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Viability and Invariance Kernels of Impulse Differential Inclusions

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Abstract

Impulse differential inclusions provide a framework for modelling hybrid phenomena. In the context of impulse differential inclusions, verification for safety specifications and safe controller synthesis can be formulated as viability and invariance questions for appropriate sets of states. In this paper, a characterisation of viability and invariance kernels (i.e. the largest subsets of a given set that are viable or invariant) is presented. In the process, a method for computing these sets using standard viability and invariance tools is developed.

Keywords: Hybrid systems; Viability Theory; Non-smooth analysis; Viability and Invariance Kernels.

1 Introduction

A substantial part of the literature on hybrid systems has been devoted to the problem of *reachability* and *safety*, that is the question of whether, under the dynamics of a hybrid system, a given set of states can be reached from a given set of initial conditions. Techniques have been developed for establishing whether the set of reachable states is contained in a certain set, either algorithmically (see [1] and the references therein) or deductively [2]. In the case of hybrid control systems, methods have been developed for synthesising controllers that satisfy such *safety* specifications (see, for example [3, 4] and the references therein). Since the reachability problem quickly becomes computationally infeasible, approximation techniques have been proposed to facilitate the analysis [5, 6].

For continuous dynamical systems described by differential inclusions, questions of reachability can be addressed in the context of *viability theory* [7]. Viability theory deals with two fundamental properties of sets of states of a dynamical system. Roughly speaking, a set of states, K, is called *viable* if for all initial conditions in K there exists a solution of the dynamical system that remains in K; it is called *invariant* if for all initial conditions in K all solutions of the system remain in K. In the case where a set, K, is not viable (respectively invariant), viability theory techniques can also be used to establish the largest subset of K which is viable (respectively, invariant), which is known as the *viability kernel* (respectively, *invariance kernel*) of K. Numerical algorithms have been developed to compute these kernels [8], and have been used to compute things such as basins of attraction for equilibria [9].

In [10], viability theory concepts were extended to a fairly large class of hybrid systems, known as *impulse differential inclusions.* Methods from non-smooth analysis were used to characterise sets of states that are viable or invariant under the dynamics of an impulse differential inclusion, and study the existence of runs. In this paper we pursue this direction further, by providing a characterisation for the viability and invariance kernels of sets of states of an impulse differential inclusion. It is easy to see that viability and invariance kernels are closely related to maximal invariant and maximal controlled invariant sets, therefore their characterisation is important for verification and controller synthesis for safety specifications.

The material is arranged in four sections. In Section 2, we review the impulse differential inclusion framework, and summarise the viability and invariance conditions derived in [10]. Procedures for es-

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tablishing the viability and invariance kernels of a set are then developed in Section 3. Current research topics are highlighted in 4. The proofs of the various facts stated in the paper are rather technical, and have been omitted to maintain the flow of the paper. The interested reader is referred to [11].

2 Impulse Differential Inclusions

We start by summarising the notation used in the subsequent development. The paper assumes some familiarity with the basic concepts of non-smooth and set valued analysis (upper and lower semicontinuity, Lipschitz maps etc.) and differential inclusions (solution concepts, etc.). The reader is referred [7] to for a thorough treatment.

For an arbitrary set, K, 2^K is used to denote the power set of K, i.e. the set of all subsets of K. For a set valued map $R : X \rightarrow 2^Y$ and a set $K \subseteq Y$ we use $R^{-1}(K)$ to denote the *inverse image* of K under R and $R^{\ominus 1}(K)$ to denote the *extended core* of K under R, defined by

$$\begin{aligned} R^{-1}(K) &= \{x \in X \mid R(x) \cap K \neq \emptyset\}, \\ R^{\ominus 1}(K) &= \{x \in X \mid R(x) \subseteq K\} \\ &\quad \cup \{x \in X \mid R(x) = \emptyset\}. \end{aligned}$$

Notice that $R^{-1}(Y)$ is the set of $x \in X$ such that $R(x) \neq \emptyset$. We call the set $R^{-1}(Y)$ the *domain* of R and the set $\{(x, y) \in X \times Y \mid y \in R(x)\}$ the *graph* of R.

We use X to denote a finite dimensional vector space with the standard Euclidean metric, denoted by d. For a closed subset, $K \subseteq X$, of a finite dimensional vector space, and a point $x \in K$, we use $T_K(x)$ to denote the *contingent cone* to K at x, i.e. the set of $v \in X$ such that there exists a sequence of real numbers $h_n > 0$ converging to 0 and a sequence of $v_n \in X$ converging to v satisfying

$$\forall\, n \geq 0, \;\; x + h_n v_n \;\in\; K.$$

We say that a map $F : X \rightarrow 2^X$ is *Marchaud* if and only if (1) the graph and the domain of F are nonempty and closed; (2) for all $x \in X$, $F(x)$ is convex, compact and nonempty; and, (3) the growth of F is linear, that is there exists $c > 0$ such that for all $x \in X$

$$\sup\{\|v\| \mid v \in F(x)\} \;\leq\; c(\|x\| + 1).$$

We will consider hybrid phenomena, in the sense of dynamical phenomena that involve both continuous evolution and discrete transitions. To distinguish the times at which discrete transitions take place we recall the notion of a hybrid time trajectory [4, 12].

Definition 1 *A* **hybrid time trajectory**, $\tau = \{I_i\}_{i=0}^N$ *is a finite or infinite sequence of intervals of the real line, such that*

- *for* $i < N$, $I_i = [\tau_i, \tau_i']$, *and, if* $N < \infty$, *either* $I_N = [\tau_N, \tau_N']$, *or* $I_N = [\tau_N, \tau_N'[$;
- *for all* i, $\tau_i \leq \tau_i' = \tau_{i+1}$.

Since the dynamical systems we will consider will be time invariant, we assume, without loss of generality, that $\tau_0 = 0$. The interpretation is that τ_i are the times at which discrete transitions take place. Notice that discrete transitions are assumed to be instantaneous, and therefore multiple discrete transitions may take place at the same time instant (since it is possible for $\tau_i = \tau_{i+1}$). Each hybrid time trajectory, τ, is fully ordered by the relation $\prec$, which for $t \in [\tau_i, \tau_i'] \in \tau$ and $t' \in [\tau_j, \tau_j'] \in \tau$ is defined by $t \prec t'$ if and only if $t < t'$ or $i < j$; we use $t \preceq t'$ to denote $t \prec t'$ or $t = t'$ and $i = j$. For $t \in \mathbb{R}$, we use $t \in \tau$ as a shorthand notation for "there exists a j such that $t \in [\tau_j, \tau_j'] \in \tau$". For a topological space K we use $k : \tau \rightsquigarrow K$ as a shorthand notation for a map assigning values from K to all $t \in \tau$. Notice that $k : \tau \rightsquigarrow K$ is *not a function* over the interval $\bigcup_i I_i$, since it assigns multiple values to the times $t = \tau_i = \tau_{i-1}'$.

We are now ready to introduce the class of dynamical systems considered in this paper (see also [10]).

Definition 2 *An* **impulse differential inclusion** *is a collection* (X, F, R, J), *consisting of a finite dimensional vector space* X, *a set valued map* $F : X \rightarrow 2^X$, *regarded as a differential inclusion* $\dot{x} \in F(x)$, *a set valued map* $R : X \rightarrow 2^X$, *regarded as a reset map, and a set* $J \subseteq X$, *regarded as a forced transition set.*

We call $x \in X$ the *state* of the impulse differential inclusion. Subsequently, $I = X \setminus J$ will be used to denote the complement of J. The set I is sometimes referred to as the "domain", or the "invariant" in the hybrid systems literature.

Impulse differential inclusions can be used to describe hybrid phenomena in the following sense.

Definition 3 *A* **run** *of an impulse differential inclusion,* (X, F, R, J), *is a pair,* (τ, x), *consisting of*

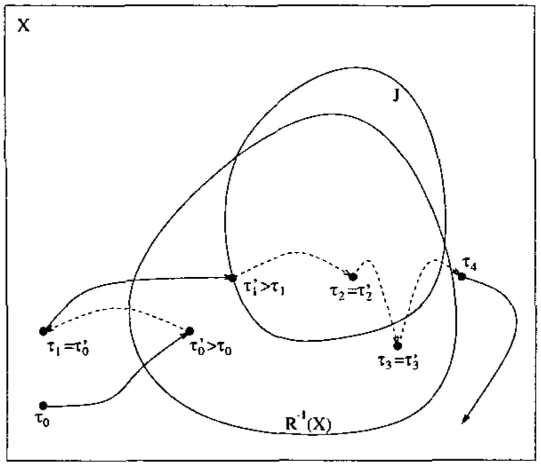

Figure 1: A run of an impulse differential inclusion (X, F, R, J)

a hybrid time trajectory τ and a map $x : \tau \rightsquigarrow X$, that satisfies:

- Discrete Evolution*: for all i, $x(\tau_{i+1}) \in R(x(\tau_i'))$*
- Continuous Evolution*: if $\tau_i < \tau_i'$, $x(\cdot)$ is a solution to the differential inclusion $\dot{x} \in F(x)$ over the interval $[\tau_i, \tau_i']$ starting at $x(\tau_i)$, with $x(t) \notin J$ for all $t \in [\tau_i, \tau_i'[$.*

We will use $\mathcal{R}_{(X,F,R,J)}(x_0)$ to denote the set of all runs of an impulse differential inclusion (X, F, R, J) starting at a state $x(\tau_0) = x_0 \in X$. An example of a run of an impulse differential inclusion is shown in Figure 1; the solid arrows indicate continuous evolution while the dotted arrows indicate discrete transitions. According to Definition 3, R *enables* discrete transitions (transitions may happen when $R(x) \neq \emptyset$ but do not have to), while J *forces* discrete transitions (transitions must happen when $x \in J$). It is easy to see that by appropriately embedding discrete states into finite dimensional vector spaces impulse differential inclusions can be used to model a very wide class of hybrid phenomena.

A run of an impulse differential inclusion is called:

- *finite*, if τ is a finite sequence ending with a compact interval,
- *infinite*, if either τ is an infinite sequence, or $\sum_i(\tau_i' - \tau_i) = \infty$,
- *Zeno*, if it is infinite and $\sum_i(\tau_i' - \tau_i) < \infty$.

We will use $\mathcal{R}^{\infty}_{(X,F,R,J)}(x_0)$ to denote the set of all infinite runs of (X, F, R, J) starting at x_0 (some of which may be Zeno while others not). In [10] conditions were developed to guarantee that infinite runs exist, using the viability concepts introduced below.

Definition 4 *A run, (τ, x) of an impulse differential inclusion, (X, F, R, J), is called* **viable** *in a set $K \subseteq X$ if for all $t \in \tau$, $x(t) \in K$.*

Notice that the definition of a viable run requires the state to remain in the set K throughout the run, along continuous evolution up until and including the state before a discrete transition, as well as after the discrete transition. Based on the notion of a viable run, one can define two different classes of sets.

Definition 5 *A set $K \subseteq X$ is called* **viable** *under an impulse differential inclusion, (X, F, R, J), if for all $x_0 \in K$ there exists an infinite run, $(\tau, x) \in \mathcal{R}^{\infty}_{(X,F,R,J)}(x_0)$, viable in K. K is called* **invariant** *under the impulse differential inclusion, if for all $x_0 \in K$ all runs $(\tau, x) \in \mathcal{R}_{(X,F,R,J)}(x_0)$ are viable in K.*

It should be easy to appreciate the concept of invariance, and its implications for safety verification for hybrid systems. Careful examination of Definition 5 reveals that viability is closely related to the concept of controlled invariance in the absence of uncontrollable disturbances [4]. It is easy to see that if a set is viable under an impulse differential inclusion, then there exists a controller (involving feedback during both continuous evolution and discrete transitions) that ensures that if the state starts in that set it remains there for ever.

The characterisation of viable and invariant sets presented in [10], led to the following conditions.

Theorem 1 *Consider an impulse differential inclusion (X, F, R, J) such that F is Marchaud, R is upper semicontinuous with closed domain and J is closed*[1]*. A closed set $K \subseteq X$ is viable under (X, F, R, J) if and only if*

1. *$K \cap J \subseteq R^{-1}(K)$, and*
2. *$\forall\, x \in K \backslash R^{-1}(K)$, $F(x) \cap T_K(x) \neq \emptyset$*

[1]Similar conditions characterise viability when the set J is open, or, in other words, the set $I = X \setminus J$ is closed.

The conditions can be interpreted as requiring that if for some state in K continuous evolution keeping the state in K is impossible, a discrete transition back into K must be possible.

Theorem 2 *Consider an impulse differential inclusion (X, F, R, J) such that F is Marchaud and Lipschitz and J is closed. A closed set $K \subseteq X$ is invariant under (X, F, R, J) if and only if*

1. $R(K) \subseteq K$, *and*
2. $\forall\, x \in K \setminus J,\ \ F(x) \subseteq T_K(x)$.

The conditions can be interpreted as requiring that for any state in K and any possible type of evolution from that state (continuous or discrete), the state must remain in K. It can be shown that these very intuitive conditions reduce to the standard viability and invariance conditions for differential inclusions and discrete time systems, if the impulse differential inclusion allows only continuous or only discrete dynamics.

In the cases where an impulse differential inclusion fails to satisfy a given viability or invariance requirement, one would like to establish sets of initial conditions (if any) for which the requirement will be satisfied. This notion can be characterised by the viability and invariance kernels.

Definition 6 *The* **viability kernel**, $\mathrm{Viab}_{(X,F,R,J)}(K)$ *of a set $K \subseteq X$ under an impulse differential inclusion (X, F, R, J) is the set of states $x_0 \in X$, for which there exists an infinite run, $(\tau, x) \in \mathcal{R}^{\infty}_{(X,F,R,J)}(x_0)$, viable in K. The* **invariance kernel**, $\mathrm{Inv}_{(X,F,R,J)}(K)$ *of K under (X, F, R, J) is the set of states $x_0 \in X$, for which all runs $(\tau, x) \in \mathcal{R}_{(X,F,R,J)}(x_0)$ are viable in K.*

The viability (respectively invariance) kernel of a set K can be thought of as the unique maximal controlled invariant (respectively invariant) subset of the set K.

3 Viability and Invariance Kernels

The viability kernel of an impulse differential inclusion can be characterised in terms of the notion of the viability kernel with target for a continuous differential inclusion. This notion was first introduced in [13], in the context of optimal control problems with terminal constraints. For a differential inclusion $\dot{x} \in F(x)$, the viability kernel of a set $K \subseteq X$ with target $C \subseteq X$, $Viab_F(K, C)$, is defined as the set of states for which there exists a solution to the differential inclusion that remains in K either for ever, of until it reaches C. The following lemma summarises the basic properties of the viability kernel with target.

Lemma 1 *Consider a Marchaud map $F : X \to 2^X$ and two closed sets K and C. $\mathrm{Viab}_F(K, C)$ is the largest closed subset of K such that for all $x \in K \setminus C$, $F(x) \cap T_K(x) \neq \emptyset$.*

The proof for the case where F is Lipschitz is given in [13]. The proof for the more general case can be found in [11]. Notice that, by definition

$$K \cap C \subseteq Viab_F(K, C) \subseteq K.$$

Using this notion, one can give an alternative characterisation of the sets that are viable under an impulse differential inclusion, as fixed points of an appropriate operator. For an impulse differential inclusion (X, F, R, J), consider the operator $Pre^{\exists}_{(X,F,R,J)} : 2^X \to 2^X$ defined by

$$\begin{aligned} Pre^{\exists}_{(X,F,R,J)}(K) = {} & Viab_F(K \cap I, R^{-1}(K)) \\ & \cup (K \cap R^{-1}(K)) \end{aligned}$$

Recall that $I = X \setminus J$.

Lemma 2 *Consider an impulse differential inclusion (X, F, R, J) such that F is Marchaud, R is upper semicontinuous with closed domain, and J is open. A closed set $K \subseteq X$ is viable under (X, F, R, J) if and only if it is a fixed point of the operator $\mathrm{Pre}^{\exists}_{(X,F,R,J)}$.*

The lemma follows from the observation that $Pre^{\exists}_{(X,F,R,J)}(K)$ is effectively the set of states in K for which there exists a piece of continuous evolution (possibly trivial or infinite) followed by a discrete transition that remains in K.

Theorem 3 *Consider an impulse differential inclusion (X, F, R, J) such that F is Marchaud, R is upper semicontinuous with closed domain and compact images, and J is open. The viability kernel of a closed set $K \subseteq X$ under (X, F, R, J) is the largest closed subset of K viable under (X, F, R, J), that is, the largest closed fixed point of $\mathrm{Pre}^{\exists}_{(X,F,R,J)}$ contained in K.*

It should be stressed that the conditions of Theorem 3 ensure that for all initial conditions in the viability kernel infinite runs of the impulse differential inclusion exist, but do not ensure that these runs will extend over an infinite time horizon; all runs starting at certain initial conditions in the viability kernel may turn out to be Zeno.

The proof of Theorem 3 is based on the following procedure for approximating the viability kernel.

Algorithm 1 (Viability Kernel Approx.)

> **initialisation:** $K_{-1} = \emptyset$, $K_0 = K$, $i = 0$
> **while** $K_i \neq K_{i-1}$
> **begin**
> $K_{i+1} = Pre^{\exists}_{(X,F,R,J)}(K_i)$
> $i = i + 1$
> **end**

One can use existing software tools for computing viability kernels for differential inclusions to approximate $Viab_F(K \cap I, R^{-1}(K))$ at each step of the above algorithm.

The invariance kernel can be characterised using the notion of the invariance kernel with target for continuous differential inclusions. For a differential inclusion $\dot{x} \in F(x)$, the invariance kernel of a set K with target C, $Inv_F(K, C)$ is defined as the set of states for which all solutions to the differential inclusion remain in K either for ever, of until they reach C. The following lemma summarises the basic properties of the invariance kernel with target.

Lemma 3 *Consider a Marchaud and Lipschitz map $F : X \to 2^X$ and two closed sets K and C. $\mathrm{Inv}_F(K, C)$ is the largest closed subset of K such that for all $x \in K \setminus C$, $F(x) \subseteq T_K(x)$.*

Notice that, by definition

$$K \cap C \subseteq Inv_F(K, C) \subseteq K.$$

Using the notion of invariance kernel with target, one can give an alternative characterisation of the sets that are invariant under an impulse differential inclusion, as fixed points of an operator. Given an impulse differential inclusion (X, F, R, J), consider the operator $Pre^{\forall}_{(X,F,R,J)} : 2^X \to 2^X$ defined by

$$Pre^{\forall}_{(X,F,R,J)}(K) = Inv_F(K, J) \cap R^{\ominus 1}(K)$$

Lemma 4 *Consider an impulse differential inclusion (X, F, R, J) such that F is Marchaud and Lipschitz, R is lower semicontinuous, and J is closed. A closed set $K \subseteq X$ is invariant under (X, F, R, J) if and only if it is a fixed point of the operator $\mathrm{Pre}^{\forall}_{(X,F,R,J)}$.*

$Pre^{\forall}_{(X,F,R,J)}(K)$ is effectively the set of states in K for which all pieces of continuous evolution and all discrete transitions remain in K.

Theorem 4 *Consider an impulse differential inclusion (X, F, R, J) such that F is Marchaud and Lipschitz, R is lower semicontinuous and J is closed. The invariance kernel of a closed set $K \subseteq X$ under (X, F, R, J) is the largest closed subset of K invariant under (X, F, R, J), that is, the largest, closed fixed point of $\mathrm{Pre}^{\forall}_{(X,F,R,J)}$ contained in K.*

Again the proof of Theorem 4 makes use of the sequence of nested sets generated by the following algorithm.

Algorithm 2 (Invariance Kernel Approx.)

> **initialisation:** $K_{-1} = \emptyset$, $K_0 = K$, $i = 0$
> **while** $K_i \neq K_{i-1}$
> **begin**
> $K_{i+1} = Pre^{\forall}_{(X,F,R,J)}(K_i)$
> $i = i + 1$
> **end**

Standard viability theory computational tools can be used to implement each step of the algorithm, and hence systematically approximate the invariance kernel of the set K.

4 Concluding Remarks

We presented conditions for characterising the viability and invariance kernels of sets of states under the action of an impulse differential inclusion. The conditions were based on constructive procedures for obtaining successively better estimates of the kernels. We are currently investigating how software tools for studying the viability of sets under the action of differential inclusions can be used to compute these estimates, and how the results compare with different methods that have been proposed for numerically approximating invariant and controlled invariant sets.

The results presented in this paper are part of an extensive study of hybrid control through the framework of viability theory. A number of interesting extensions are currently under investigation. They include optimal control of impulse differential inclusions (value functions and their characterisations in terms of quasi-variational inequalities or viability kernels), stability (Lyapunov functions their characterisation as viability kernels) and a study of the initialisation map, which can be used to convert a hybrid system to a discrete time system by abstracting away the continuous dynamics.

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