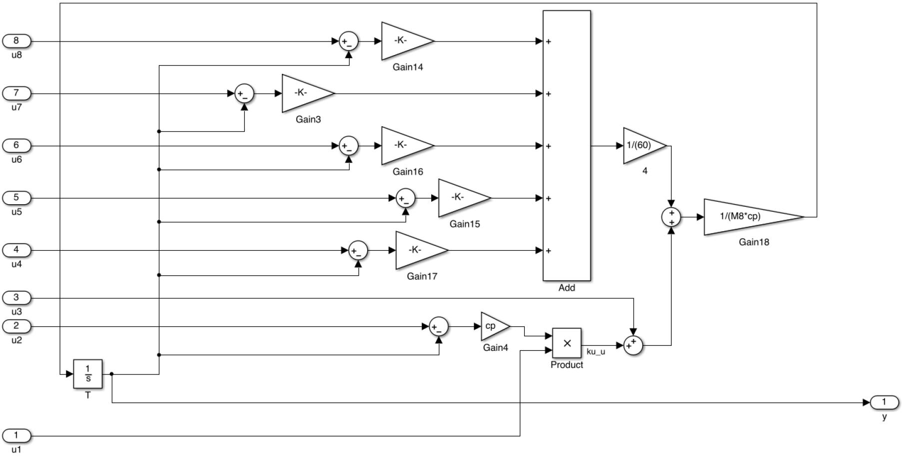


Solutions of Linear Systems

Recall that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t=0) = \mathbf{x}_0$
admits the solution $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$
Where $e^{\mathbf{A}t}$ is matrix exponentiation

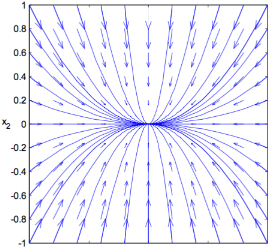
$$e^{t\mathbf{A}} = \mathbf{I}_n + \mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \dots$$

The stability depends on the Eigenvalues of A

Eigenstructure of Linear Systems

Real e-values

$\mathrm{Re}(\lambda_i) < 0$



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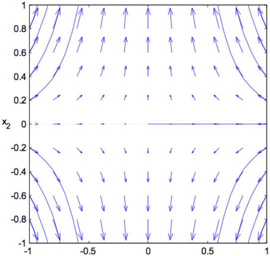

Complex e-values

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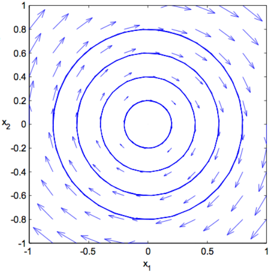

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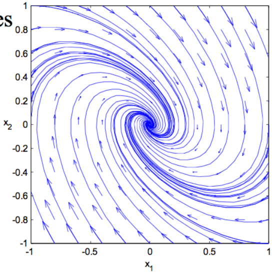
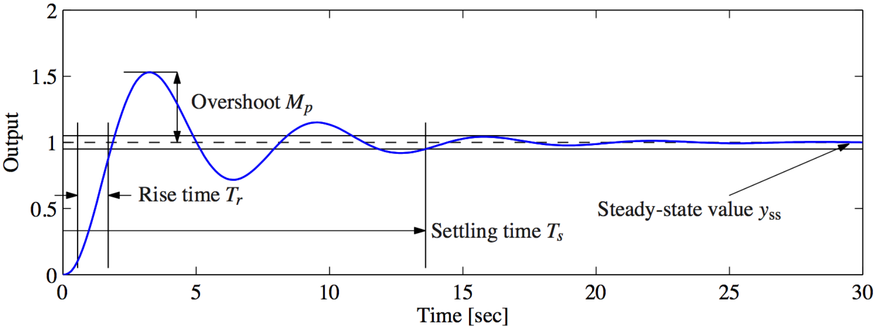


Step response for LTI systems

$$u(t) = \begin{cases} O & t = 0 \\ 1 & t > 0 \end{cases}$$

Equivalent to $\dot{\mathbf{z}} = A\mathbf{z}$ with $\mathbf{z} = \mathbf{x} + A^{-1}B$



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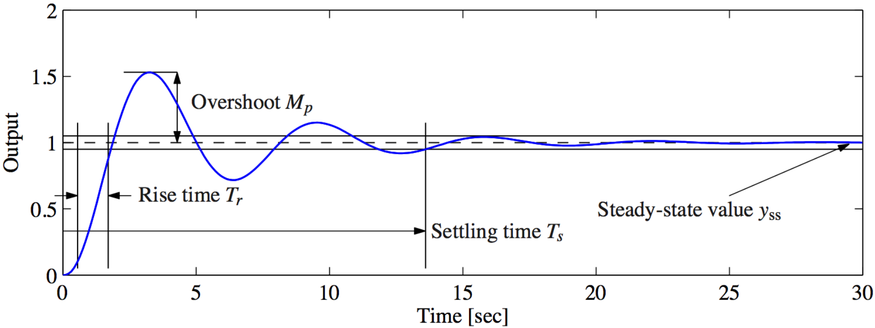
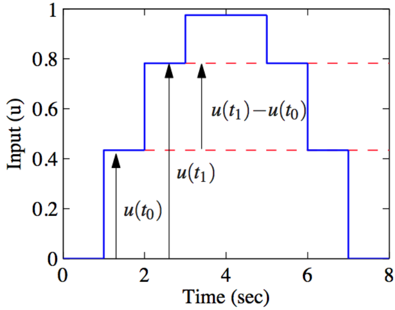
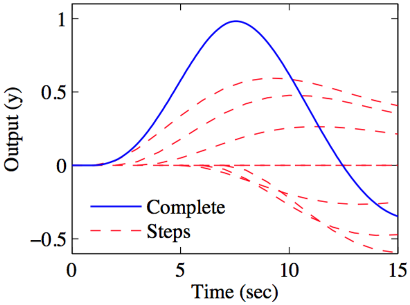


Piecewise Constant Inputs

Superposition of step responses



(a) Piecewise constant input



(b) Output response

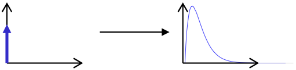
LTI Systems Solution

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

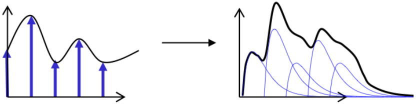
Impulse response, h(t) = $\boldsymbol{Ce^{At}B}$

- Response to input "impulse"
- Equivalent to "Green's function"



Linearity ⇒ compose response to arbitrary *u*(*t*) using *convolution*

- Decompose input into "sum" of shifted impulse functions
- Compute impulse response for each
- "Sum" impulse response to find *y*(*t*)



Complete solution: use integral instead of "sum"

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input ⇒ 2X output when $x(0) = 0$

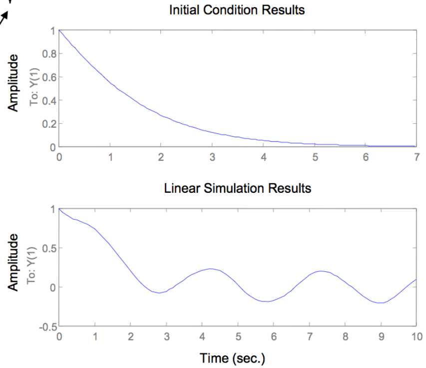
Matlab Tools for Linear Systems

$$y(t) = \underbrace{Ce^{At}x(0)}_{} + \int_{\tau=0}^{t} \underbrace{Ce^{A(t-\tau)}}_{} Bu(\tau)d\tau + Du(t)$$

```
A = [-1 1; 0 -1];   B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```



Other MATLAB commands

- gensig, square, sawtooth – produce signals of diff. types
- step, impulse, initial, lsim – time domain analysis
- bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots

Controllability

Consider the SISO LTI system:

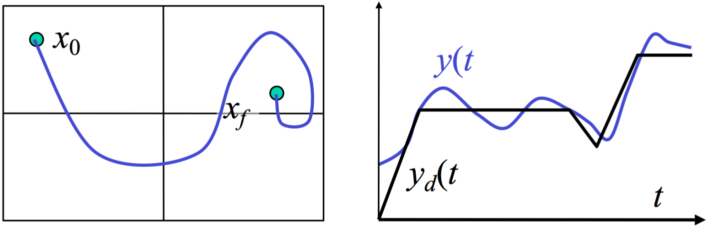
$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \qquad \mathbf{x} \in \mathcal{R}^n, u \in \mathcal{R}$$
$$y = C\mathbf{x} + Du \qquad y \in \mathcal{R}$$

is controllable if there exists $u(t)$ such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}$.

Note

- In 

, controllability is referred to as *reachability*
- Controllability enables the more general problem of tracking



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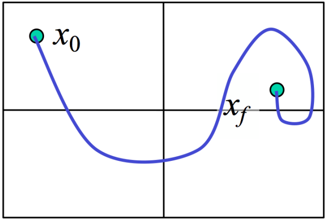
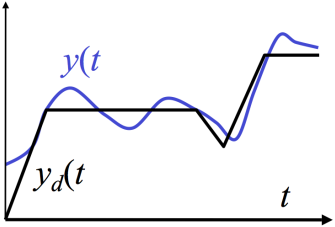




Controllability Matrix

The controllability matrix for a linear system is given by:

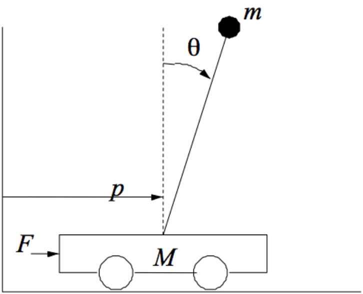
$$W_r = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

A linear system is controllable if and only if the controllability matrix is *invertible*

Note

- In the general MIMO case, the condition becomes W_r has rank n
- Very simple test to do (e.g. in Matlab, concat matrices, run `rank`)

Example: Linearized pendulum on a cart



Question: can we locally control the position of the cart by proper choice of input?

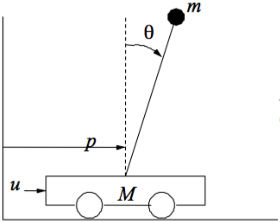
Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{M_tJ_t - m^2l^2} & \frac{-cJ_t}{M_tJ_t - m^2l^2} & \frac{-\gamma J_t lm}{M_tJ_t - m^2l^2} \\ 0 & \frac{M_t mgl}{M_tJ_t - m^2l^2} & \frac{-clm}{M_tJ_t - m^2l^2} & \frac{-\gamma M_t}{M_tJ_t - m^2l^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_tJ_t - m^2l^2} \\ \frac{lm}{M_tJ_t - m^2l^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x,$$

Note here u is F.

Example: Linearized pendulum on a cart (cont'd)



$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & \cancelto{0}{\frac{-cJ_t}{\mu}} & \cancelto{0}{\frac{-\gamma J_t l m}{\mu}} \\ 0 & \frac{M_t m g l}{\mu} & \cancelto{0}{\frac{-clm}{\mu}} & \cancelto{0}{\frac{-\gamma M_t}{\mu}} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{lm}{\mu} \end{bmatrix} u$$

$$\mu = M_t J_t - m^2 l^2$$

- Simplify by setting c, $\gamma = 0$

Reachability matrix

$$W_r = \left[\begin{array}{c|c|c|c} 0 & \frac{J_t}{\mu} & 0 & \frac{g l^3 m^3}{\mu^2} \\ 0 & \frac{lm}{\mu} & 0 & \frac{g l^2 m^2 (m+M)}{\mu^2} \\ \frac{J_t}{\mu} & 0 & \frac{g l^3 m^3}{\mu^2} & 0 \\ \frac{lm}{\mu} & 0 & \frac{g l^2 m^2 (m+M)}{\mu^2} & 0 \end{array}\right].$$

$B \quad AB \quad A^2B \quad A^3B$

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- ⇒ reachable as long as

$$\det(W_r) = \frac{g^2 l^4 m^4}{\mu^4} \neq 0$$

- ⇒ can "steer" linearization between points by proper choice of input

State Feedback

A state feedback law has the form $\mathbf{u} = -Kx + k_r r$ where r is the reference value for the output.

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The stability of the system is determined by the stability of the matrix $A - BK$. The equilibrium point and steady state output are given by

$$\mathbf{x}_e = -(A - BK)^{-1}Bk_r r \qquad \mathbf{y}_e = C\mathbf{x}_e.$$

Choosing k_r as $k_r = -1/\left(C(A - BK)^{-1}B\right)$ gives $\mathbf{y}_e = r$.

Controllability

If a system is controllable, then there exists a feedback law of the form

$$u = -Kx + k_r r$$

the gives a closed loop system with an arbitrary characteristic polynomial.

Hence the eigenvalues of a controllable linear system can be placed arbitrarily through the use of an appropriate feedback control law.

Limits of poles placement

Saturation of inputs, transient behaviors, etc.

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Popular state feedback control laws in practice:

PID controller

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)\, d\tau + k_d \frac{de}{dt},$$

where $e(t)$ is the difference between $y(t)$ and the reference value $r(t)$.

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Linear quadratic regulator

Minimizes

$$\tilde{J} = \int_0^\infty \left(\mathbf{x}^T Q_x \mathbf{x} + \mathbf{u}^T Q_u \mathbf{u}\right)\, dt.$$

for (user)-given matrices Q_x and Q_u.

Laplace Transform

The Laplace Transform of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is the function $F = \mathcal{L}\{f\} : \mathbb{C} \to \mathbb{C}$ such that

$$\forall s \in \mathbb{C},\ F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t) dt$$

Main properties:

- Linear: $\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$
- Derivative: $\mathcal{L}\{\frac{df}{dt}\}(s) = s \cdot \mathcal{L}\{f\} - f(0)$

In particular, if $f(0) = 0$, we get $\boxed{\mathcal{L}\{\frac{df}{dt}\} = sF(s)}$

Solving differential equations with Laplace Transform

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u$$

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Assuming that initial conditions are 0 and applying the Laplace transfom, we get:

$$(s^n + a_1 s^{n-1} + \cdots + a_n) Y(s) = (b_0 s^m + b_1 s^{m-1} + \cdots + b_m) U(s)$$

We note $d(s) = (s^n + a_1 s^{n-1} + \cdots + a_n)$ and $n(s) = (b_0 s^m + b_1 s^{m-1} + \cdots + b_m)$.

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We note $d(s) = (s^n + a_1 s^{n-1} + \cdots + a_n)$ and $n(s) = (b_0 s^m + b_1 s^{m-1} + \cdots + b_m)$.

$$Y(s) = \frac{n(s)}{d(s)} U(s) = H(s) U(s)$$

H(s) is called the transfer function of the system

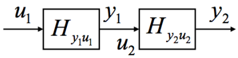
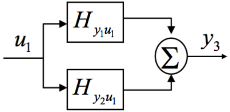
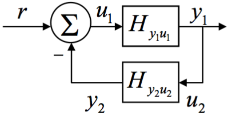
Transfer Functions and State space representation

More generally, the transfer function for a linear system (A, B, C, D) is given by

$$H(s) = \frac{n(s)}{d(s)} = C(sI - A)^{-1}B + D, \ s \in \mathbb{C}$$

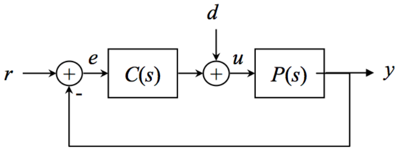
- the roots of $d(s)$ are called the poles of the system and correspond to the eigenvalues of the dynamics matrix A
- the roots of the polynomial b(s) are called the zeros of the system and correspond to exponential signals whose transmission is blocked by the system.

Transfer functions and block diagram Algebra

Type	**Diagram**	**Transfer function**
Series		$H_{y_2u_1} = H_{y_2u_2} H_{y_1u_1} = \frac{n_1 n_2}{d_1 d_2}$
Parallel		$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \frac{n_1 d_2 + n_2 d_1}{d_1 d_2}$
Feedback		$H_{y_1r} = \frac{H_{y_1u_1}}{1 + H_{y_1u_1} H_{y_2u_2}} = \frac{n_1 d_2}{n_1 n_2 + d_1 d_2}$

Transfer functions and block diagram Algebra

Basic idea: treat transfer functions as multiplication, write down equations



$$y = P(s)u$$
$$u = d + C(s)e$$
$$e = r - y$$

Manipulate equations to compute desired signals

$$\begin{aligned} e &= r - y \\ &= r - P(s)u \\ &= r - P(s)(d + C(s)e) \end{aligned}$$

$$(1 + P(s)C(s))e = r - P(s)d$$

$$e = \underbrace{\frac{1}{1 + P(s)C(s)}}_{H_{er}} r - \underbrace{\frac{P(s)}{1 + P(s)C(s)}}_{H_{ed}} d$$

Note: linearity gives superposition of terms

Usual Transfer Functions

Type	ODE	Transfer Function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
PID controller	$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
Time delay	$y(t) = u(t-\tau)$	$e^{-\tau s}$

References

Feedback Systems: An introduction for scientists and engineers
Karl J. Astrom and Richard M. Murray

http://www.cds.caltech.edu/~murray/amwiki/index.php/Main_Page