Blanket Algebra for Decomposition of Functions

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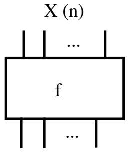
Why Decompose?

- Design with FPGAs
  - fixed number of cell inputs and outputs
    - e. g. five inputs, one output
- Design with PLAs
  - smaller area
- Decision tables (multiple-valued functions)
  - less memory

Decomposition

- $f(X) = h(U, g(V))$
- $r + p < n, \; s < n$

X (n)

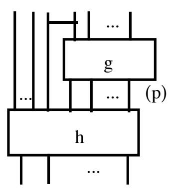


Y (m)

$X = U \cup V$
U,V may overlap

X (n)

U (r) V (s)



Y (m)

Compact Representation of Boolean Functions: Cubes

	x_1	x_2	x_3	y_1	y_2	y_1	
1	0	0	—	1	1	{1, 2, 3}	on
2	1	0	—	1	0	{4}	off
3	—	0	0	1	—	{5}	dc
4	—	1	1	0	—		
5	0	1	—	—	1	110	dc

don't care is really {0, 1}

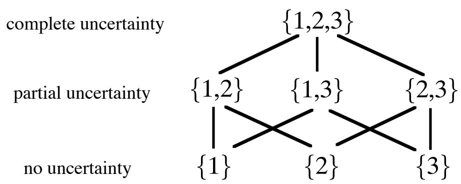
Decision Table (multi-valued)

temp	**outlook**	**humidity**	**wind**	**sport**
hot	rain	---	none	swim
cool/mild	rain	---	none	workout
---	sunny	---	---	tennis
---	overcast	high/normal	---	swim/tennis
---	rain	---	med/high	windsurf

Partial "don't cares"

- **element** set E
- **values**: D = nonempty subsets of E
- example: E = {0, 1, 2}
- D ={{0},{1},{2},{0,1},{0,2},{1,2},{0,1,2}}
  {0}: **definite value,** 0
  {1,2}: **partial don't care,** 1 or 2, not 0
  {0,1,2}: **complete don't care**

Uncertainty Partial Order



Example: **Set Matrix M**

E = {0,1,2}, n = 3 inputs, m = 2 outputs,
set T of 4 rows, drop { }

Row	x_1	x_2	x_3	y_1	y_2
1	0	0	0,2	0,1	0
2	1	1,2	0,2	0,2	1
3	2	0,1,2	0	1,2	2
4	1,2	0,2	0,2	0,1	1,2

Input and Output Projections

- (n+m)-tuple t of T
- $t = t_1, ..., t_n, ..., t_{n+1}, ..., t_{n+m}$
- $t_\downarrow = t_1, ..., t_n$ **input projection**
- $t^\uparrow = t_{n+1}, ..., t_{n+m}$ **output projection**

Consistency of Matrix M

Row	x_1	x_2	x_3	y_1	y_2
1	0	0	0,2	0,1	0
2	1	1,2	0,2	0,2	1
3	**2**	**0**,1,2	**0**	**1**,2	**2**
4	1,**2**	**0**,2	**0**,2	0,**1**	1,**2**

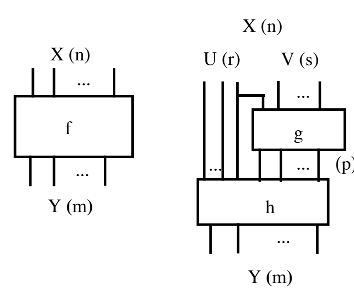
if input projections of a set of rows cover an input combination,
then corresponding output projections must have a common value

Set Function of M $\quad f: E^n \rightarrow D^m$

Row	x_1	x_2	x_3	y_1	y_2	x_1	x_2	x_3	y_1	y_2
1	0	0	0,2	0,1	0	0	0	0	0,1	0
2	**1**	1,**2**	0,**2**	**0,2**	**1**	*0*	*0*	*1*	*0,1,2*	*0,1,2*
3	2	0,1,2	0	1,2	2	0	0	2	0,1	0
4	**1**,2	0,**2**	0,**2**	**0,1**	**1,2**	....	....	....	....	....
						1	**2**	**2**	**0**	**1**
						....	....	....	....	....

- **singleton vector e** ({0},{0},{0}) identified with (0,0,0), etc
- for f(e) take **intersection** of y values in all rows covering e

Decomposition



Example of Decomposition

Row	x_1	x_2	x_3	f_1	f_2
1	**0**	**0**	**0**,2	**0,1**	**0**
2	*1*	*1*, 2	*0*, 2	*0, 2*	*1*
3	2	0,1,2	0	1,2	2
4	1,2	0,2	0,2	0,1	1,2

$U = \{x_1\}$

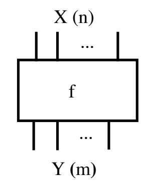
$V = \{x_2, x_3\}$

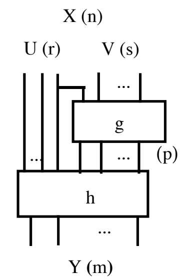
x_2	x_3	g
0	**0**,2	**2**
1	*0*, 2	*1*
2	0	1
2	2	0

x_1	g	h_1	h_2
0	**2**	**0,1**	**0**
1	0,*1*	*0*	*1*
1	2	0,1	1,2
2	0	0,1	1,2
2	1,2	1	2

Decomposition with don't cares

- r + p < n, s < n
- **f(X) ⊇ h(U, g(V))**





Decomposition Problem

- can f(X) given by M be decomposed?
- solution:

  find sets U and V (or pick arbitrarily)

  find **blanket** β_g (from M - described later)

  ⇒ **construct g from** $\boldsymbol{\beta_g}$

  ⇒ **construct h from** $\boldsymbol{\beta_g}$

Blankets

- set S
- **blanket** $\beta = \{B_1, ..., B_k\}$ or just $\{B_i\}$
- $B_1, ..., B_k$ nonempty, distinct subsets of S whose union is S
- examples: S = {1, 2, 3}

$\beta = \{\{1,2\},\{2,3\}\}$, $\beta' = \{\{1\},\{1,2\},\{2,3\}\}$

Set Systems (Hartmanis &Stearns, 1966)

- **blanket**

  {{1,2,3},{2,3},{1,4}}
- **set system**: blanket where $B_i \supseteq B_j$ implies i=j

  {{1,2,3},{1,4}}
- **partition**: set system with disjoint blocks

  {{1,2},{3,4}}

Blanket Product

- **nonempty operator** $ne\{B_i\} = \{B_i\} - \{\emptyset\}$
- **blanket product**

  $\beta * \beta' = ne\{ B_i \cap B_j \mid B_i \in \beta,\ B_j \in \beta' \}$

- {{1,2,3},{3,4,5}} ∗ {{1,3,4},{1,5},{2,3,4}}

  = {{1,3},{1},{2,3},{3,4},{5}}

Relation ≤ on Blankets

- $\beta \leq \beta'$
- for each $B_i \in \beta$, there is a $B_j \in \beta'$ such that $B_i \subseteq B_j$

$\{\{1\},\{2\},\{3\}\} \leq \{\{1,2\},\{2,3\}\}$

$\{\{1,2\},\{2,3\}\} \leq \{\{1,2\},\{2,3\},\{1,3\}\}$

$\{\{1\},\{1,2\}\} \leq \{\{1,2\}\}$ and $\{\{1,2\}\} \leq \{\{1\},\{1,2\}\}$

Input and Output Blankets

Row	x_1	x_2	x_3	y_1	y_2
1	0	0	0,2	0,1	0
2	1	1,2	0,2	0,2	1
3	2	0,1,2	0	1,2	2
4	1,2	0,2	0,2	0,1	1,2

- $U = \{x_1\}$; $\beta_U = \{\{1\},\{2,4\},\{3,4\}\}$
- $V = \{x_2, x_3\}$; $\beta_V = \{\{1,3,4\},\{1,4\},\{2,3\},\{2\},\{2,3,4\},\{2,4\}\}$
- $\beta_Y = \{\{1\},\{2,4\},\{4\},\{3,4\},\{2\},\{3\}\}$

β "corresponds" to g wrt V

for all tuples t, u in T,
if there exist minterms d and e such that
$t_{\downarrow}{}^{V} \supseteq d$, $u_{\downarrow}{}^{V} \supseteq e$ and $g(d) = g(e)$,
then t and u appear in the same block of β

Theorem

- f(X) a set function specified by set T of tuples
- (U, V) a pair of subsets of X, $U \cup V = X$
- For every blanket β_g satisfying

  $\beta_V \leq \beta_g$ and $\beta_U * \beta_g \leq \beta_f$

  there is a decomposition (g, h) of f such that β_g corresponds to g wrt V
- (construction of β_g postponed)

Constructing g: Relevant Inputs

Row	x_1	x_2	x_3	f_1	f_2
1	0	0	0,2	0,1	0
2	1	1,2	0,2	0,2	1
3	2	0,1,2	0	1,2	2
4	1,2	0,2	0,2	0,1	1,2

x_2	x_3	rows
0	0	{1,3,4}
0	2	{1,4}
1	0	{2,3}
1	2	{2}
2	0	{2,3,4}
2	2	{2,4}

- $V = \{x_2, x_3\}$
- $\beta_V = \{\{1,3,4\},\{1,4\},\{2,3\},\{2\},\{2,3,4\},\{2,4\}\}$
- Suppose we are **given** $\beta_g = \{\{1,2,4\},\{2,3,4\},\{1,3,4\}\}$
- $\beta_V \leq \beta_g$

Constructing g from β_g

$\beta_g = \{\{1,2,4\},\{2,3,4\},\{1,3,4\}\}$

encode 0 1 2

x_2	x_3	β_V	≤	β_g	options	g
0	0	{1,3,4}		{1,3,4}	2	2
0	2	{1,4}		{1,2,4}, {1,3,4}	0, 2	2
1	0	{2,3}		{2,3,4}	1	1
1	2	{2}		{1,2,4}, {2,3,4}	0, 1	1
2	0	{2,3,4}		{2,3,4}	1	1
2	2	{2,4}		{1,2,4}, {2,3,4}	0, 1	0

Finding h: Relevant Inputs

Row	x_1	x_2	x_3	f_1	f_2	x_2	x_3	g	
1	**0**	**0**	**0**,2	0,1	0	**0**	**0**,2	**2**	(merged)
2	1	1,2	0,2	0,2	1	1	0,2	1	(merged)
3	2	0,1,2	0	1,2	2	2	0	1	
4	1,2	0,2	0,2	0,1	1,2	2	2	0	

x_1	g	h_1	h_2
0	**2**		
1	2		
1	0		
1	1		
2	2		
2	0		
2	1		

Constructing h from β_g

x_1	β_U	g	β_g	$\beta_U * \beta_g$	≤	β_Y	h_1	h_2
0	{1}	2	{1,3,4}	{1}		{1}	0,1	0
1	{2,4}	2	{1,3,4}	{4}		{2,4},**{4}**,{3,4}	0,1	1,2
1	{2,4}	0	{1,2,4}	{2,4}		{2,4}	0	1
1	{2,4}	1	{2,3,4}	{2,4}		{2,4}	0	1
2	{3,4}	2	{1,3,4}	{3,4}		{3,4}	1	2
2	{3,4}	0	{1,2,4}	{4}		{2,4},**{4}**,{3,4}	0,1	1,2
2	{3,4}	1	{2,3,4}	{3,4}		{3,4}	1	2

β_g = {{1,2,4},{2,3,4},{1,3,4}}

encoding	0	1	2

β_Y = {	{1},	{2,4},	{4},	{3,4},	{2},	{3} }
y	{0,1}{0}	{0}{1}	{0,1}{1,2}	{1}{2}	{0,2}{1}	{ 1,2}{2}

Final Decomposition

Row	x_1	x_2	x_3	f_1	f_2	x_2	x_3	g
1	0	0	0,2	0,1	0	0	0,2	2
2	1	1,2	0,2	0,2	1	1	0,2	1
3	2	0,1,2	0	1,2	2	2	0	1
4	1,2	0,2	0,2	0,1	1,2	2	2	0

x_1	g	h_1	h_2	
0	2	0,1	0	
1	0, 1	0	1	merged
1	2	0,1	1,2	
2	0	0,1	1,2	
2	1,2	1	2	merged

Selecting U and V

- a difficult problem
- some necessary conditions exist to eliminate certain pairs
- trial and error

Finding β_g

- start with β_V
- merge blocks of β_V to get β', such that $\beta_V \leq \beta'$
- check if $\beta_U * \beta' \leq \beta_Y$
- graph-theoretic methods
  - maximal cliques in a graph
  - coloring problems

Decomposition Using Cubes

- compact representation, not truth table
- methods exist for Boolean functions
- we are working on multiple-valued case

"Demain"

- logic synthesis tool
- developed by T. Luba and collaborators
- Warsaw University of Technology
- reports good results

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Shestakov's Work (1994)

- "double decomposition"
  $f(X) = h(g_1(U), g_2(V))$
- methods very similar to Luba's
- equivalent to two steps of "single decomposition"

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Main References

- **J. A. Brzozowski and T. Luba**, *Decompositions of Boolean Functions Specified by Cubes,* Research Report CS-97-01, January 1997, submitted for publication
- **J. A. Brzozowski and J. J. Lou**, *Blanket Algebra for Multiple-Valued Function Decomposition,* manuscript, June 1997, to be published by World Scientific
- **J. J. Lou and J. A. Brzozowski,** *A Formalization of Shestakov's Decomposition***,** Research Report CS-98-03, February 1998
- **http://maveric.uwaterloo.ca**

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Conclusions

- **blankets** generalize set systems and partitions
- characterization of **decompositions** by **blankets**
- generalization to **multi-valued** functions
- applies to **nondisjunctive** decompositions where $U \cap V \neq \emptyset$
- some challenging problems remain

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