# Algorithms for Solving Higher Index DAEs

**Lecture 12b in EECS 144/244**

University of California, Berkeley
April 30, 2013

**David Broman**
broman@eecs.berkeley.edu

EECS Department
University of California, Berkeley, USA
and
Linköping University, Sweden



## Agenda

**Part I**
DAE Basics

**Part II**
Matching

**Part III**
BLT Sorting

**Part IV**
Pantelides

**Part V**
Dummy Derivatives

broman@eecs.berkeley.edu

# Part I

## DAE Basics



## DAE


broman@eecs.berkeley.edu

**System of Differential algebraic equation (DAE) in general form:**

$$F(x, \dot{x}, y, t) = 0$$

where $x, \in \mathbb{R}^n, \dot{x} \in \mathbb{R}^n, y \in \mathbb{R}^m, F : G \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{n+m}$.

**An DAE is distinguish from an ODE in that its differentiated variables cannot be completely solved for in terms of other variables.**

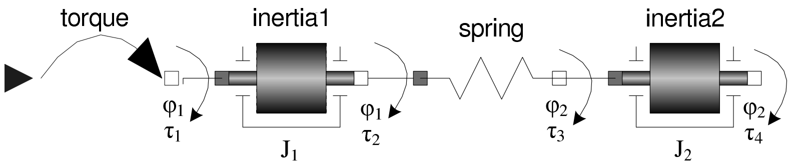
**<u>Definition</u>: The index of an DAE is the minimum number of times that all or part of the DAE must be differentiated with respect to *t* in order to determine *x'* as a continuous function of *x* and *t*.**

**(Brenan, Campbell, Petzold, 1989)**

Part I DAE Basics | Part II Matching | Part III BLT Sorting | Part IV Pantelides | Part V Dummy Derivatives

# Drive Shaft Example

broman@eecs.berkeley.edu




Is this an DAE?

$$
\begin{aligned}
\dot{\varphi}_1 &= \omega_1 \\
\dot{\varphi}_2 &= \omega_2 \\
\dot{\omega}_1 &= \frac{\tau_1 + \tau_2}{J_1} \\
\dot{\omega}_2 &= \frac{\tau_3 + \tau_4}{J_2} \\
\tau_1 &= u \\
\tau_2 &= c \cdot (\varphi_2 - \varphi_1) \\
\tau_3 &= -c \cdot (\varphi_2 - \varphi_1) \\
\tau_4 &= 0
\end{aligned}
$$

**Variables:** $(\varphi_1, \varphi_2, \omega_1, \omega_2, \tau_1, \tau_2, \tau_3, \tau_4)$

**Appearing differentiated:** $(\dot{\varphi}_1, \dot{\varphi}_2, \dot{\omega}_1, \dot{\omega}_2)$

Incidence matrix. Differentiated variables and the algebraic variables are unknown.

| | $\dot{\omega}_1$ | $\dot{\omega}_2$ | $\dot{\phi}_1$ | $\dot{\phi}_2$ | $\tau_1$ | $\tau_2$ | $\tau_3$ | $\tau_4$ |
|---|---|---|---|---|---|---|---|---|
| $f_1$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $f_2$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $f_3$ | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $f_4$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $f_5$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $f_6$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $f_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $f_8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

| | $\tau_4$ | $\tau_3$ | $\tau_2$ | $\tau_1$ | $\dot{\omega}_2$ | $\dot{\omega}_1$ | $\dot{\phi}_2$ | $\dot{\phi}_1$ |
|---|---|---|---|---|---|---|---|---|
| $f_8$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $f_7$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $f_6$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $f_5$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $f_4$ | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| $f_3$ | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| $f_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $f_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Index-0 DAE (by substitution we get directly an ODE)

Matching: Find a unique mapping between variables and equations.

Sorting: Sort equations (permute matrix)





broman@eecs.berkeley.edu

# Part II

# Matching

# Example: Matching

**System of equations**

$$f_1(y) = 0$$
$$f_2(\dot{x}_1, \dot{x}_2, y) = 0$$
$$f_3(\dot{x}_2) = 0$$

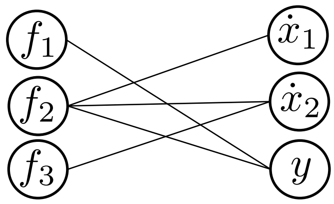
**Construct a bipartite graph**

$$G = (F, V, E)$$

$$F = \{f_1, f_2, f_3\} \qquad E = \{(f_1, y), (f_2, \dot{x}_1), (f_2, \dot{x}_2), (f_2, y), (f_3, \dot{x}_2)\}$$
$$V = \{\dot{x}_1, \dot{x}_2, y\}$$

**Incidence Matrix**

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} \dot{x}_1 & \dot{x}_2 & y \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$








# Algorithm: Matching

MATCH($G$)
$assign \leftarrow \emptyset$
**for** each $f \in G.F$
  **do** $C \leftarrow \emptyset$
    **if** not MATCH-EQUATION($G, f, \underline{C}, \underline{assign}, \emptyset$)
      **then return** (FALSE, $assign$)
**return** (TRUE, $assign$)

**Color visited vertices**

$$C \subseteq G.F \cup G.V$$

**Assigns variables to equations**

$$assign[v] = \begin{cases} f & \text{if } f \text{ matches } v \\ \text{NIL} & \text{otherwise} \end{cases}$$

Underline means call by reference.

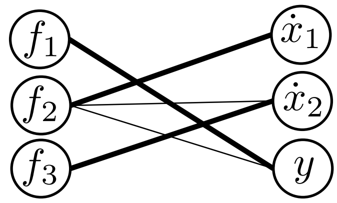
*vmap* and *equation coloring* is not used until in Part IV.

MATCH-EQUATION($G, f, \underline{C}, \underline{assign}, vmap$)
$C \leftarrow C \cup \{f\}$
**if** there exits a $v \in G.V$ such that $(f, v) \in G.E$
  and $assign[v] = \text{NIL}$ and $vmap[v] = \text{NIL}$
  **then** $assign[v] \leftarrow f$
    **return** TRUE
  **else** **for** each $v$ where $(f, v) \in G.E$ and $v \notin C$
      and $vmap[v] = \text{NIL}$
      **do** $C \leftarrow C \cup \{v\}$
        **if** MATCH-EQUATION($G, assign[v], \underline{C}, \underline{assign}, vmap$)
          **then** $assign[v] \leftarrow f$
            **return** TRUE
**return** FALSE

# Example: Matching

broman@eecs.berkeley.edu

```
MATCH(G)
  assign ← ∅
  for each f ∈ G.F
      do C ← ∅
         if not MATCH-EQUATION(G, f, C, assign, ∅)
            then return (FALSE, assign)
  return (TRUE, assign)
```




**Exercise**

Do each step of the algorithms and keep track of $C$ and *assign*.

```
MATCH-EQUATION(G, f, C, assign, vmap)
  C ← C ∪ {f}
  if there exits a v ∈ G.V such that (f, v) ∈ G.E
     and assign[v] = NIL and vmap[v] = NIL
    then assign[v] ← f
         return TRUE
    else for each v where (f, v) ∈ G.E and v ∉ C
             and vmap[v] = NIL
            do C ← C ∪ {v}
               if MATCH-EQUATION(G, assign[v], C, assign, vmap)
                  then assign[v] ← f
                       return TRUE
  return FALSE
```

**Case A: For f1, use x1.**

$$assign = \{y \mapsto f_1, \dot{x}_1 \mapsto f_2, \dot{x}_2 \mapsto f_3\}$$
$$C = \{f_1, f_2, f_3\}$$

**Case B: For f1, first use x2 (Reassignment of x2)**

$$assign = \{y \mapsto f_1, \dot{x}_1 \mapsto f_2, \dot{x}_1 \mapsto f_3\}$$
$$C = \{f_1, f_2, f_3, \dot{x}_2, \}$$



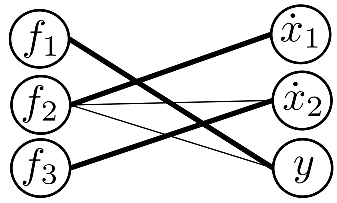


# Example: Matching

broman@eecs.berkeley.edu

**System of equations**

$$f_1(y) = 0$$
$$f_2(\dot{x}_1, \dot{x}_2, y) = 0$$
$$f_3(\dot{x}_2) = 0$$




**Incidence Matrix**

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} \dot{x}_1 & \dot{x}_2 & y \end{array} \\ \begin{pmatrix} 0 & 0 & \boxed{1} \\ \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 0 \end{pmatrix} \end{array}$$

**We may now permute the matrix**

$$\begin{array}{c} \\ f_3 \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} \dot{x}_2 & y & \dot{x}_1 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

The matching problem solves the problem of finding a permutation such that the matrix has a nonzero diagonal. Also called *maximum traversal*.

# Part III

# BLT Sorting



## Sorting into Lower Triangular Form



| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $f_1$ | 0 | 0 | **1** | 0 | 0 | **1** | 0 |
| $f_2$ | **1** | **1** | 0 | 0 | **1** | **1** | 0 |
| $f_3$ | 0 | 0 | 0 | **1** | **1** | **1** | 0 |
| $f_4$ | 0 | **1** | 0 | 0 | 0 | 0 | 0 |
| $f_5$ | 0 | 0 | **1** | 0 | **1** | 0 | 0 |
| $f_6$ | 0 | **1** | 0 | 0 | 0 | **1** | 0 |
| $f_7$ | 0 | **1** | **1** | 0 | 0 | 0 | **1** |

**Unsorted Matrix**

| | $x_2$ | $x_6$ | $x_3$ | $x_7$ | $x_5$ | $x_4$ | $x_1$ |
|---|---|---|---|---|---|---|---|
| $f_4$ | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| $f_6$ | **1** | **1** | 0 | 0 | 0 | 0 | 0 |
| $f_1$ | 0 | **1** | **1** | 0 | 0 | 0 | 0 |
| $f_7$ | **1** | 0 | **1** | **1** | 0 | 0 | 0 |
| $f_5$ | 0 | 0 | **1** | 0 | **1** | 0 | 0 |
| $f_3$ | 0 | **1** | 0 | 0 | **1** | **1** | 0 |
| $f_2$ | **1** | **1** | 0 | 0 | **1** | 0 | **1** |

**Sorting (permutation of Matrix) into Lower Triangular Matrix Form**

We have now causal form; solving the equation system is straight forward.

But, we cannot always permute the matrix into lower triangular form…

# Sorting into Block Lower Triangular (BLT) Form

broman@eecs.berkeley.edu

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $f_1$ | 0 | 0 | **1** | **1** | 0 | **1** | 0 |
| $f_2$ | **1** | **1** | 0 | 0 | **1** | **1** | 0 |
| $f_3$ | 0 | 0 | 0 | **1** | **1** | 0 | 0 |
| $f_4$ | 0 | **1** | 0 | 0 | 0 | 0 | 0 |
| $f_5$ | **1** | 0 | **1** | 0 | **1** | 0 | 0 |
| $f_6$ | 0 | **1** | 0 | 0 | 0 | **1** | 0 |
| $f_7$ | 0 | 0 | **1** | 0 | 0 | 0 | **1** |

**Another unsorted Matrix**

| | $x_2$ | $x_6$ | $x_3$ | $x_5$ | $x_1$ | $x_4$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| $f_4$ | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| $f_6$ | **1** | **1** | 0 | 0 | 0 | 0 | 0 |
| $f_1$ | 0 | **1** | **1** | 0 | 0 | **1** | 0 |
| $f_5$ | 0 | 0 | **1** | **1** | **1** | 0 | 0 |
| $f_2$ | **1** | **1** | 0 | **1** | **1** | 0 | 0 |
| $f_3$ | 0 | 0 | 0 | **1** | 0 | **1** | 0 |
| $f_7$ | 0 | 0 | **1** | 0 | 0 | 0 | **1** |

**Sorting (permutation of Matrix) into Block Lower Triangular (BLT ) Form**

We have identified an **algebraic loop**.

At each time step, the algebraic loops may be solved using Guassian elimination (if linear) or a Newton's method (if nonlinear).

Our goal in Part III is to show a combination of algorithms for sorting into BLT form.

An DAE in BLT form with algebraic loops is Index 1





# Algorithm: BLT Sort

broman@eecs.berkeley.edu

Input: a bipartite graph G

```
BLT(G)
(match, assign) ← MATCH(G)
if not match
   then return error "Singular"

D.V ← G.F
D.E ← ∅
for each (f, v) ∈ G.E where f ∈ G.F and assign[v] ≠ f
    do D.E ← D.E ∪ {(assign[v], f)}

MAKEEMPTY(O)
MAKEEMPTY(S)
i ← 0
lowlink ← ∅
number ← ∅
for each v ∈ D.V
    do if number[v] = NIL
          then STRONGCONNECT(v, D, S, i, lowlink, number, O)
return O
```

Output: a stack of sets of equation vertices, where each set represents an equation block in the BLT matrix.

# Algorithm: BLT Sort

Input: a bipartite graph G

```
BLT(G)
(match, assign) ← MATCH(G)
if not match
   then return error "Singular"

D.V ← G.F
D.E ← ∅
for each (f, v) ∈ G.E where f ∈ G.F and assign[v] ≠ f
    do D.E ← D.E ∪ {(assign[v], f)}

MAKEEMPTY(O)
MAKEEMPTY(S)
i ← 0
lowlink ← ∅
number ← ∅
for each v ∈ D.V
    do if number[v] = NIL
          then STRONGCONNECT(v, D, S, i, lowlink, number, O)
return O
```

**Part 1** Find matching

**Part 2** Construct equation dependency graph

**Part 3** Sort into blocks of equations using Tarjan's strongly connected component algorithm

Output: a stack of sets of equation vertices, where each set represents an equation block in the BLT matrix.





# Example: BLT Sort

$$
\begin{array}{c|cccccc}
 & \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 & y_1 & y_2 \\
\hline
f_1 & 0 & 0 & 1 & \boxed{1} & 0 & 1 \\
f_2 & \boxed{1} & 1 & 0 & 0 & 1 & 1 \\
f_3 & 0 & \boxed{1} & 1 & 0 & 0 & 0 \\
f_4 & 1 & 0 & 1 & 0 & \boxed{1} & 0 \\
f_5 & 0 & 1 & 0 & 0 & 0 & \boxed{1} \\
f_6 & 0 & 0 & \boxed{1} & 0 & 0 & 1
\end{array}
$$

$$G = (F, V, E)$$

$$F = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

$$V = \{\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, y_1, y_2\}$$

**In Part 1 of BLT - matching**

```
1  (match, assign) ← MATCH(G)
2  if not match
3     then return error "Singular"
```

Returns TRUE (steps omitted) with assignment

$$assign = \{\dot{x}_1 \mapsto f_2, \dot{x}_2 \mapsto f_3, \dot{x}_3 \mapsto f_6, \dot{x}_4 \mapsto f_1, y_1 \mapsto f_4, y_2 \mapsto f_5\}$$

# Algorithm: BLT Sort

broman@eecs.berkeley.edu

Input: a bipartite graph G

```
BLT(G)
(match, assign) ← MATCH(G)
if not match
   then return error "Singular"

D.V ← G.F
D.E ← ∅
for each (f, v) ∈ G.E where f ∈ G.F and assign[v] ≠ f
    do D.E ← D.E ∪ {(assign[v], f)}

MAKEEMPTY(O)
MAKEEMPTY(S)
i ← 0
lowlink ← ∅
number ← ∅
for each v ∈ D.V
    do if number[v] = NIL
          then STRONGCONNECT(v, D, S, i, lowlink, number, O)
return O
```

**Part 2**
Construct equation dependency graph

Output: a stack of sets of equation vertices, where each set represents an equation block in the BLT matrix.





# Example: BLT Sort

broman@eecs.berkeley.edu

$$
\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 & y_1 & y_2 \end{array} \\
\begin{pmatrix}
0 & 0 & \mathbf{1} & \boxed{\mathbf{1}} & 0 & \mathbf{1} \\
\boxed{\mathbf{1}} & \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} \\
0 & \boxed{\mathbf{1}} & \mathbf{1} & 0 & 0 & 0 \\
\mathbf{1} & 0 & \mathbf{1} & 0 & \boxed{\mathbf{1}} & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \boxed{\mathbf{1}} \\
0 & 0 & \boxed{\mathbf{1}} & 0 & 0 & \mathbf{1}
\end{pmatrix}
\end{array}
$$

$$G = (F, V, E)$$

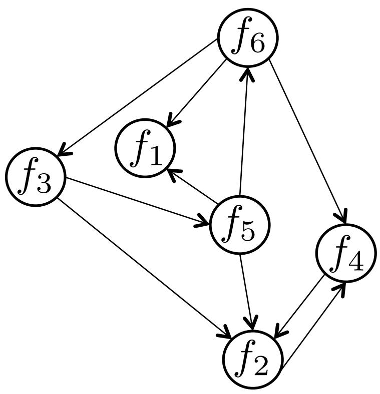
$$F = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

$$V = \{\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, y_1, y_2\}$$

$$assign = \{\dot{x}_1 \mapsto f_2, \dot{x}_2 \mapsto f_3, \dot{x}_3 \mapsto f_6, \dot{x}_4 \mapsto f_1, y_1 \mapsto f_4, y_2 \mapsto f_5\}$$

**In Part 2 of BLT – construct equation dependency graph (digraph)**

```
5  D.V ← G.F
6  D.E ← ∅
7  for each (f, v) ∈ G.E where f ∈ G.F and assign[v] ≠ f
8      do D.E ← D.E ∪ {(assign[v], f)}
```




$$D = (V, E)$$

$$V = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

$$E = \{f_2 \mapsto f_4, f_3 \mapsto f_2, f_3 \mapsto f_5, f_4 \mapsto f_2, f_5 \mapsto f_1, f_5 \mapsto f_2, f_5 \mapsto f_6, f_6 \mapsto f_1, f_6 \mapsto f_3, f_6, \mapsto f_4\}$$

**Exercise**
Create D graphically

# Algorithm: BLT Sort

broman@eecs.berkeley.edu

Input: a bipartite graph G

```
BLT(G)
(match, assign) ← MATCH(G)
if not match
   then return error "Singular"

D.V ← G.F
D.E ← ∅
for each (f, v) ∈ G.E where f ∈ G.F and assign[v] ≠ f
    do D.E ← D.E ∪ {(assign[v], f)}

MAKEEMPTY(O)
MAKEEMPTY(S)
i ← 0
lowlink ← ∅
number ← ∅
for each v ∈ D.V
    do if number[v] = NIL
          then STRONGCONNECT(v, D, S, i, lowlink, number, O)
return O
```

**Part 3**
Sort into blocks of equations using Tarjan's strongly connected component algorithm

Output: a stack of sets of equation vertices, where each set represents an equation block in the BLT matrix.





# Algorithm: StrongConnect (Tarjan)

broman@eecs.berkeley.edu

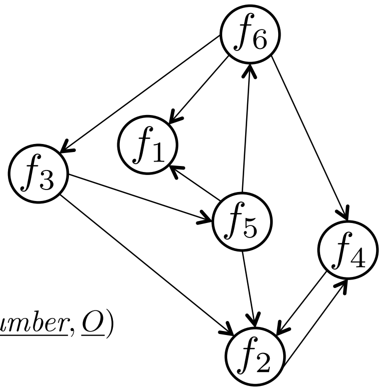
```
STRONGCONNECT(v, D, S, i, lowlink, number, O)
i ← i + 1
lowlink[v] ← i
number[v] ← i
PUSH(S, v)
for each w ∈ D.V where (v, w) ∈ D.E
    do if number[w] = NIL
          then STRONGCONNECT(w, D, S, i, lowlink, number, O)
               lowlink[v] ← MIN(lowlink[v], lowlink[w])
          else if w ∈ S and number[w] < number[v]
                  then lowlink[v] ← MIN(lowlink[v], number[w])
if lowlink[v] = number[v]
   then eqset ← ∅
        while not ISEMPTY(S) and number[TOP(S)] ≥ number[v]
            do eqset ← eqset ∪ {POP(S)}
        PUSH(O, eqset)
return
```

**Part I** DAE Basics | **Part II** Matching | **Part III** BLT Sorting | **Part IV** Pantelides | **Part V** Dummy Derivatives

# Algorithm: StrongConnect (Tarjan)

broman@eecs.berkeley.edu

```
STRONGCONNECT(v, D, S, i, lowlink, number, O)
  i ← i + 1
  lowlink[v] ← i
  number[v] ← i
  PUSH(S, v)
  for each w ∈ D.V where (v, w) ∈ D.E
      do if number[w] = NIL
            then STRONGCONNECT(w, D, S, i, lowlink, number, O)
                 lowlink[v] ← MIN(lowlink[v], lowlink[w])
            else if w ∈ S and number[w] < number[v]
                    then lowlink[v] ← MIN(lowlink[v], number[w])
  if lowlink[v] = number[v]
     then eqset ← ∅
          while not ISEMPTY(S) and number[TOP(S)] ≥ number[v]
             do eqset ← eqset ∪ {POP(S)}
          PUSH(O, eqset)
  return
```




**Exercise**
Construct stack O

**Top of the stack is to the left**
$O = [\{f_3, f_5, f_6\}, \{f_2, f_4\}, \{f_1\}]$





# Example: BLT Sort

broman@eecs.berkeley.edu

$$\begin{array}{c|cccccc} & \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 & y_1 & y_2 \\ \hline f_1 & 0 & 0 & 1 & 1 & 0 & 1 \\ f_2 & 1 & 1 & 0 & 0 & 1 & 1 \\ f_3 & 0 & 1 & 1 & 0 & 0 & 0 \\ f_4 & 1 & 0 & 1 & 0 & 1 & 0 \\ f_5 & 0 & 1 & 0 & 0 & 0 & 1 \\ f_6 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

$$G = (F, V, E)$$

$$F = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$
$$V = \{\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, y_1, y_2\}$$

$$assign = \{\dot{x}_1 \mapsto f_2, \dot{x}_2 \mapsto f_3, \dot{x}_3 \mapsto f_6, \dot{x}_4 \mapsto f_1, y_1 \mapsto f_4, y_2 \mapsto f_5\}$$

$$O = [\{f_3, f_5, f_6\}, \{f_2, f_4\}, \{f_1\}]$$

We can now create the sorted BLT matrix

$$\begin{array}{c|cccccc} & \dot{x}_2 & y_2 & \dot{x}_3 & \dot{x}_1 & y_1 & \dot{x}_4 \\ \hline f_3 & 1 & 0 & 1 & 0 & 0 & 0 \\ f_5 & 1 & 1 & 0 & 0 & 0 & 0 \\ f_6 & 0 & 1 & 1 & 0 & 0 & 0 \\ f_2 & 1 & 1 & 0 & 1 & 1 & 0 \\ f_4 & 0 & 0 & 1 & 1 & 1 & 0 \\ f_1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}$$

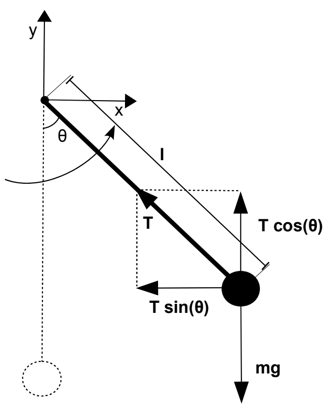
# Part IV

## Pantelides





# Example: Pendulum




**Pendulum in Cartesian coordinate system**

$$-T \cdot \frac{x}{l} = m\ddot{x}$$
$$-T \cdot \frac{y}{l} - mg = m\ddot{y}$$
$$x^2 + y^2 = l^2$$

Simplified using

$$-T/l = \lambda$$
$$m = 1$$
$$l^2 = L$$

**Simplified**

$$\ddot{x} = \lambda \cdot x$$
$$\ddot{y} = \lambda \cdot y - g$$
$$x^2 + y^2 = L$$

Is this an DAE?

Can we solve it?

Can we create BLT?

IDEA: Symbolically differentiate equations to get derivatives.

**Rewritten in first order**

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{u} = \lambda \cdot x$$
$$\dot{v} = \lambda \cdot y - g$$
$$x^2 + y^2 = L$$

**Incidence Matrix**

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{matrix} \dot{x} & \dot{y} & \dot{u} & \dot{v} & \lambda \end{matrix} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

No, we cannot find a matching.

# Pendulum: Graph Construction

broman@eecs.berkeley.edu

**System of equations**

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= \lambda \cdot x \\ \dot{v} &= \lambda \cdot y - g \\ x^2 + y^2 &= L\end{aligned}$$

$$\begin{aligned}f_1(\dot{x}, u) &= 0 \\ f_2(\dot{y}, v) &= 0 \\ f_3(\dot{u}, \lambda, x) &= 0 \\ f_4(\dot{v}, \lambda, y) &= 0 \\ f_5(x, y) &= 0\end{aligned}$$

Note that we include both differentiated and not differentiated variables.

**Construct a bipartite graph**

$G = (F, V, E)$

$$F = \{f_1, f_2, f_3, f_4, f_5\}$$

$$V = \{x, y, u, v, \dot{x}, \dot{y}, \dot{u}, \dot{v}, \lambda\}$$

$$\begin{aligned}E = \{&(f_1, \dot{x}), (f_1, u), \\ &(f_2, \dot{y}), (f_2, v), \\ &(f_3, \dot{u}), (f_3, \lambda), (f_3, x), \\ &(f_4, \dot{v}), (f_4, \lambda), (f_4, y), \\ &(f_5, x), (f_5, y)\}\end{aligned}$$

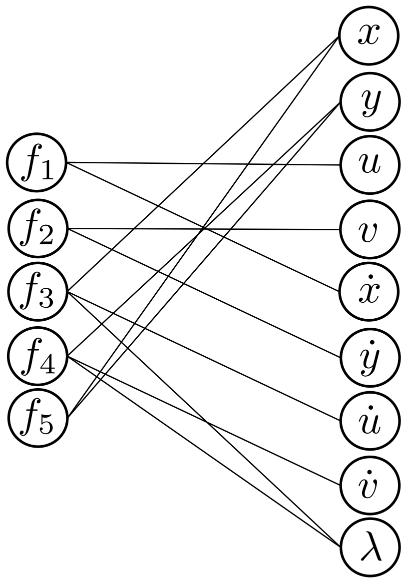




# Pendulum

broman@eecs.berkeley.edu

$$\begin{aligned}f_1(\dot{x}, u) &= 0 \\ f_2(\dot{y}, v) &= 0 \\ f_3(\dot{u}, \lambda, x) &= 0 \\ f_4(\dot{v}, \lambda, y) &= 0 \\ f_5(x, y) &= 0\end{aligned}$$



# Algorithm: Pantelides

```
PANTELIDES(G, vmap, eqmap)
 1  assign ← ∅
 2  for each e ∈ G.F
 3      do f ← e
 4          repeat
 5              C ← ∅
 6              match ← MATCH-EQUATION(G, f, C, assign, vmap)
 7              if not match
 8                then for each v ∈ C where v ∈ G.V
 9                       do let v′ be a vertex, such that v′ ∉ G.V
10                          vmap[v] ← v′
11                          G.V ← G.V ∪ {v′}
12                     for each f ∈ C where f ∈ G.F
13                       do let f′ be a vertex, such that f′ ∉ G.F
14                          eqmap[f] ← f′
15                          G.F ← G.F ∪ {f′}
16                          for each v ∈ G.V where (f, v) ∈ G.E
17                            do G.E ← G.E ∪ {(f′, v), (f′, vmap[v])}
18                     for each v ∈ C where v ∈ G.V
19                       do assign[vmap[v]] ← eqmap[assign[v]]
20                     f ← eqmap[f]
21          until match
22  return assign
```

**Mapping variables to differentiated variables**

$$vmap[v] = \begin{cases} v' & \text{if } \frac{dv}{dt} = v' \\ \text{NIL} & \text{otherwise} \end{cases}$$

**Mapping equations to their differentiated version**

$$eqmap[f] = \begin{cases} f' & \text{if } \frac{df}{dt} = f' \\ \text{NIL} & \text{otherwise} \end{cases}$$

**Assigns variables to equations**

$$assign[v] = \begin{cases} f & \text{if } f \text{ matches } v \\ \text{NIL} & \text{otherwise} \end{cases}$$





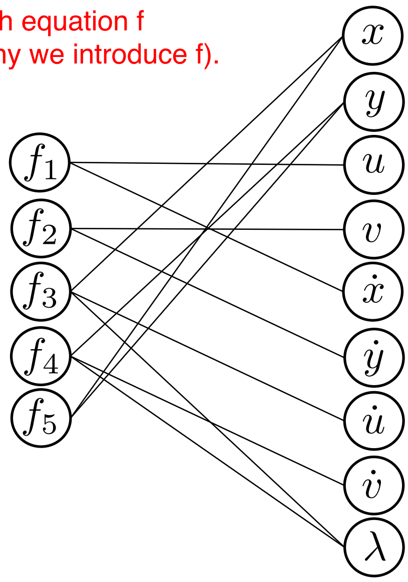
# Pendulum

We start with no variable to equation assignments.

Iterate over each equation f (we see later why we introduce f).

```
PANTELIDES(G, vmap, eqmap)
 1  assign ← ∅
 2  for each e ∈ G.F
 3      do f ← e
 4          repeat
 5              C ← ∅
```

Preparation for matching algorithm. Set all vertices to be uncolored.




**Initial state after step 5.**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$$
$$eqmap = \{\}$$
$$assign = \{\}$$
$$C = \{\}$$

# Algorithm: Pantelides

broman@eecs.berkeley.edu

PANTELIDES($G$, $\underline{vmap}$, $\underline{eqmap}$)

- $assign \leftarrow \emptyset$
- **for** each $e \in G.F$
  - **do** $f \leftarrow e$
    - **repeat**
      - $C \leftarrow \emptyset$
      - $match \leftarrow$ MATCH-EQUATION($G, f, \underline{C}, \underline{assign}, vmap$)
      - **if** not $match$
        - **then for** each $v \in C$ where $v \in G.V$
          - **do** let $v'$ be a vertex, such that $v' \notin G.V$
          - $vmap[v] \leftarrow v'$
          - $G.V \leftarrow G.V \cup \{v'\}$
        - **for** each $f \in C$ where $f \in G.F$
          - **do** let $f'$ be a vertex, such that $f' \notin G.F$
          - $eqmap[f] \leftarrow f'$
          - $G.F \leftarrow G.F \cup \{f'\}$
          - **for** each $v \in G.V$ where $(f, v) \in G.E$
            - **do** $G.E \leftarrow G.E \cup \{(f', v), (f', vmap[v])\}$
        - **for** each $v \in C$ where $v \in G.V$
          - **do** $assign[vmap[v]] \leftarrow eqmap[assign[v]]$
        - $f \leftarrow eqmap[f]$
    - **until** $match$
- **return** $assign$

Try to find a match for equation ***f***.





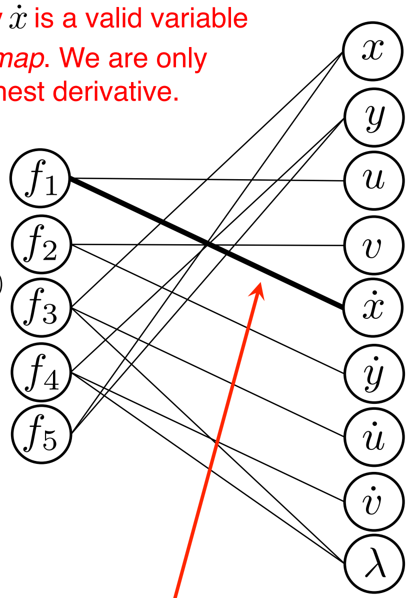
# Pendulum

broman@eecs.berkeley.edu

MATCH-EQUATION($G$, $f$, $\underline{C}$, $\underline{assign}$, $vmap$)

- $C \leftarrow C \cup \{f\}$
- **if** there exits a $v \in G.V$ such that $(f, v) \in G.E$
  - and $assign[v] = $ NIL and $vmap[v] = $ NIL
  - **then** $assign[v] \leftarrow f$
    - **return** TRUE
  - **else** **for** each $v$ where $(f, v) \in G.E$ and $v \notin C$
    - and $vmap[v] = $ NIL
    - **do** $C \leftarrow C \cup \{v\}$
      - **if** MATCH-EQUATION($G$, $assign[v]$, $\underline{C}$, $\underline{assign}$, $vmap$)
        - **then** $assign[v] \leftarrow f$
          - **return** TRUE
- **return** FALSE

Note that only $\dot{x}$ is a valid variable because of *vmap*. We are only match for highest derivative.




**State when returning from Match-Equation.**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$$
$$eqmap = \{\}$$
$$assign = \{\dot{x} \mapsto f_1\}$$
$$C = \{f_1\}$$

Matched variable to equation

Colored one equation

# Algorithm: Pantelides

broman@eecs.berkeley.edu

```
PANTELIDES(G, vmap, eqmap)
assign ← ∅
for each e ∈ G.F
    do f ← e
        repeat
                C ← ∅
                match ← MATCH-EQUATION(G, f, C, assign, vmap)
                if not match
                  then for each v ∈ C where v ∈ G.V
                           do let v′ be a vertex, such that v′ ∉ G.V
                              vmap[v] ← v′
                              G.V ← G.V ∪ {v′}
                       for each f ∈ C where f ∈ G.F
                           do let f′ be a vertex, such that f′ ∉ G.F
                              eqmap[f] ← f′
                              G.F ← G.F ∪ {f′}
                              for each v ∈ G.V where (f, v) ∈ G.E
                                  do G.E ← G.E ∪ {(f′, v), (f′, vmap[v])}
                       for each v ∈ C where v ∈ G.V
                           do assign[vmap[v]] ← eqmap[assign[v]]
                       f ← eqmap[f]
          until match
return assign
```

Function Match-Equation returns TRUE. Consequently, we break out of the repeat-until loop and proceeds with the next equation.



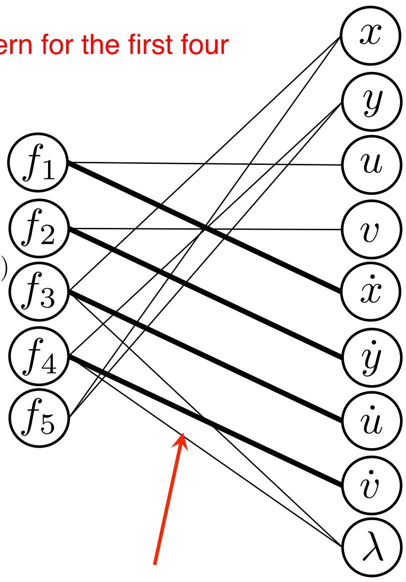


# Pendulum

broman@eecs.berkeley.edu

```
MATCH-EQUATION(G, f, C, assign, vmap)
C ← C ∪ {f}
if there exits a v ∈ G.V such that (f, v) ∈ G.E
   and assign[v] = NIL and vmap[v] = NIL
  then assign[v] ← f
       return TRUE
  else for each v where (f, v) ∈ G.E and v ∉ C
         and vmap[v] = NIL
          do C ← C ∪ {v}
             if MATCH-EQUATION(G, assign[v], C, assign, vmap)
               then assign[v] ← f
                    return TRUE
return FALSE
```

Same pattern for the first four equations.




**State after matching for** $f_1, f_2, f_3, f_4$

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$$
$$eqmap = \{\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$$
$$C = \{f_4\}$$

Matched 4 equations (could also have matched lambda).

Note that only the last equation is colored because colors are cleared before matching.

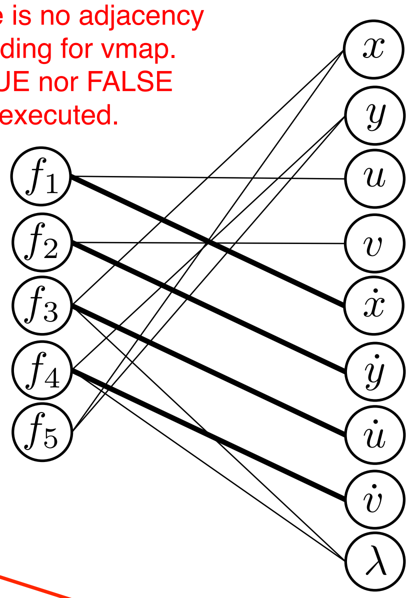
# Pendulum

broman@eecs.berkeley.edu

MATCH-EQUATION$(G, f, \underline{C}, \underline{assign}, vmap)$

$C \leftarrow C \cup \{f\}$
**if** there exits a $v \in G.V$ such that $(f, v) \in G.E$
  and $assign[v] = \text{NIL}$ and $vmap[v] = \text{NIL}$
  **then** $assign[v] \leftarrow f$
    **return** TRUE
  **else** **for** each $v$ where $(f, v) \in G.E$ and $v \notin C$
      and $vmap[v] = \text{NIL}$
      **do** $C \leftarrow C \cup \{v\}$
        **if** MATCH-EQUATION$(G, assign[v], \underline{C}, \underline{assign}, vmap)$
          **then** $assign[v] \leftarrow f$
            **return** TRUE
**return** FALSE

For $f_5$, there is no adjacency variable holding for vmap. Neither TRUE nor FALSE branch are executed.




**State after matching $f_5$**

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$
$eqmap = \{\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{f_5\}$

Algorithm Match-Equation returns FALSE and returns with $f_5$ colored.





# Algorithm: Pantelides

broman@eecs.berkeley.edu

PANTELIDES$(G, \underline{vmap}, \underline{eqmap})$

$assign \leftarrow \emptyset$
**for** each $e \in G.F$
  **do** $f \leftarrow e$
    **repeat**
      $C \leftarrow \emptyset$
      $match \leftarrow$ MATCH-EQUATION$(G, f, \underline{C}, \underline{assign}, vmap)$
      **if** not $match$
        **then** **for** each $v \in C$ where $v \in G.V$
            **do** let $v'$ be a vertex, such that $v' \notin G.V$
              $vmap[v] \leftarrow v'$
              $G.V \leftarrow G.V \cup \{v'\}$
          **for** each $f \in C$ where $f \in G.F$
            **do** let $f'$ be a vertex, such that $f' \notin G.F$
              $eqmap[f] \leftarrow f'$
              $G.F \leftarrow G.F \cup \{f'\}$
              **for** each $v \in G.V$ where $(f, v) \in G.E$
                **do** $G.E \leftarrow G.E \cup \{(f', v), (f', vmap[v])\}$
          **for** each $v \in C$ where $v \in G.V$
            **do** $assign[vmap[v]] \leftarrow eqmap[assign[v]]$
          $f \leftarrow eqmap[f]$
    **until** $match$
**return** $assign$

We have match = FALSE

No colored variables.

But we have one colored equation.

No colored variables.

# Pendulum

broman@eecs.berkeley.edu

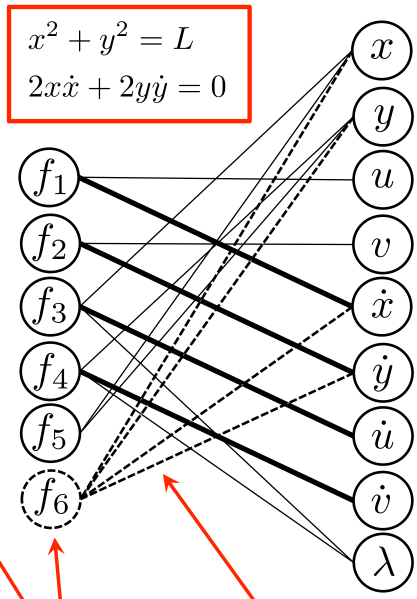
**State after matching $f_5$**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$$
$$eqmap = \{\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$$
$$C = \{f_5\}$$

```
for each f ∈ C where f ∈ G.F
    do let f′ be a vertex, such that f′ ∉ G.F
       eqmap[f] ← f′
       G.F ← G.F ∪ {f′}
       for each v ∈ G.V where (f, v) ∈ G.E
           do G.E ← G.E ∪ {(f′, v), (f′, vmap[v])}
```

**State after creating differentiated equation.**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$$
$$eqmap = \{f_5 \mapsto f_6\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$$
$$C = \{f_5\}$$




Create a new equation node ***f*$_6$** by differentiating ***f*$_5$**.

Create edges to variables and their derivatives.





# Algorithm: Pantelides

broman@eecs.berkeley.edu

```
PANTELIDES(G, vmap, eqmap)
 assign ← ∅
 for each e ∈ G.F
   do f ← e
      repeat
            C ← ∅
            match ← MATCH-EQUATION(G, f, C, assign, vmap)
            if not match
              then for each v ∈ C where v ∈ G.V
                      do let v′ be a vertex, such that v′ ∉ G.V
                         vmap[v] ← v′
                         G.V ← G.V ∪ {v′}
                   for each f ∈ C where f ∈ G.F
                      do let f′ be a vertex, such that f′ ∉ G.F
                         eqmap[f] ← f′
                         G.F ← G.F ∪ {f′}
                         for each v ∈ G.V where (f, v) ∈ G.E
                             do G.E ← G.E ∪ {(f′, v), (f′, vmap[v])}
                   for each v ∈ C where v ∈ G.V
                      do assign[vmap[v]] ← eqmap[assign[v]]
                   f ← eqmap[f]
        until match
 return assign
```

Repeat again (match was FALSE), but now with the differentiated equation ***f*$_6$**.

$$eqmap = \{f_5 \mapsto f_6\}$$

## Pendulum

broman@eecs.berkeley.edu

**State before calling Match-Equation**

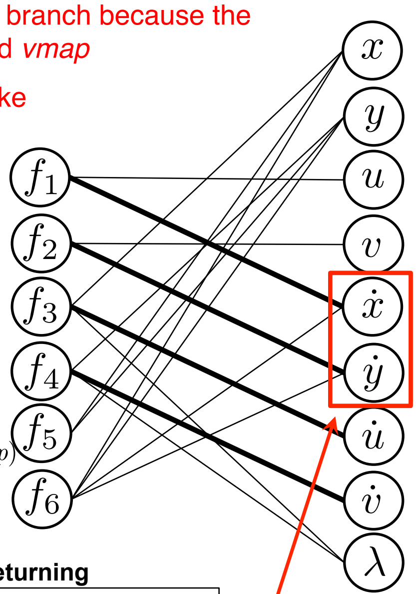
$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$
$eqmap = \{f_5 \mapsto f_6\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{\}$

MATCH-EQUATION$(G, f, \underline{C}, \underline{assign}, vmap)$

1 $C \leftarrow C \cup \{f\}$
2 **if** there exits a $v \in G.V$ such that $(f, v) \in G.E$
3 and $assign[v] = \text{NIL}$ and $vmap[v] = \text{NIL}$
4 **then** $assign[v] \leftarrow f$
5 **return** TRUE
6 **else for** each $v$ where $(f, v) \in G.E$ and $v \notin C$
7 and $vmap[v] = \text{NIL}$
8 **do** $C \leftarrow C \cup \{v\}$
9 **if** MATCH-EQUATION$(G, assign[v], \underline{C}, \underline{assign}, vmap)$
10 **then** $assign[v] \leftarrow f$
11 **return** TRUE
12 **return** FALSE

Cannot take TRUE branch because the states of *assign* and *vmap*

Color and make recursive call.




Two variables are applicable for the FALSE branch.

**Before first recursive call at line 9**

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$
$eqmap = \{f_5 \mapsto f_6\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{f_6, \dot{x}\}$

**After recursive when returning**

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$
$eqmap = \{f_5 \mapsto f_6\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$





## Algorithm: Pantelides

broman@eecs.berkeley.edu

PANTELIDES$(G, \underline{vmap}, \underline{eqmap})$

$assign \leftarrow \emptyset$
**for** each $e \in G.F$
**do** $f \leftarrow e$
**repeat**
$C \leftarrow \emptyset$
$match \leftarrow$ MATCH-EQUATION$(G, f, \underline{C}, \underline{assign}, vmap)$
**if** not $match$
**then for** each $v \in C$ where $v \in G.V$
**do** let $v'$ be a vertex, such that $v' \notin G.V$
$vmap[v] \leftarrow v'$
$G.V \leftarrow G.V \cup \{v'\}$
**for** each $f \in C$ where $f \in G.F$
**do** let $f'$ be a vertex, such that $f' \notin G.F$
$eqmap[f] \leftarrow f'$
$G.F \leftarrow G.F \cup \{f'\}$
**for** each $v \in G.V$ where $(f, v) \in G.E$
**do** $G.E \leftarrow G.E \cup \{(f', v), (f', vmap[v])\}$
**for** each $v \in C$ where $v \in G.V$
**do** $assign[vmap[v]] \leftarrow eqmap[assign[v]]$
$f \leftarrow eqmap[f]$
**until** $match$
**return** $assign$

$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$

Implicitly differentiating equation two times

$$x^2 + y^2 = L$$
$$2x\dot{x} + 2y\dot{y} = 0$$
$$2x\ddot{x} + 2\dot{x}^2 + 2y\ddot{y} + 2\dot{y}^2 = 0$$

First step: create new differentiated variables

# Pendulum

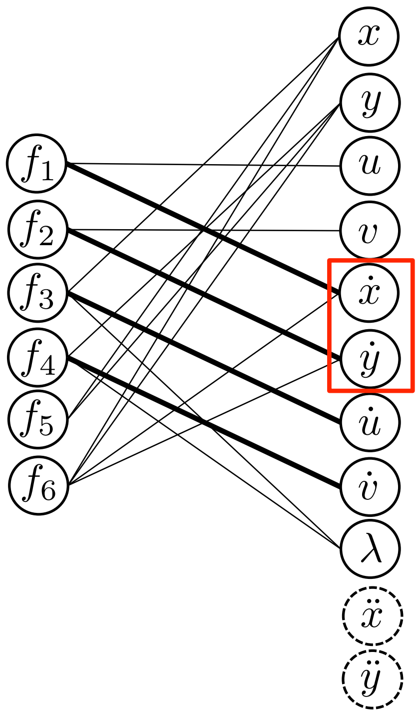
broman@eecs.berkeley.edu

**State before creating new variables**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}\}$$
$$eqmap = \{f_5 \mapsto f_6\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$$
$$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$$

**for** each $v \in C$ where $v \in G.V$
 **do** let $v'$ be a vertex, such that $v' \notin G.V$
  $vmap[v] \leftarrow v'$
  $G.V \leftarrow G.V \cup \{v'\}$

New variables and mapping




**After adding new variables**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$$
$$eqmap = \{f_5 \mapsto f_6\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$$
$$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$$





# Algorithm: Pantelides

broman@eecs.berkeley.edu

PANTELIDES($G$, $\underline{vmap}$, $\underline{eqmap}$)

```
assign ← ∅
for each e ∈ G.F
    do f ← e
       repeat
            C ← ∅
            match ← MATCH-EQUATION(G, f, C, assign, vmap)
            if not match
              then for each v ∈ C where v ∈ G.V
                       do let v' be a vertex, such that v' ∉ G.V
                          vmap[v] ← v'
                          G.V ← G.V ∪ {v'}
                   for each f ∈ C where f ∈ G.F
                       do let f' be a vertex, such that f' ∉ G.F
                          eqmap[f] ← f'
                          G.F ← G.F ∪ {f'}
                          for each v ∈ G.V where (f, v) ∈ G.E
                              do G.E ← G.E ∪ {(f', v), (f', vmap[v])}
                   for each v ∈ C where v ∈ G.V
                       do assign[vmap[v]] ← eqmap[assign[v]]
                   f ← eqmap[f]
          until match
return assign
```

Second step: create new differentiated equation nodes

# Pendulum

broman@eecs.berkeley.edu

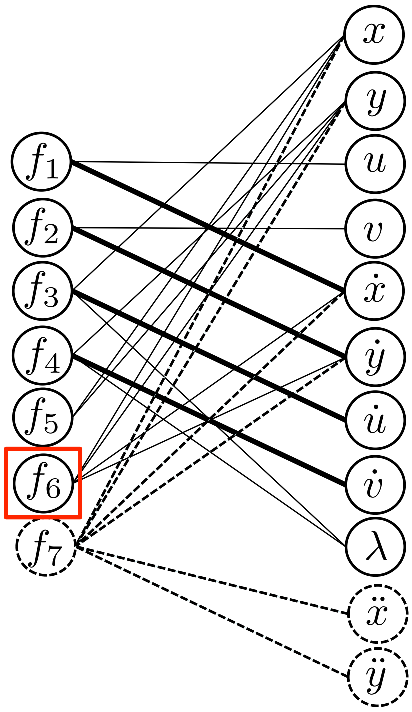
**State before creating equation nodes**

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$
$eqmap = \{f_5 \mapsto f_6\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$

**for** each $f \in C$ where $f \in G.F$
  **do** let $f'$ be a vertex, such that $f' \notin G.F$
    $eqmap[f] \leftarrow f'$
    $G.F \leftarrow G.F \cup \{f'\}$
    **for** each $v \in G.V$ where $(f, v) \in G.E$
      **do** $G.E \leftarrow G.E \cup \{(f', v), (f', vmap[v])\}$

**After adding equation $f_6$**

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$
$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$







# Pendulum

broman@eecs.berkeley.edu

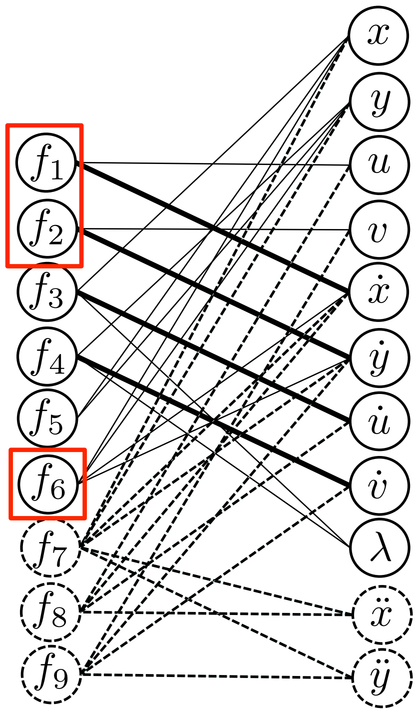
**State before creating equation nodes**

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$
$eqmap = \{f_5 \mapsto f_6\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$

**for** each $f \in C$ where $f \in G.F$
  **do** let $f'$ be a vertex, such that $f' \notin G.F$
    $eqmap[f] \leftarrow f'$
    $G.F \leftarrow G.F \cup \{f'\}$
    **for** each $v \in G.V$ where $(f, v) \in G.E$
      **do** $G.E \leftarrow G.E \cup \{(f', v), (f', vmap[v])\}$

**After adding all equations**

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$
$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$
$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$



# Algorithm: Pantelides

broman@eecs.berkeley.edu

```
PANTELIDES(G, vmap, eqmap)
  assign ← ∅
  for each e ∈ G.F
      do f ← e
         repeat
                C ← ∅
                match ← MATCH-EQUATION(G, f, C, assign, vmap)
                if not match
                  then for each v ∈ C where v ∈ G.V
                           do let v′ be a vertex, such that v′ ∉ G.V
                              vmap[v] ← v′
                              G.V ← G.V ∪ {v′}
                       for each f ∈ C where f ∈ G.F
                           do let f′ be a vertex, such that f′ ∉ G.F
                              eqmap[f] ← f′
                              G.F ← G.F ∪ {f′}
                              for each v ∈ G.V where (f, v) ∈ G.E
                                  do G.E ← G.E ∪ {(f′, v), (f′, vmap[v])}
                       for each v ∈ C where v ∈ G.V
                           do assign[vmap[v]] ← eqmap[assign[v]]
                       f ← eqmap[f]
           until match
  return assign
```

Third step: assign variables to equations for new variables.





# Pendulum

broman@eecs.berkeley.edu

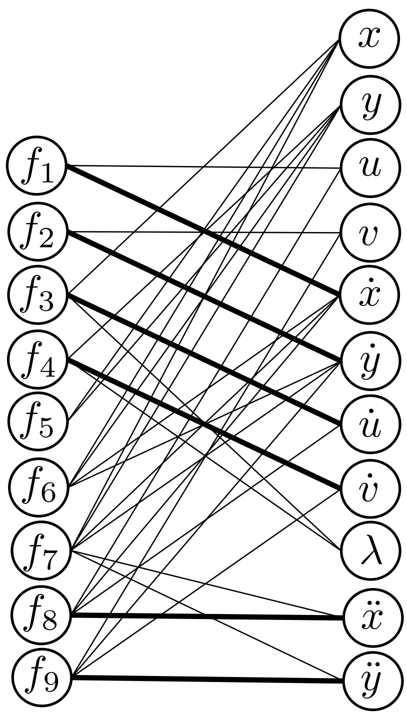
**After adding all equations**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$$
$$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4\}$$
$$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$$

**for** each $v \in C$ where $v \in G.V$
  **do** $assign[vmap[v]] \leftarrow eqmap[assign[v]]$

**After adding new assignments**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$$
$$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4, \ddot{x} \mapsto f_8, \ddot{y} \mapsto f_9\}$$
$$C = \{f_6, \dot{x}, f_1, \dot{y}, f_2\}$$



# Algorithm: Pantelides

broman@eecs.berkeley.edu

PANTELIDES($G$, $\underline{vmap}$, $\underline{eqmap}$)

$assign \leftarrow \emptyset$
**for** each $e \in G.F$
  **do** $f \leftarrow e$
    **repeat**
      $C \leftarrow \emptyset$
      $match \leftarrow$ MATCH-EQUATION($G, f, \underline{C}, \underline{assign}, vmap$)
      **if** not $match$
        **then for** each $v \in C$ where $v \in G.V$
          **do** let $v'$ be a vertex, such that $v' \notin G.V$
            $vmap[v] \leftarrow v'$
            $G.V \leftarrow G.V \cup \{v'\}$
        **for** each $f \in C$ where $f \in G.F$
          **do** let $f'$ be a vertex, such that $f' \notin G.F$
            $eqmap[f] \leftarrow f'$
            $G.F \leftarrow G.F \cup \{f'\}$
            **for** each $v \in G.V$ where $(f, v) \in G.E$
              **do** $G.E \leftarrow G.E \cup \{(f', v), (f', vmap[v])\}$
        **for** each $v \in C$ where $v \in G.V$
          **do** $assign[vmap[v]] \leftarrow eqmap[assign[v]]$
        $f \leftarrow eqmap[f]$
    **until** $match$
**return** $assign$

Repeat again with second differentiated version of equation five.





# Pendulum

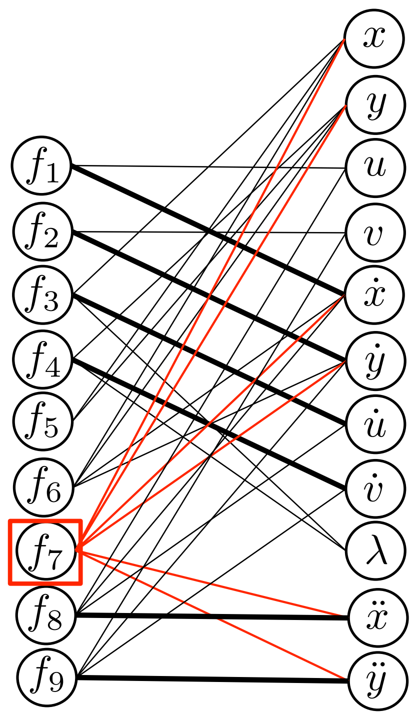
broman@eecs.berkeley.edu

**Before matching**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$$
$$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4, \ddot{x} \mapsto f_8, \ddot{y} \mapsto f_9\}$$

MATCH-EQUATION($G$, $f$, $\underline{C}$, $\underline{assign}$, $vmap$)

$C \leftarrow C \cup \{f\}$
**if** there exits a $v \in G.V$ such that $(f, v) \in G.E$
  and $assign[v] =$ NIL and $vmap[v] =$ NIL
  **then** $assign[v] \leftarrow f$
    **return** TRUE
  **else for** each $v$ where $(f, v) \in G.E$ and $v \notin C$
      and $vmap[v] =$ NIL
      **do** $C \leftarrow C \cup \{v\}$
        **if** MATCH-EQUATION($G$, $assign[v]$, $\underline{C}$, $\underline{assign}$, $vmap$)
          **then** $assign[v] \leftarrow f$
            **return** TRUE
**return** FALSE




For clarity: view variables and edges where vmap[v] = NIL

# Pendulum

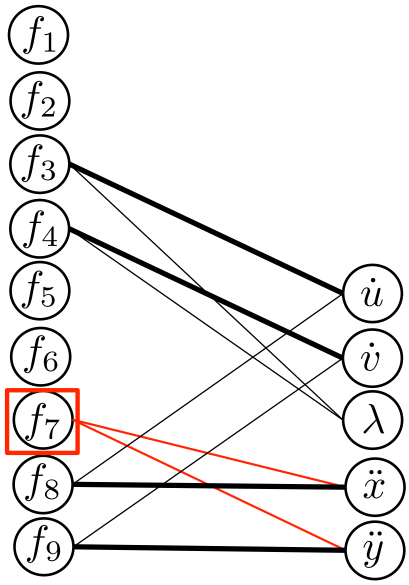
broman@eecs.berkeley.edu

**Before matching**

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$$
$$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4, \ddot{x} \mapsto f_8, \ddot{y} \mapsto f_9\}$$

MATCH-EQUATION$(G, f, \underline{C}, \underline{assign}, vmap)$

$C \leftarrow C \cup \{f\}$
**if** there exits a $v \in G.V$ such that $(f, v) \in G.E$
  and $assign[v] =$ NIL and $vmap[v] =$ NIL
  **then** $assign[v] \leftarrow f$
    **return** TRUE
  **else** **for** each $v$ where $(f, v) \in G.E$ and $v \notin C$
    and $vmap[v] =$ NIL
      **do** $C \leftarrow C \cup \{v\}$
        **if** MATCH-EQUATION$(G, assign[v], \underline{C}, \underline{assign}, vmap)$
          **then** $assign[v] \leftarrow f$
            **return** TRUE
**return** FALSE







# Pendulum

broman@eecs.berkeley.edu

**Before matching**

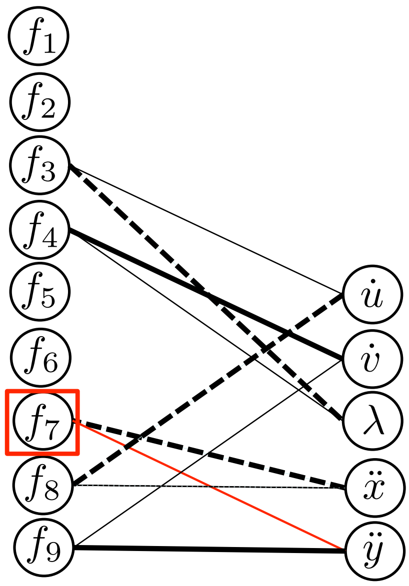
$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$$
$$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_3, \dot{v} \mapsto f_4, \ddot{x} \mapsto f_8, \ddot{y} \mapsto f_9\}$$

MATCH-EQUATION$(G, f, \underline{C}, \underline{assign}, vmap)$

$C \leftarrow C \cup \{f\}$
**if** there exits a $v \in G.V$ such that $(f, v) \in G.E$
  and $assign[v] =$ NIL and $vmap[v] =$ NIL
  **then** $assign[v] \leftarrow f$
    **return** TRUE
  **else** **for** each $v$ where $(f, v) \in G.E$ and $v \notin C$
    and $vmap[v] =$ NIL
      **do** $C \leftarrow C \cup \{v\}$
        **if** MATCH-EQUATION$(G, assign[v], \underline{C}, \underline{assign}, vmap)$
          **then** $assign[v] \leftarrow f$
            **return** TRUE
**return** FALSE

$$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$$
$$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$$
$$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_8, \dot{v} \mapsto f_4, \lambda \mapsto f_3, \ddot{x} \mapsto f_7, \ddot{y} \mapsto f_9\}$$
$$C = \{f_7, \ddot{x}, f_8, \dot{u}, f_3\}$$

Successful match!



# Algorithm: Pantelides

broman@eecs.berkeley.edu

PANTELIDES$(G, \underline{vmap}, \underline{eqmap})$

$assign \leftarrow \emptyset$
**for** each $e \in G.F$
  **do** $f \leftarrow e$
    **repeat**
      $C \leftarrow \emptyset$
      $match \leftarrow$ MATCH-EQUATION$(G, f, \underline{C}, \underline{assign}, vmap)$
      **if** not $match$
        **then for** each $v \in C$ where $v \in G.V$
          **do** let $v'$ be a vertex, such that $v' \notin G.V$
            $vmap[v] \leftarrow v'$
            $G.V \leftarrow G.V \cup \{v'\}$
        **for** each $f \in C$ where $f \in G.F$
          **do** let $f'$ be a vertex, such that $f' \notin G.F$
            $eqmap[f] \leftarrow f'$
            $G.F \leftarrow G.F \cup \{f'\}$
            **for** each $v \in G.V$ where $(f, v) \in G.E$
              **do** $G.E \leftarrow G.E \cup \{(f', v), (f', vmap[v])\}$
        **for** each $v \in C$ where $v \in G.V$
          **do** $assign[vmap[v]] \leftarrow eqmap[assign[v]]$
        $f \leftarrow eqmap[f]$
    **until** $match$
**return** $assign$

Last equation and successful match.
Algorithm terminates.



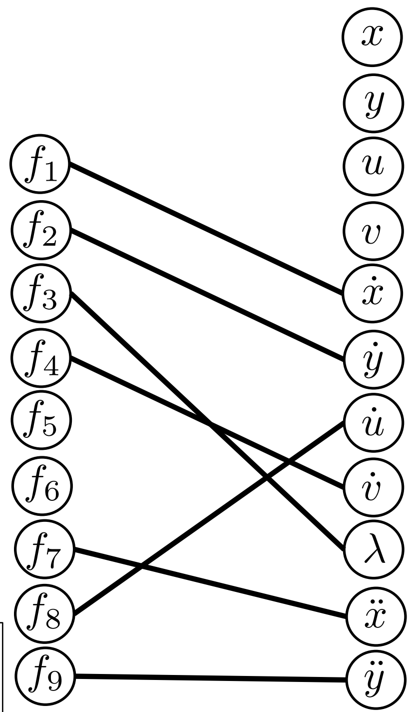


# Result of Pantelides on Pendulum

broman@eecs.berkeley.edu

$vmap = \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\}$
$eqmap = \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\}$
$assign = \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_8, \dot{v} \mapsto f_4, \lambda \mapsto f_3, \ddot{x} \mapsto f_7, \ddot{y} \mapsto f_9\}$

| | | |
|---|---|---|
| (1) | $\dot{x} = u$ | $f_1(\dot{x}, u) = 0$ |
| (2) | $\dot{y} = v$ | $f_2(\dot{y}, v) = 0$ |
| (3) | $\dot{u} = \lambda \cdot x$ | $f_3(\dot{u}, \lambda, x) = 0$ |
| (4) | $\dot{v} = \lambda \cdot y - g$ | $f_4(\dot{v}, \lambda, y) = 0$ |
| (5) | $x^2 + y^2 = L$ | $f_5(x, y) = 0$ |
| (6) | $2x\dot{x} + 2y\dot{y} = 0$ | $f_6(x, \dot{x}, y, \dot{y}) = 0$ |
| (7) | $2x\ddot{x} + 2\dot{x}^2 + 2y\ddot{y} + 2\dot{y}^2 = 0$ | $f_7(x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y}) = 0$ |
| (8) | $\ddot{x} = \dot{u}$ | $f_8(\ddot{x}, \dot{u}) = 0$ |
| (9) | $\ddot{y} = \dot{v}$ | $f_9(\ddot{y}, \dot{v}) = 0$ |



$|G.F| = 9 \qquad |G.V| = 11$

**Two variables out of the set G.V can be given arbitrary initialization values, as long as all constraints above are satisfied.**

## Result of Pantelides on Pendulum

broman@eecs.berkeley.edu

$$\begin{aligned} vmap &= \{x \mapsto \dot{x}, y \mapsto \dot{y}, u \mapsto \dot{u}, v \mapsto \dot{v}, \dot{x} \mapsto \ddot{x}, \dot{y} \mapsto \ddot{y}\} \\ eqmap &= \{f_5 \mapsto f_6, f_6 \mapsto f_7, f_1 \mapsto f_8, f_2 \mapsto f_9\} \\ assign &= \{\dot{x} \mapsto f_1, \dot{y} \mapsto f_2, \dot{u} \mapsto f_8, \dot{v} \mapsto f_4, \\ & \quad \lambda \mapsto f_3, \ddot{x} \mapsto f_7, \ddot{y} \mapsto f_9\} \end{aligned}$$

Is the system of equations solvable if we replace the old equations with their differentiated version?

$$\begin{aligned} &(1) \quad \dot{x} = u \\ &(2) \quad \dot{y} = v \\ &(3) \quad \dot{u} = \lambda \cdot x \\ &(4) \quad \dot{v} = \lambda \cdot y - g \\ &(5) \quad x^2 + y^2 = L \\ &(6) \quad 2x\dot{x} + 2y\dot{y} = 0 \\ &(7) \quad 2x\ddot{x} + 2\dot{x}^2 + 2y\ddot{y} + 2\dot{y}^2 = 0 \\ &(8) \quad \ddot{x} = \dot{u} \\ &(9) \quad \ddot{y} = \dot{v} \end{aligned}$$

By substituting (8) and (9) we have

$$\ddot{x} = \lambda \cdot x$$

$$\ddot{y} = \lambda \cdot y - g$$

$$2x\ddot{x} + 2\dot{x}^2 + 2y\ddot{y} + 2\dot{y}^2 = 0$$

Which is solvable for highest derivative

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} \lambda & \ddot{y} & \ddot{x} \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 \\ 0 & \mathbf{1} & \mathbf{1} \end{pmatrix} \end{array}$$

Same result if converted into order one equation

$$\begin{array}{c} \\ f_2 \\ f_1 \\ f_5 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \dot{y} & \dot{x} & \dot{u} & \lambda & \dot{v} \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \end{pmatrix} \end{array}$$





## Termination of Pantelides

broman@eecs.berkeley.edu

Does Pantelides algorithm terminate?

Depends on the input graph.

Before Pantelides, check that matching can be found on a matrix where variables do not distinguish if they appear differentiated or not.

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= \lambda \cdot x \\ \dot{v} &= \lambda \cdot y - g \\ x^2 + y^2 &= L \end{aligned}$$

$$\begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & u & v & \lambda \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \end{pmatrix} \end{array}$$

**Matrix to check**

$$\begin{array}{c} \\ f_1 \\ f_3 \\ f_4 \\ f_2 \\ f_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u & \lambda & v & y & x \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \end{pmatrix} \end{array}$$

Yes, match was found. Hence the problem is not structurally singular.

# Part IV

## Dummy Derivatives





# Index Reduction

Should differentiated equations from Pantelides be used for index reduction?

$$\ddot{x} = \lambda \cdot x$$
$$\ddot{y} = \lambda \cdot y - g$$
$$2x\ddot{x} + 2\dot{x}^2 + 2y\ddot{y} + 2\dot{y}^2 = 0$$

The reduced problem (index-1) is mathematically correct, but since equation

$$x^2 + y^2 = L$$

is not present, numerical approximation gives a "drifting problem". In our example, the pendulum's length will grow...

# Dummy Derivative

broman@eecs.berkeley.edu

Basic Idea:
- Include all differentiated equations
- For each equation, introduce a “dummy derivative” variable.

$$\ddot{x} = \lambda \cdot x$$
$$y'' = \lambda \cdot y - g$$
$$x^2 + y^2 = L$$
$$2x\dot{x} + 2yy' = 0$$
$$2x\ddot{x} + 2\dot{x}^2 + 2yy'' + 2y'^2 = 0$$

All constraints are present and the number of equations and unknowns match.

The actual algorithm is presented by Mattson and Söderlind (1993)





# Conclusions

broman@eecs.berkeley.edu

**Matching**
Finds a unique mapping between variables and equations. Used both in BLT sorting and Pantelides algorithm

**BLT Sorting**
Sort blocks of equation, where each block represents an algebraic loop. Uses matching and Tarjan’s algorithm

**Pantelides**
Determine the subset of equations that needs to be differentiated.

**Dummy Derivative**
Method that uses Pantelides to perform correct index reduction.

**Thank you for listening!**

# References and Further Reading




- Iain S. Duff. On Algorithms for Obtaining a Maximum Transversal. ACM Transactions on Mathematical Software, 7(3):315–330, 1981.
- Iain S. Duff and John K. Reid. An Implementation of Tarjan's Algorithm for the Block Triangularization of a Matrix. ACM Transactions on Mathe- matical Software, 4(2):137–147, 1978.
- S. E. Mattsson, H. Olsson, and H. Elmqvist. Dynamic selection of states in fymola. In Proceedings of the Modelica Workshop, pages 61–67, 2000.
- S. E. Mattsson and G. Söderlind. Index reduction in differential-algebraic equations using dummy derivatives. SIAM Journal on Scientific Computing, 14(3):677–692, 1993.
- C. C. Pantelides. The Consistent Initialization of Differential-Algebraic Sys- tems. SIAM Journal on Scientific and Statistical Computing, 9(2):213–231, 1988.
- R. Tarjan. Depth-first search and linear graph algorithms. SIAM Journal on Computing, 1(2):146–160, 1972.