Petri Nets

eee249

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Most slides borrowed from
Luciano Lavagno’s lecture ee249 (1998)

Models Of Computation
for reactive systems

- Main MOCs:
  - Communicating Finite State Machines
  - Dataflow Process Networks
  - Discrete Event
  - Codesign Finite State Machines
  - Petri Nets

- Main languages:
  - StateCharts
  - Esterel
  - Dataflow networks
Outline

- Petri nets
  - introduction
  - examples
  - properties
  - analysis techniques

Petri Nets (PNs)

- Model introduced by C.A. Petri in 1962
  - Ph.D. Thesis: “Communication with Automata”
- Applications: distributed computing, manufacturing, control, communication networks, transportation…
- PNs describe explicitly and graphically:
  - sequencing/causality
  - conflict/non-deterministic choice
  - concurrency
- Asynchronous model (partial ordering)
- Main drawback: no hierarchy
Petri Net Graph

- Bipartite weighted directed graph:
  - Places: circles
  - Transitions: bar or boxes
  - Arcs: arrows labeled with weights
- Tokens: black dots

A PN \((N, M_0)\) is a Petri Net Graph \(N\)

- places: represent distributed state by holding tokens
  - marking (state) \(M\) is an \(n\)-vector \((m_1, m_2, m_3, \ldots)\), where \(m_i\) is the non-negative number of tokens in place \(p_i\).
  - initial marking \((M_0)\) is initial state
- transitions: represent actions/events
  - enabled transition: enough tokens in predecessors
  - firing transition: modifies marking
- \(\ldots\) and an initial marking \(M_0\).

Places/Transition: conditions/events
Transition firing rule

- A marking is changed according to the following rules:
  - A transition is enabled if there are enough tokens in each input place
  - An enabled transition may or may not fire
  - The firing of a transition modifies marking by consuming tokens from the input places and producing tokens in the output places

Concurrency, causality, choice
Concurrency, causality, choice

Concurrent processes:

1. $t_1$
2. $t_2$
3. $t_3$
4. $t_4$
5. $t_5$
6. $t_6$

Causality, sequencing:

- $t_1$, $t_2$, $t_3$, $t_4$, $t_5$, $t_6$
Concurrency, causality, choice

Concurrency, causality, choice
Confusion

- t1 and t2 are concurrent but their firing order is not irrelevant for conflict resolution (not local choice)
- From (1,1,0,0) to (0,0,1,1)
  - solving a conflict (t1,t2)
  - not solving a conflict (t2,t1)

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Producer-Consumer Problem

Produce

Buffer

Consume

Produce

Buffer

Consume
Producer-Consumer Problem

Produce
Buffer
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Produce

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Producer-Consumer Problem

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Producer-Consumer Problem
PN properties

- Behavioral: depend on the initial marking (most interesting)
  - Reachability
  - Boundedness
  - Schedulability
  - Liveness
  - Conservation

- Structural: do not depend on the initial marking (often too restrictive)
  - Consistency
  - Structural boundedness

Reachability

- Marking $M$ is reachable from marking $M_0$ if there exists a sequence of firings $\sigma = M_0 t_1 M_1 t_2 M_2 \ldots M$ that transforms $M_0$ to $M$.

- The reachability problem is decidable.
Liveness

- *Liveness*: from any marking any transition can become fireable
  - Liveness implies deadlock freedom, not vice versa

Not live

Liveness

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  - Liveness implies deadlock freedom, not vice versa

Not live
Boundedness

- **Boundedness**: the number of tokens in any place cannot grow indefinitely
  - (1-bounded also called *safe*)
  - Application: places represent buffers and registers (check there is no overflow)

Unbounded
Boundedness

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**Boundedness**

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![Diagram of boundedness](image)

**Conservation**

- *(Strict) Conservation*: the total number of tokens in the net is constant

![Diagram of conservation](image)
(Strict) Conservation: the total number of tokens in the net is constant
Analysis techniques

- Structural analysis techniques
  - Incidence matrix
  - T- and S- Invariants

- State Space Analysis techniques
  - Coverability Tree
  - Reachability Graph

Incidence Matrix

Necessary condition for marking $M$ to be reachable from initial marking $M_0$:
there exists firing vector $v$ s.t.:

$$M = M_0 + A v$$
State equations

- E.g. reachability of $M = [0 \ 0 \ 1]^T$ from $M_0 = [1 \ 0 \ 0]^T$

\[ A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \]

\[ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

\[ v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \]

\[ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \]

but also $v_2 = [1 \ 1 \ 2]^T$ or any $v_k = [1 \ (k) \ (k+1)]^T$

Necessary Condition only

Firing vector: (1,2,2)  
Deadlock!!
State equations and invariants

- Solutions of $Ax = 0$ (in $M = M_0 + Ax, M = M_0$)

**T-invariants**
- sequences of transitions that (if fireable) bring back to original marking
- periodic schedule in SDF
- e.g. $x = [0 \ 1 \ 1]^T$

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
1 & 1 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]

Application of T-invariants

- Scheduling
  - Cyclic schedules: need to return to the initial state

T-invariant: $(1,1,1,1,1)$
Schedule: $i \ast k2 \ast k1 + o$
State equations and invariants

- Solutions of \( yA = 0 \)

**S-invariants**
- sets of places whose total token count does not change after the firing of any transition (\( yM = yM' \))
- e.g. \( y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \)

\[
A^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\]

Application of S-invariants

- **Structural Boundedness**: bounded for any finite initial marking \( M_0 \)
- **Existence of a positive S-invariant** is CS for structural boundedness
  - initial marking is finite
  - weighted token count does not change
Summary of algebraic methods

- Extremely efficient
  (polynomial in the size of the net)
- Generally provide only necessary or sufficient information
- Excellent for ruling out some deadlocks or otherwise dangerous conditions
- Can be used to infer structural boundedness

Coverability Tree

- Boundedness is decidable
  with coverability tree
Coverability Tree

■ Boundedness is decidable
with coverability tree

```
  p1  t1  p2  t2  p3  t3  p4
  1000
  0100
  0011
```

Coverability Tree

■ Boundedness is decidable
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```
  p1  t1  p2  t2  p3  t3  p4
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```
Coverability Tree

- Boundedness is decidable with coverability tree

Cannot solve the reachability and liveness problems
For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings.
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Subclasses of Petri nets

- Reachability analysis is too expensive
- State equations give only partial information
- Some properties are preserved by reduction rules
  e.g. for liveness and safeness

- Even reduction rules only work in some cases
- Must restrict class in order to prove stronger results

Subclasses of Petri nets: SMs

- State machine: every transition has at most 1 predecessor and 1 successor
- Models only causality and conflict
  - (no concurrency, no synchronization of parallel activities)
Subclasses of Petri nets: MGs

- Marked Graph: every place has at most 1 predecessor and 1 successor
- Models only causality and concurrency (no conflict)
- Same as underlying graph of SDF
- Studied by Commoner et al. ('71)

Subclasses of Petri nets: FC nets

- Free-Choice net: every transition after choice has exactly 1 predecessor
Free-Choice Petri Nets (FCPN)

Free-Choice (FC)

Confusion (not-Free-Choice)  Extended Free-Choice

- Free-Choice: the outcome of a choice depends on the value of a token (abstracted non-deterministically) rather than on its arrival time.
  - Easy to analyze

Free-Choice nets

- Introduced by Hack (‘72)
- Extensively studied by Best (‘86) and Desel and Esparza (‘95)
- Can express concurrency, causality and choice without confusion
- Very strong structural theory
  - necessary and sufficient conditions for liveness and safeness, based on decomposition
  - concurrency, causality and choice relations are mutually exclusive
  - exploits duality between MG and SM
**MG (& SM) decomposition**

- An *Allocation* is a control function that chooses which transition fires among several conflicting ones (A: P → T).
- Eliminate the subnet that would be inactive if we were to use the allocation...

**Reduction Algorithm**
- Delete all unallocated transitions
- Delete all places that have all input transitions already deleted
- Delete all transitions that have at least one input place already deleted
- Obtain a *Reduction* (one for each allocation) that is a conflict free subnet

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**MG reduction and cover**

- Choose one successor for each conflicting place:
MG reduction and cover

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MG reduction and cover

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**MG reductions**

- The set of all reductions yields a **cover of MG components** (T-invariants)

**SM reduction and cover**

- Choose one predecessor for each transition:
Choose one predecessor for each transition:

The set of all reductions yields a cover of SM components (S-invariants)
Hack’s theorem (‘72)

Let N be a Free-Choice PN:

- N has a live and safe initial marking (well-formed) if and only if
  - every MG reduction is strongly connected and not empty, and
  - the set of all reductions covers the net
- every SM reduction is strongly connected and not empty, and
  - the set of all reductions covers the net

Example of non-live (but safe) FCN
Hack’s theorem

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Hack’s theorem

- Example of non-live (but safe) FCN

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Hack’s theorem

- Example of non-live (but safe) FCN

Hack’s theorem

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![Diagram of FCN example](image-url)
Hack’s theorem

- Example of non-live (but safe) FCN
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Other results for LSFC nets

- Let $t_1$ and $t_2$ be two transitions of a live and safe Free-Choice net.
  
  Then $t_1$ and $t_2$ are:
  
  - **sequential** if
    
    there exists a simple cycle to which both belong
  
  - **concurrent** if
    
    they are not ordered, and
    
    there exists an MG component to which both belong
  
  - **conflicting** otherwise
Summary of LSFC nets

- Largest class for which structural theory really helps
- Structural component analysis may be expensive
  (exponential number of MG and SM components in the worst case)
- But…
  - number of MG components is generally small
  - FC restriction simplifies characterization of behavior

Summary of Petri Nets

- Graphical formalism
- Distributed state (including buffering)
- Concurrency, sequencing and choice made explicit
- Structural and behavioral properties
- Analysis techniques based on
  - linear algebra (only sufficient)
  - structural analysis (necessary and sufficient only for FC)
Petri Net extensions

- Add interpretation to tokens and transitions
  - Colored nets (tokens have value)
- Add time
  - time/timed Petri Nets (deterministic delay)
    - type (duration, delay)
    - where (place, transition)
    - control (weak, strong)
  - Stochastic PNs (probabilistic delay)
  - Generalized Stochastic PNs (timed and immediate transitions)
- Add hierarchy
  - Place Chart Nets

PNs and SDF

IIR 2nd order filter
\[ o(n) = k_1 o(n-1) + k_2 i(n) \]

**Homogeneous SDF network**

**Marked Graph**

**Fast Fourier Transform**

**Weighted T-System**
PNs and BDF

BDF network

Petri Net

Switch/Select vs. choice/merge
PNs: No correlation between different choices

PNs and BDF

BDF network

Petri Net

PNs are not-determinate
PNs and BDF

PNs are not-determinate

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