Dynamical Systems Revisited: Hybrid Systems with Zeno Executions*

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Abstract

Results from classical dynamical systems are generalized to hybrid dynamical systems. The concept of ω limit set is introduced for hybrid systems and is used to prove new results on invariant sets and stability, where Zeno and non-Zeno hybrid systems can be treated within the same framework. As an example, LaSalle's Invariance Principle is extended to hybrid systems. Zeno hybrid systems are discussed in detail. The ω limit set of a Zeno execution is characterized for classes of hybrid systems.

1 Introduction

Systems with interacting continuous-time and discrete-time dynamics are used as models in a large variety of applications. The rich structure of such hybrid systems allow them to accurately predict the behavior of quite complex systems. However, the continuous–discrete nature of the system calls for new system theoretical tools for modeling, analysis, and design. Intensive recent activity have provided a few such tools, for instance, Lyapunov stability results. However, as will be shown in this paper, in many cases the results come with assumptions that are not only hard to check but also unnecessary. There are several fundamental properties of hybrid systems that have not been sufficiently studied in the literature. These include questions on existence and uniqueness of executions, which have only recently been addressed [11, 8]. Another question is when a hybrid system exhibits an infinite number of discrete transitions during a finite time interval, which is referred to as Zeno. The significance of these questions has been pointed out by many researchers, e.g., He and Lemmon [4] write "An important issue which is not addressed in this paper concerns necessary and sufficient conditions for a switched system to be live, deadlock free, or nonZeno."

The main contribution of the paper is to carefully generalize concepts from classical dynamical systems like ω limit sets and invariant sets, in a way so that Zeno executions are treated within the same framework as regular non-Zeno executions. It is then straightforward to extend existing results,

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for instance, Lyapunov stability theorems for hybrid systems [2, 12]. We illustrate this by proving LaSalle's Invariance Principle for hybrid systems.

Zeno is an interesting mathematical property of some hybrid systems, which does not occur in smooth dynamical systems. Real physical systems are not Zeno. Models of physical systems may, however, be Zeno due to a too high level of abstraction. In the latter part of the paper, we characterize Zeno executions and their Zeno states, where the Zeno states are defined as the ω limit points of a Zeno execution. We are able to completely characterize the set of Zeno states for a few classes of hybrid systems. It is shown that the features of the reset maps are important. For example, if the resets are identity maps or the resets are contractions, the continuous part of the Zeno state is a singleton.

The outline of the paper is as follows. Some preliminaries together with the definition of a hybrid system and its executions are given in Section 2. The relation between ω limit sets and invariant sets in Section 3 leads to LaSalle's Invariance Principle for hybrid systems. Section 4 discusses Zeno hybrid systems. Some conclusions are finally given in Section 5. All proofs are omitted due to limited space.

2 Hybrid Automata and Executions

2.1 Notation

For a finite collection V of variables, let $\mathbf{V}$ denote the set of valuations of these variables. We use lower case letters to denote both a variable and its valuation. We refer to variables whose set of valuations is finite or countable as *discrete* and to variables whose set of valuations is a subset of a Euclidean space as *continuous*. For a set of continuous variables X with $\mathbf{X} = \mathbb{R}^n$ for $n \geq 0$, we assume that $\mathbf{X}$ is given the Euclidean metric topology, and use $\|\cdot\|$ to denote the Euclidean norm. For a set of discrete variables Q, we assume that $\mathbf{Q}$ is given the discrete topology (every subset is an open set), generated by the metric $d_D(q, q') = 0$ if $q = q'$ and $d_D(q, q') = 1$ if $q \neq q'$. We denote the valuations of the union $Q \cup X$ by $\mathbf{Q} \times \mathbf{X}$, which is given the product topology, generated by the metric $d((q, x), (q', x')) = d_D(q, q') + \|x - x'\|$. Using the metric d, we define the distance between two sets $U_1, U_2 \subseteq \mathbf{Q} \times \mathbf{X}$ by $d(U_1, U_2) = \inf_{(q_i, x_i) \in U_i} d((q_1, x_1), (q_2, x_2))$. We assume that a subset U of a topological space is given the induced topology, and we use $\overline{U}$ to denote its closure, U^o its interior, ∂U its boundary, U^c its complement, $|U|$ its cardinality, and $P(U)$ the set of all subsets of U.

2.2 Basic Definitions

The following definitions are based on [9, 5, 8].

Definition 1 (Hybrid Automaton) *A hybrid automaton H is a collection $H = (Q, X,$ Init, $f,$ Inv, Reset$)$, where*

- *Q is a finite collection of discrete variables;*
- *X is a finite collection of continuous variables with $\mathbf{X} = \mathbb{R}^n$;*
- Init $\subseteq \mathbf{Q} \times \mathbf{X}$ *is a set of initial states;*
- *$f : \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$ is a vector field;*

- $\text{Inv} \subseteq \mathbf{Q} \times \mathbf{X}$ *is the domain of* H;[1] *and,*

- $\text{Reset} : \mathbf{Q} \times \mathbf{X} \to P(\mathbf{Q} \times \mathbf{X})$ *is a reset relation.*

We refer to $(q, x) \in \mathbf{Q} \times \mathbf{X}$ as the *state* of H. Unless otherwise stated, we introduce the following assumption, to prevent some obvious pathological cases.

Assumption 1 *$|\mathbf{Q}| < \infty$ and f is Lipschitz continuous in its second argument.*

Note that, under the discrete topology on $\mathbf{Q}$, f is trivially continuous in its first argument. Under this assumption, a hybrid automaton can be represented by a directed graph $(\mathbf{Q}, E)$, with vertices $\mathbf{Q}$ and edges $E = \{(q, q') \in \mathbf{Q} \times \mathbf{Q} \ : \ \exists x, x' \in \mathbf{X}, (q', x') \in \text{Reset}(q, x)\}$. With each vertex $q \in \mathbf{Q}$, we associate a set of continuous initial states $\text{Init}(q) = \{x \in \mathbf{X} \ : \ (q, x) \in \text{Init}\}$, a vector field $f(q, \cdot)$, and a set $I(q) = \{x \in \mathbf{X} \ : \ (q, x) \in \text{Inv}\}$. With each edge $e = (q, q') \in E$, we associate a guard $G(e) = \{x \in \mathbf{X} \ : \ \exists x' \in \mathbf{X}, (q', x') \in \text{Reset}(q, x)\}$, and a reset relation $R(e, x) = \{x' \in \mathbf{X} \mid (q', x') \in \text{Reset}(q, x)\}$. Since there is a unique graphical representation for each hybrid automaton, we will use the corresponding graphs as formal definitions for hybrid automata in most examples.

Definition 2 (Hybrid Time Trajectory) *A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a finite or infinite sequence of intervals, such that*

- *$I_i = [\tau_i, \tau_i']$ for $i < N$, and, if $N < \infty$, $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$;*

- *$\tau_i \leq \tau_i' = \tau_{i+1}$ for $i \geq 0$.*

A hybrid time trajectory is a sequence of intervals of the real line, whose end points overlap. The interpretation is that the end points of the intervals are the times at which discrete transitions take place. Note that $\tau_i = \tau_i'$ is allowed, therefore multiple discrete transitions may take place at the same "time". Since the dynamical systems we will be concerned with will be time invariant we can, without loss of generality, assume $\tau_0 = 0$. Hybrid time trajectories can extend to infinity if τ is an infinite sequence or if it is a finite sequence ending with an interval of the form $[\tau_N, \infty)$. We denote by $\mathcal{T}$ the set of all hybrid time trajectories and use $t \in \tau$ as shorthand notation for that there exists i such that $t \in I_i \in \tau$. For a topological space K we use $k : \tau \to K$ as a short hand notation for a map assigning a value from K to each $t \in \tau$; note that k is not a function on the real line, as it assigns multiple values to the same $t \in \mathbb{R}$: $t = \tau_i' = \tau_{i+1}$ for all $i \geq 0$. Each $\tau \in \mathcal{T}$ is fully ordered by the relation $\prec$ defined by $t_1 \prec t_2$ for $t_1 \in [\tau_i, \tau_i']$ and $t_2 \in [\tau_j, \tau_j']$ if and only if $i < j$, or $i = j$ and $t_1 < t_2$.

Definition 3 (Execution) *An execution χ of a hybrid automaton H is a collection $\chi = (\tau, q, x)$ with $\tau \in \mathcal{T}$, $q : \tau \to \mathbf{Q}$, and $x : \tau \to \mathbf{X}$, satisfying*

- $\big(q(\tau_0), x(\tau_0)\big) \in \text{Init}$ *(initial condition);*

- *for all i with $\tau_i < \tau_i'$, $q(\cdot)$ is constant and $x(\cdot)$ is a solution[2] to the differential equation $dx/dt = f(q, x)$ over $[\tau_i, \tau_i']$, and for all $t \in [\tau_i, \tau_i')$, $(q(t), x(t)) \in \text{Inv}$ (continuous evolution); and*

[1] The set Inv is called the invariant set in the hybrid system literature in computer science. Note that Inv is not invariant in the usual dynamical systems sense.

[2] "Solution" is interpreted in the sense of Caratheodory.

- *for all i, $(q(\tau_{i+1}), x(\tau_{i+1})) \in \text{Reset}\big(q(\tau_i'), x(\tau_i')\big)$ (discrete evolution).*

We say a hybrid automaton *accepts* an execution χ or not. For an execution $\chi = (\tau, q, x)$, we use $(q_0, x_0) = \big(q(\tau_0), x(\tau_0)\big)$ to denote the initial state of χ. The *execution time* $\tau_\infty(\chi)$ is defined as $\tau_\infty(\chi) = \sum_{i=0}^{N}(\tau_i' - \tau_i)$, where $N+1$ is the number of intervals in the hybrid time trajectory. An execution is *finite* if τ is a finite sequence ending with a compact interval, it is called *infinite* if τ is either an infinite sequence or if $\tau_\infty(\chi) = \infty$, and it is called *Zeno* if it is infinite but $\tau_\infty(\chi) < \infty$. The execution time of a Zeno execution is also called the Zeno time. We use $\mathcal{E}_H(q_0, x_0)$ to denote the set of all executions of H with initial condition $(q_0, x_0) \in \text{Init}$, $\mathcal{E}_H^\infty(q_0, x_0)$ to denote the set of all infinite executions of H with initial condition $(q_0, x_0) \in \text{Init}$. We define $\mathcal{E}_H = \bigcup_{(q_0,x_0)\in\text{Init}} \mathcal{E}_H(q_0, x_0)$ and $\mathcal{E}_H^\infty = \bigcup_{(q_0,x_0)\in\text{Init}} \mathcal{E}_H^\infty(q_0, x_0)$. To simplify the notation, we will drop the subscript H whenever the automaton is clear from the context.

2.3 Classes of Automata

Definition 4 (Non-Blocking and Deterministic Automaton) *A hybrid automaton H is non-blocking if $\mathcal{E}_H^\infty(x_0)$ is non-empty for all $x_0 \in \text{Init}$. It is deterministic if $\mathcal{E}_H^\infty(x_0)$ contains at most one element for all $x_0 \in \text{Init}$.*

Note that if a hybrid automaton is both non-blocking and deterministic, then it accepts a unique infinite execution for each initial condition. In [8] conditions were established that determine whether an automaton is non-blocking and deterministic. The conditions require one to argue about the set of states reachable by a hybrid automaton, and the set of states from which continuous evolution is impossible. A state $(q, x) \in \mathbf{Q} \times \mathbf{X}$ is called *reachable by H*, if there exists a finite execution $\chi = (\tau, q, x)$ with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^{N}$ and $(q(\tau_N'), x(\tau_N')) = (q, x)$. We use Reach_H to denote the set of states reachable by a hybrid automaton, and $\text{Reach}_H(q)$ the projection of Reach_H to discrete state q. We will drop the subscript H whenever the automaton is clear from the context. The set Reach is in general difficult to compute. Fortunately, the conditions of subsequent theorems will not require us to do so: any over-approximation of the reachable set will be sufficient. In [1, 8] methods for computing such over-approximations using simple induction arguments are outlined.

The set of states from which continuous evolution is impossible is given by

$$\text{Out}_H = \{(q^0, x^0) \in \mathbf{Q} \times \mathbf{X} \,|\, \forall\, \epsilon > 0,\ \exists\, t \in [0, \epsilon),\ (q^0, x(t)) \notin \text{Inv}\},$$

where $x(\cdot)$ is the solution to $dx/dt = f(q^0, x)$ with $x(0) = x^0$. Note that if Inv is an open set, then Out is simply Inv^c. If Inv is closed, then Out may also contain parts of the boundary of Inv. In [8] methods for computing Out were proposed, under appropriate smoothness assumptions on f and the boundary of Inv. As before, we will use $\text{Out}_H(q)$ to denote the projection of Out to discrete state q, and drop the subscript H whenever the automaton is clear from the context. With these two pieces of notation one can show the following [8] results.

Lemma 1 *A (deterministic) hybrid automaton H is non-blocking if (and only if) for all $(q, x) \in \text{Out} \cap \text{Reach}$, $\text{Reset}(q, x) \neq \emptyset$.*

Lemma 2 *A hybrid automaton is deterministic if and only if for all $(q, x) \in \text{Reach}$, $|\text{Reset}(q, x)| \leq 1$ and, if $\text{Reset}(q, x) \neq \emptyset$, then $(q, x) \in \text{Out}$.*

Definition 5 (Invariant Preserving) *A hybrid automaton is invariant preserving if* $\text{Reach} \subseteq \overline{\text{Inv}}$.

In other words, H is invariant preserving if the state remains in the closure of the invariant along all executions.

Proposition 1 *An automaton H is invariant preserving if and only if* $\text{Init} \subseteq \text{Inv}$ *and for all* $(q, x) \in \text{Inv} \cap \text{Reach}$, $\text{Reset}(q, x) \subseteq \text{Inv}$.

Note that the use of Reach is again not limiting. Note also that the conditions of the lemma do not depend on the vector field f. This is because, by the definition of an execution, the state can never end up outside the closure of the invariant along continuous evolution.

Definition 6 (Transverse Invariants) *A hybrid automaton H is said to have transverse invariants if there exists a function $\sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ continuously differentiable in its second argument, such that* $\text{Inv} = \{(q, x) \in \mathbf{Q} \times \mathbf{X} \; : \; \sigma(q, x) \geq 0\}$ *and for all (q, x) with $\sigma(q, x) = 0$, $L_f\sigma(q, x) \neq 0$.*

Here $L_f\sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ denotes the Lie derivative of σ along f defined as

$$L_f\sigma(q, x) = \frac{\partial \sigma}{\partial x}(q, x) \cdot f(q, x)$$

In other words, an automaton has transverse invariants if the set Inv is closed, its boundary is differentiable, and the vector field f is pointing either inside or outside of Inv along the boundary.[3] If H has transverse invariants the set Out_H admits a fairly simple characterization.

Proposition 2 *If H has transverse invariants, then*

$$\text{Out}_H = \{(q, x) \in \mathbf{Q} \times \mathbf{X} \; : \; \sigma(q, x) < 0\} \cup \{(q, x) \in \mathbf{Q} \times \mathbf{X} \; : \; \sigma(q, x) = 0 \text{ and } L_f\sigma(q, x) < 0\}.$$

3 Invariant Sets and Stability

3.1 Basic Definitions

We first recall some standard concepts from dynamical system theory, and discuss how they generalize to hybrid automata.

Definition 7 (Invariant Set) *A set $M \subseteq \mathbf{Q} \times \mathbf{X}$ is called invariant if for all $(q_0, x_0) \in M$, $(\tau, q, x) \in \mathcal{E}_H(q_0, x_0)$, and $t \in \tau$, it holds that $(q(t), x(t)) \in M$.*

The class of invariant sets is closed under arbitrary unions and intersections. Invariant sets are such that all executions starting in the set remain in the set for ever.[4] We are interested in studying the stability of invariant sets, i.e., determine whether all trajectories that start close to an invariant set remain close to it.

[3] Under appropriate smoothness assumptions on σ and f the definition of transverse invariants can be relaxed somewhat by allowing $L_f\sigma(q, x) = 0$ on the boundary of Inv and taking higher-order Lie derivatives, until one that is non-zero is found. Even though many of the results presented here extend to this relaxed definition, the proofs are slightly more technical. We will therefore limit ourselves to the notion of transverse invariants given in Definition 6.

[4] Strictly speaking, we need to assume that $M \subseteq \text{Init}$.

Definition 8 (Stable Invariant Set) *An invariant set $M \subseteq \mathbf{Q} \times \mathbf{X}$ is called stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all (q_0, x_0) with $d((q_0, x_0), M) < \delta$, for all $(\tau, q, x) \in \mathcal{E}_H(q_0, x_0)$ and for all for all $t \in \tau$, $d((q(t), x(t)), M) < \epsilon$. An invariant set is called (locally) asymptotically stable if it is stable and in addition there exists $\Delta > 0$ such that for all (q_0, x_0) with $d((q_0, x_0), M) < \Delta$ and all $(\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$, $\lim_{t \to \tau_\infty} d((q(t), x(t)), M) = 0$.*

Note that since τ is fully ordered the above limit is well defined. The asymptotic behavior of an infinite execution is captured in terms of its ω limit set.

Definition 9 (ω limit set) *The ω limit point $(\hat{q}, \hat{x}) \in \mathbf{Q} \times \mathbf{X}$ of an execution $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty$ is a point for which there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in \tau$ such that as $n \to \infty$, $\theta_n \to \tau_\infty$ and $(q(\theta_n), x(\theta_n)) \to (\hat{q}, \hat{x})$. The ω limit set $S_\chi \subseteq \mathbf{Q} \times \mathbf{X}$ is the set of all ω limit points of an execution χ.*

The following lemma establishes a relation between ω limit sets and invariant sets.

Lemma 3 *Consider a hybrid automaton H and assume it is deterministic, invariant preserving, has transverse invariants, $R(e, \cdot)$ continuous for all $e \in E$, and $d(G(e), G(e')) > 0$ for all $e = (q_1, q_2)$, $e' = (q_1, q_2')$ with $q_2 \neq q_2'$. Then, for any execution $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty$, if $x(\cdot)$ is bounded, then S_χ is a (i) nonempty, (ii) compact, and (iii) invariant set. Further, (iv) for all $\epsilon > 0$ there exists $T \in \tau$ such that $d((q(t), x(t)), S_\chi) < \epsilon$ for all $t \geq T$.*

The conditions of the lemma are sufficient. They can also be shown to be tight: one can construct hybrid automata that violate any one of the conditions of the lemma that accept infinite executions whose ω limit set is not invariant. The conditions of the lemma are not sufficient to establish continuity of executions with respect to initial conditions. We conjecture however that a non-blocking automaton that satisfies the conditions of the lemma has the continuity property.[5]

LaSalle's principle is a useful tool when studying the stability of conventional, continuous dynamical systems. Lemma 3 allows us to extend this tool to hybrid systems.

Theorem 1 (LaSalle's Invariance Principle) *Consider a hybrid automaton H that satisfies the conditions of Lemma 3. Assume there exists a compact invariant set $\Omega \subseteq \mathbf{Q} \times \mathbf{X}$ and let $\Omega_1 = \Omega \cap \mathrm{Out}^c$ and $\Omega_2 = \Omega \cap \mathrm{Out}$. Furthermore, assume there exists a continuous function $V : \Omega \to \mathbb{R}$, such that*

- *for all $(q, x) \in \Omega_1$, V is continuously differentiable with respect to x and $L_f V(q, x) \leq 0$; and*
- *for all $(q, x) \in \Omega_2$, $V(\mathrm{Reset}(q, x)) \leq V(q, x)$.*

Define $S_1 = \{(q, x) \in \Omega_1 : \ L_f V(q, x) = 0\}$ and $S_2 = \{(q, x) \in \Omega_2 : \ V(\mathrm{Reset}(q, x)) = V(q, x)\}$ and let M be the largest invariant subset of $S_1 \cup S_2$. Then, for all $(q_0, x_0) \in \Omega$ every execution $(\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$ approaches M as $t \to \tau_\infty$.

4 Zeno Hybrid Automata

Zeno hybrid automata accept executions with infinitely many discrete transitions within a finite time interval. Such systems are hard to analysis and simulate in a way that gives constructive information about the behavior of the real system. It is therefore important to be able to determine if a model is

[5]Because of the presence of discrete transitions, continuity is interpreted in the Skorohod metric [3].

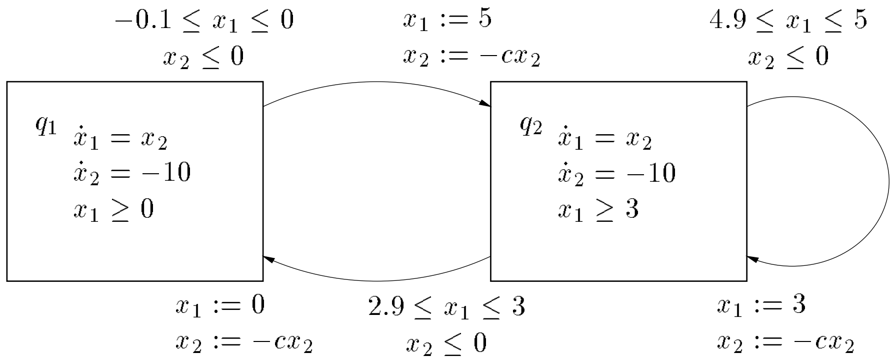

Figure 1: An example of a Zeno hybrid automaton.

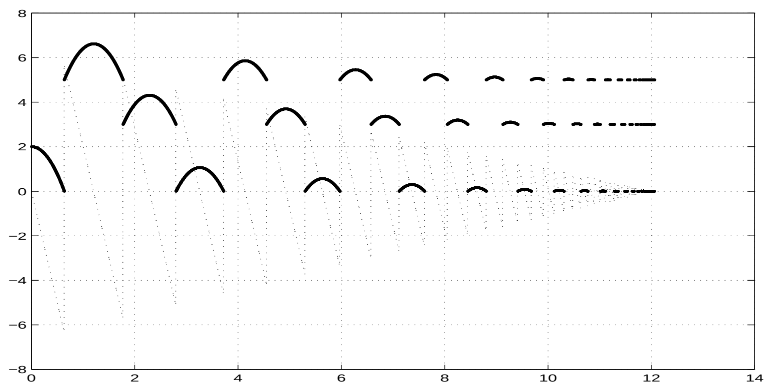

Figure 2: An example of an execution for the hybrid automaton in Example 1. The continuous part of the state is shown: x_1 (solid) and x_2 (dotted).

Zeno and in applicable cases remove Zenoness. These problems have been discussed in [5, 6]. In this section, some further characterization of Zeno executions are made. Recall that an infinite execution χ is Zeno if $\tau_\infty(\chi) = \sum_{i=0}^{\infty}(\tau_i' - \tau_i)$ is bounded.

Definition 10 (Zeno Hybrid Automaton) *A hybrid automaton H is Zeno if there exists $(q_0, x_0) \in \text{Init}$ such that all executions in $\mathcal{E}_H^\infty(q_0, x_0)$ are Zeno.*

Example 1 The hybrid automaton in Figure 1 is Zeno. This is easily checked by explicitly deriving the time intervals $\tau_i' - \tau_i$, which in this case gives a converging geometric series. Figure 2 shows an execution accepted by the automaton.

We make the following two straightforward observations.

Proposition 3 *A hybrid automaton is Zeno only if (Q, E) is a cyclic graph.*

Proposition 4 *If there exists a finite collection of states $\{(q_i, x_i)\}_{i=1}^N$ such that*

- $(q_1, x_1) = (q_N, x_N)$*;*
- $(q_i, x_i) \in \text{Reach}_H$ *for some $i = 1, \ldots, N$; and*

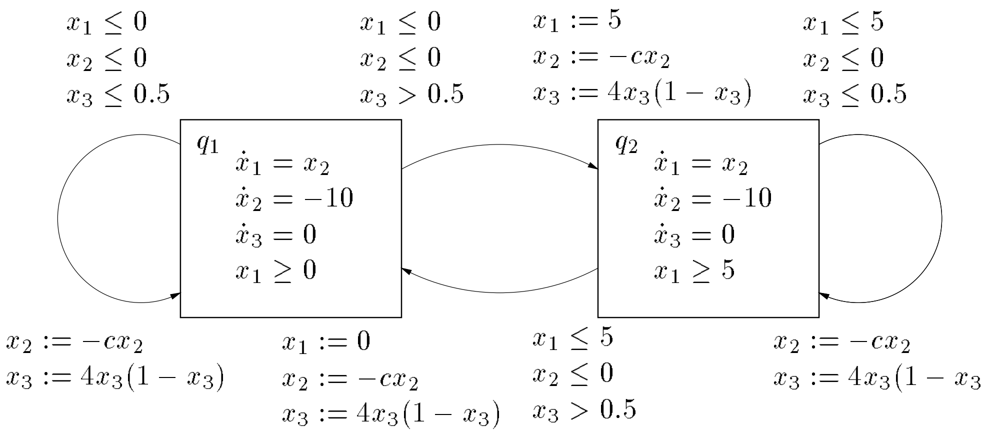

Figure 3: A hybrid automaton that accepts Zeno executions that do not periodically jump between the discrete states as the Zeno time is approached.

- $(q_{i+1}, x_{i+1}) = \text{Reset}(q_i, x_i)$ *for all* $i = 1, \dots, N-1$;

then there exists a Zeno execution.

Zenoness is critically dependent on the reset relations. For example, if in Example 1 the resets $x_2 := -cx_2$ are replaced by $x_2 := x_2/(dx_2 - 1)$, where $d = 1/\sqrt{20x_1(\tau_0)}$, then the time intervals $\tau_i' - \tau_i$ decrease as $\{1/i\}_{i=0}^{\infty}$. This is a diverging series, so the new hybrid automaton is not Zeno.

If the continuous part of the Zeno execution is bounded, then it has an ω limit point. We introduce the term *Zeno state* for such a point.

Definition 11 (Zeno State) *The ω limit point of a Zeno execution is called the Zeno state.*

We use $Z_\infty \subset \mathbf{Q} \times \mathbf{X}$ to denote the set of Zeno states, so Z_∞ is the ω limit set of the Zeno execution. We write $\mathbf{Q}_\infty$ for the discrete part of Z_∞ and E_∞ for the corresponding edges. In Example 1, we have $Z_\infty = \{(q_1, (0,0)), (q_2, (3,0)), (q_2, (5,0))\}$, $\mathbf{Q}_\infty = \{q_1, q_2\}$, and $E_\infty = E$.

It is easy to construct an example with a Zeno executions that do not have a Zeno state. The idea is to let the continuous part of the execution become unbounded as $t \to \tau_\infty(\chi)$. It is also straightforward to derive examples where the set of Zeno states have any number of elements, as well as an infinite but countable or uncountable number of elements. An interesting question is if for a Zeno execution $\chi = (q, x, \tau)$, the discrete part q must become periodic for $t \in \tau$ sufficiently close to $\tau_\infty(\chi)$, as in Example 1. The answer is no as illustrated by the following example.

Example 2 Consider the Zeno hybrid automaton in Figure 3. This system does not accept Zeno executions that periodically jump between the two discrete states. A simulation is presented in Figure 4, where x_1 and x_2 are shown. The third continuous state is initialized at $x_3(\tau_0) = 0.9$. The reason for the quasi-periodic behavior is that the reset map of x_3 is the logistic map and iteration of this map will give any value in $(0,1)$, e.g., [10].

A reset relation R is *non-expanding* if there exists $\delta \in [0,1]$ such that for all $(q, q') \in E$, $x \in G(q, q')$, and $x' \in R(q, q', x)$, it holds that $\|x'\| \leq \delta \|x\|$. A reset relation R, which is a function, is

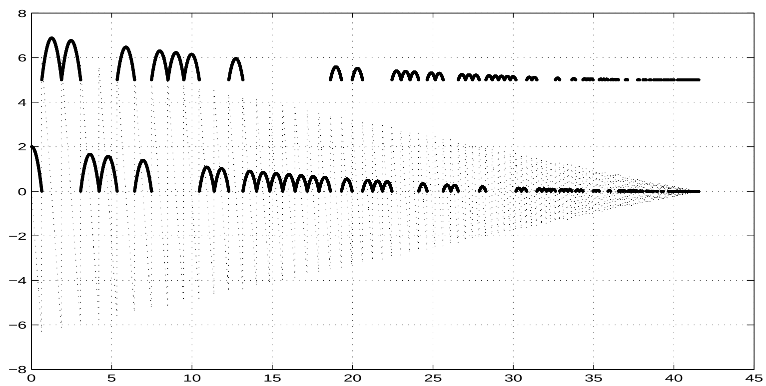

Figure 4: An example of an execution for the hybrid automaton in Example 2. The continuous states x_1 (solid) and x_2 (dotted) are shown. Note how they illustrate the quasi-periodicity.

contracting if there exists $\delta \in [0,1)$ such that for all $(q,q') \in E$ and $x,y \in G(q,q')$, it holds that $\|R(q,q',x) - R(q,q',y)\| \le \delta\|x-y\|$. Note that it is no restriction to assume that the reset relation is a function in the second case. For smooth dynamical systems, a Lipschitz assumption on the vector field excludes finite escape time. This is not a sufficient condition for hybrid systems. However, if the reset relations are non-expanding (in addition to the Lipschitz assumption on $f(q,\cdot)$), then the state is bounded along executions.

Lemma 4 *Consider a hybrid automaton with non-expanding reset relations. Then, there exists $c > 0$ such that for all executions $\chi = (\tau, q, x) \in \mathcal{E}_H$ and $t \in \tau$,*

$$\|x(t)\| \le \big(\|x(\tau_0)\| + 1\big)e^{c(t-\tau_0)} - 1.$$

When $x(\cdot)$ is bounded, the Bolzano–Weierstrass Property implies that there exists at least one Zeno state for each Zeno execution. If the resets are the identity map, then the continuous part of the Zeno state is a singleton, as stated next.

Proposition 5 *Consider a hybrid automaton with $R(q,q',x) = \{x\}$ for all $(q,q') \in E$. Then, for every Zeno execution $\chi = (\tau,q,x)$, it holds that $Z_\infty = \mathbf{Q}_\infty \times \{\hat{x}\}$ for some $\mathbf{Q}_\infty \subseteq \mathbf{Q}$ and $\hat{x} \in \mathbf{X}$.*

If all the reset relations are contracting and has the origin as a common fixed point, then the continuous part of the Zeno state is the origin.

Proposition 6 *Consider a Zeno hybrid automaton with $R(q,q',\cdot)$ contracting and $R(q,q',0) = 0$ for all $(q,q') \in E$. Then, for every Zeno execution $\chi = (\tau,q,x)$, it holds that $Z_\infty = \mathbf{Q}_\infty \times \{0\}$ for some $\mathbf{Q}_\infty \subseteq \mathbf{Q}$.*

5 Conclusions

Motivated by numerous assumptions like "In this paper, we assume that the switched system is live and nonZeno" [4] and suggestions like "Additional work is needed in determining the role that Zeno-type control might play in hybrid system supervision" [7], we have extended some classical results to hybrid systems, using tools that capture also Zeno executions. We have also tried to illustrate some of the nature of Zeno by characterizing Zeno executions and Zeno states for a few quite broad classes of hybrid systems.

References

[1] M. Branicky, E. Dolginova, and N. Lynch. A toolbox for proving and maintaining hybrid specifications. In A. Nerode P. Antsaklis, W. Kohn and S. Sastry, editors, *Hybrid Systems IV*, number 1273 in LNCS, pages 18–30. Springer Verlag, 1997.

[2] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, April 1998.

[3] M. Broucke. Regularity of solutions and homotopy equivalence for hybrid systems. In *IEEE Conference on Decision and Control*, Tampa, FL, 1998.

[4] K. X. He and M. D. Lemmon. Lyapunov stability of continuous-valued systems under the supervision of discrete-event transition systems. In *Hybrid Systems: Computation and Control*, volume 1386 of *Lecture Notes in Computer Science*. Springer-Verlag, Berling, 1998.

[5] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry. Regularization of Zeno hybrid automata. *Systems & Control Letters*, 1999. To appear.

[6] K. H. Johansson, J. Lygeros, S. Sastry, and M. Egerstedt. Simulation of Zeno hybrid automata. In *IEEE Conference on Decision and Control*, Phoenix, AZ, 1999.

[7] M. D. Lemmon, K. X. He, and I Markovsky. Supervisory hybrid systems. *IEEE Control Systems Magazine*, 19(4):42–55, 1999.

[8] J. Lygeros, K. H. Johansson, S. Sastry, and M. Egerstedt. On the existence of executions of hybrid automata. In *IEEE Conference on Decision and Control*, Phoenix, AZ, 1999.

[9] J. Lygeros, C. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3), March 1999.

[10] S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer-Verlag, New York, 1999.

[11] A. J. van der Schaft and J. M. Schumacher. Complementarity modeling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, April 1998.

[12] H. Ye, A. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Transactions on Automatic Control*, 43(4):461–474, April 1998.