Singularity-Free Dynamic Equations of Vehicle-Manipulator Systems

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Abstract

In this paper we derive the singularity-free dynamic equations of vehicle-manipulator systems using a minimal representation. These systems are normally modeled using Euler angles, which leads to singularities, or Euler parameters, which is not a minimal representation and thus not suited for Lagrange's equations. We circumvent these issues by introducing quasi-coordinates which allows us to derive the dynamics using minimal and globally valid non-Euclidean configuration coordinates. This is a great advantage as the configuration space of the vehicle in general is non-Euclidean. We thus obtain a computationally efficient and singularity-free formulation of the dynamic equations with the same complexity as the conventional Lagrangian approach. The closed form formulation makes the proposed approach well suited for system analysis and model-based control. This paper focuses on the dynamic properties of vehicle-manipulator systems and we present the explicit matrices needed for implementation together with several mathematical relations that can be used to speed up the algorithms. We also show how to calculate the inertia and Coriolis matrices and present these for several different vehicle-manipulator systems in such a way that this can be implemented for simulation and control purposes without extensive knowledge of the mathematical background. By presenting the explicit equations needed for implementation, the approach presented becomes more accessible and should reach a wider audience, including engineers and programmers.

Keywords: Robot modeling, vehicle-manipulator dynamics, singularities, quasi-coordinates.

1. Introduction

A good understanding of the dynamics of a robotic manipulator mounted on a moving vehicle is important in a wide range of applications. Especially, the use of robots in harsh and remote areas has increased the need for vehicle-robot solutions. A robotic manipulator mounted on a moving vehicle is a flexible and versatile solution well suited for these applications and will play an important role in the operation and surveillance of remotely located plants in the very near future. Recreating realistic models of for example space or deep-sea conditions is thus important. Both for simulation and for model-based control the explicit dynamic equations of vehicle-manipulator systems need to be implemented in a robust and computationally efficient way to guarantee safe testing and operation of these systems.

One example of such a system is spacecraft-manipulator systems [1, 2, 3, 4, 5] which are emerging as an alternative to human operation in space. Operations include assembling, repair, refuelling, maintenance, and operations of satellites and space stations. Due to the enormous risks and costs involved with launching humans into space, robotic solutions evolve as the most cost-efficient and reliable solution. However, space manipulation involves quite a few challenges. In this paper we focus on modeling spacecraft-manipulator systems, which is quite different from standard robot modeling. Firstly, the manipulator is mounted on a free-floating (unactuated) or free-flying (actuated) spacecraft. There is thus no obvious way to choose the inertial frame. Secondly, the motion of the manipulator affects the motion of the base, which results in a set of dynamic equations different from the fixed-base case due to the dynamic coupling. Finally, the free fall environment complicates the control and enhances the non-linearities in the Coriolis matrix. The framework presented in this paper is especially suited for modeling such systems, especially when applying the so-called dynamically equivalent manipulator approach [6, 7]. A set of minimal, singularity free dynamic equations for spacecraft-manipulator systems are presented for the first time using the proposed framework.

A second example studied in detail in this paper is the use of autonomous underwater vehicles (AUVs) with robotic arms, or underwater robotic vehicles (URVs). This is an efficient way to perform challenging tasks over a large sub-sea area. Operations at deeper water and more remote areas where humans cannot or do not want to operate, require more advanced and robust underwater systems and thus the need for continuously operating robots for surveillance, maintenance, and operation emerges [8, 9, 10, 11]. We derive the minimal, singularity free dynamic equations of AUV-manipulator systems using the proposed framework, which is presented for the first time in this paper. We also show how to add the hydrodynamic effects such as added mass and damping forces.

The use of robotic manipulators on ships is another important application [9, 12]. In From et al. [13] the dynamic equations were derived and the effects of the moving ship on the manipulator was analyzed. In the Ampelmann project [14] a Stewart platform is mounted on a ship and is used to compensate for the motion of the ship by keeping the platform still with respect to the world frame. This can be modeled as a 2-joint mechanism where one joint represents the uncontrollable ship motion and one joint the Stewart platform. There are also other relevant research areas where a robotic manipulator is mounted on a floating base. Lebans et al. [15] give a cursory description of a telerobotic shipboard handling system, and Kosuge et al. [16, 17] address the control of robots floating on the water utilizing vehicle restoring forces. Another interesting research area is macro/micro manipulators [18, 19] where the two manipulators in general have different dynamic properties.

It is a well known fact that the kinematics of a rigid body contains singularities if the Euler angles are used to represent the orientation of the body and the joint topology is not taken into account. One solution to this problem is to use a non-minimal representation such as the unit quaternion to represent the orientation. This will, however, increase the complexity of the implementation and because the unit quaternion is a covering manifold for the set of rotation matrices they are also subject to the unfortunate unwinding phenomenon [20]. Also, as the number of variables is not minimal, this representation cannot be used in Lagrange's equations. This is a major drawback when it comes to modeling vehicle-manipulator systems as most methods used for robot modeling are based on the Lagrangian approach. It is thus a great advantage if also the vehicle dynamics can be derived from the Lagrange equations.

The use of Lie groups and algebras as a mathematical basis for the derivation of the dynamics of multibody systems can be used to overcome this problem [21, 22]. We then choose the coordinates generated by the Lie algebra as local Euclidean coordinates which allows us to describe the dynamics locally. For this approach to be valid globally the total configuration space needs to be covered by an atlas of local exponential coordinate patches. The appropriate equations must then be chosen for the current configuration. The geometric approach presented in Bullo and Lewis [23] can then be used to obtain a globally valid set of dynamic equations on a single Lie group, such as an AUV or spacecraft with no robotic manipulator attached.

Even though combinations of Lie groups can be used to represent multibody systems, the formulation is very complex and not suited for implementation in a simulation environment. In Kwatny and Blankenship [24] quasi-coordinates and the Lie bracket were used to derive the dynamic equations of fixed-base robotic manipulators without singularities using Poincaré's formulation of the Lagrange equations. In Kozlowski and Herman [25, 26] several control laws using a quasi-coordinate approach were presented, but only robots with conventional 1-DoF joints were considered. Common for all these methods is, however, that the configuration space of the vehicle and robot is described as $q \in \mathbb{R}^n$. This is not a problem when dealing with 1-DoF revolute or prismatic joints but more complicated joints such as ball-joints or free-floating joints then need to be modeled as compound kinematic joints [24], i.e., a combination of 1-DoF simple kinematic joints. For joints that use the Euler angles to represent the generalized coordinates this solution leads to singularities in the representation.

In this paper we follow the generalized Lagrangian approach presented in Duindam et al. [27, 28] which allows us to combine the Euclidean joints and more general joints, i.e., joints that can be described by the Lie group $SE(3)$ or one of its ten subgroups, and we extend these ideas to vehicle-manipulator systems. There are several advantages in following this approach. The use of quasi-coordinates, i.e., velocity coordinates that are not simply the time derivative of the position coordinates, allows us to include joints (or transformations) with a different topology than that of $\mathbb{R}^n$. For example, for an AUV we can represent the transformation from the inertial frame to the AUV body frame as a free-floating joint with configuration space $SE(3)$ and we avoid the singularity-prone kinematic relations between the inertial frame and the body frame velocities that normally arise in deriving the AUV dynamics [29]. This relation is subject to the well known Euler angle singularities and the dynamics are not valid globally. With our approach we thus get improved numerical stability due to the absence of singularities and, as the dynamics are valid globally, we avoid switching between different dynamic models in the implementation. This approach differs from previous work in that it allows us to derive the dynamic equations of vehicle-manipulator systems for vehicles with a configuration space different from $\mathbb{R}^n$ and thus maintains the underlying topology of the configuration space. The dynamics are expressed (locally) in exponential coordinates ϕ, but the final equations are evaluated at $\phi = 0$. This has two main advantages. Firstly, the dynamics do not depend on the local coordinates as these are eliminated from the equations and the global position and velocity coordinates are the only state variables. This makes the equations valid globally. Secondly, evaluating the equations at $\phi = 0$ greatly simplifies the dynamics and make the equations suited for implementation in simulation software. We also note that the approach is well suited for model-based control as the equations are explicit and without constraints. The fact that the configuration space of the vehicle in general is a Lie group also simplifies the implementation. Even though the expressions in the derivation of the dynamics are somewhat complex, we have several tools from the Lie theory that allows us to write the final expressions in a very simple form. We present several examples of how we can use this to simplify the dynamic equations and speed up the implementation.

The main purpose of this paper is to study systems that consist of a moving vehicle with a robotic manipulator attached to it. To the authors' best knowledge these systems have not been studied in detail in literature using the framework presented here. There is an apparent need to be able to derive the dynamics of such systems globally and using a minimal representation, especially when it comes to formulating model-based control laws. In this paper we first present the framework, based on the approach in Duindam et al. [27, 28], and then show how to expand this to vehicle-manipulator systems. The use of quasi-coordinates to derive the dynamics in this way has mainly been applied to standard robotic manipulators with the extension to more general types of joints in [24, 28]. However, the treatment of vehicle-manipulator systems deserves a special treatment. There are several reasons for this. Firstly, the vehicle and the manipulator may possess completely different dynamic properties. One apparent example is when the vehicle possesses a forced un-controllable motion while the manipulator is controllable. This is the case for manipulators mounted on ships, as treated in [13], where the high-frequency motion of the ship is a forced motion due to the waves and wind. Spacecraft-manipulator systems are another example where the spacecraft may be unactuated and its position is determined by the robot motion. Secondly, the formulation allows us to study

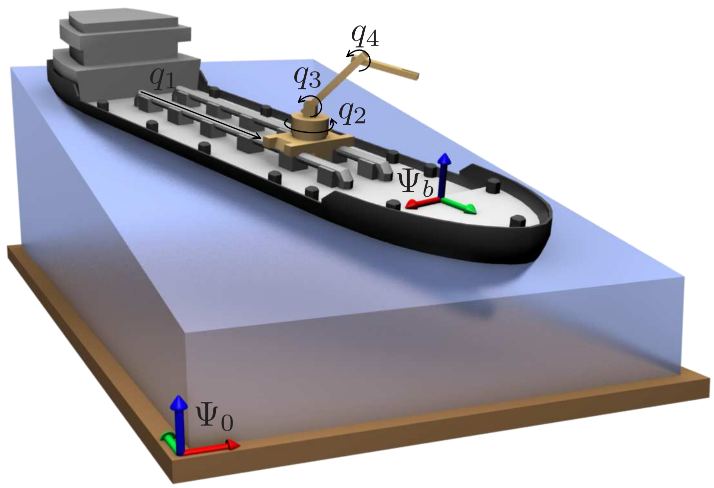

Figure 1: Model setup for a four-link robot attached to a vehicle, in this case a ship, with coordinate frame Ψ_b. Frame Ψ_0 denotes the inertial reference frame.

how the two systems, i.e., the vehicle and the manipulator, affect each other. The interaction of the two systems will depend on the inertial properties of the two systems, external forces acting on one or both systems and the type of the vehicle (floating, submerged, rolling, fixed, etc.).

The paper is organized as follows. Section 2 gives the detailed mathematical background for the proposed approach. This section can be skipped and practitioners mainly interested in implementation can go straight to Section 3 or 4. Section 3 gives the explicit dynamic equations for the AUV-manipulator dynamics along with some comments on implementing these in a simulation environment. This includes hydrodynamic and damping forces, the added mass and Coriolis matrices and other considerations that are not encountered in robot dynamics. Section 4 presents the dynamic equations for spacecraft-manipulator systems and the effects of a free-floating base in a free fall environment are treated in detail. The matrix representation of the dynamics and how to implement this is presented in great detail for several vehicles with different configuration spaces. This allows the readers to first analyse the dynamics of the system from the given equations and then implement this in a simulation or control environment without having to perform all the detailed computations themselves.

2. Dynamic Equations of Vehicle-Manipulator Systems

We extend the classical dynamic equations for a serial manipulator arm with 1-DoF joints to include the motion of the vehicle on which the manipulator is mounted. We assume that the motion of the vehicle can be described by a Lie group, i.e., $SE(3)$ or one of its ten subgroups. The most important examples of "vehicles" that have a Lie group topology are shown in Table 1.

$SE(3)$	-	AUV, 6-DoF ship, aeroplane, spacecraft
$X(z)$	-	The Schönflies group
$T(3)$	-	3-DoF gantry crane
$SO(3)$	-	Spacecraft (DEM approach), ball joint
$SE(2)$	-	Ground vehicle, 3-DoF ship
$T(2)$	-	2-DoF gantry crane

Table 1: Lie subgroups of $SE(3)$ and corresponding "vehicles". Even though some of these can be modeled as a combination of 1-DoF Euclidean joints we consider these as vehicles and group them correspondingly. The Schönflies Group $X(z)$ represent 3-DoF translation and a 1-DoF rotation about the z-axis.

2.1. Vehicle-Manipulator Kinematics

Consider the setup of Figure 1 describing a general n-link robot manipulator arm attached to a vehicle. Choose an inertial coordinate frame Ψ_0, a frame Ψ_b rigidly attached to the vehicle, and n frames Ψ_i (not shown) attached to each link i at the center of mass with axes aligned with the principal directions of inertia. Finally, choose a vector $q \in \mathbb{R}^n$ that describes the configuration of the n joints. Using standard notation [30], we can describe the pose of each frame Ψ_i relative to Ψ_0 as a homogeneous transformation matrix $g_{0i} \in SE(3)$ of the form

$$g_{0i} = \begin{bmatrix} R_{0i} & p_{0i} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{1}$$

with rotation matrix $R_{0i} \in SO(3)$ and translation vector $p_{0i} \in \mathbb{R}^3$. This pose can also be described using the vector of joint coordinates q as

$$g_{0i} = g_{0b} g_{bi} = g_{0b} g_{bi}(q). \tag{2}$$

The vehicle pose g_{0b} and the joint positions q thus fully determine the configuration state of the robot. Even though we use g_{0b} (6 DoF) to represent the vehicle configuration, the actual configuration space of the vehicle may be a subspace of $SE(3)$ of dimension $m < 6$. For ground vehicles the configuration space is $SE(2)$, with dimension $m = 3$, and the attitude of a spacecraft has configuration space $SO(3)$, also with dimension $m = 3$.

In a similar way, the spatial velocity of each link can be expressed using twists [30]:

$$V_{0i}^0 = \begin{bmatrix} v_{0i}^0 \\ \omega_{0i}^0 \end{bmatrix} = V_{0b}^0 + V_{bi}^0 = \mathrm{Ad}_{g_{0b}} \left(V_{0b}^b + J_i(q)\dot{q} \right) \tag{3}$$

where v_{0i}^0 and ω_{0i}^0 are the linear and angular velocities, respectively, of link i relative to the inertial frame, $J_i(q) \in \mathbb{R}^{6\times n}$ is the geometric Jacobian of link i relative to Ψ_b, the adjoint is defined as $\mathrm{Ad}_g := \left[\begin{smallmatrix} R & \hat{p}R \\ 0 & R \end{smallmatrix}\right] \in \mathbb{R}^{6\times 6}$, and $\hat{p} \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix such that $\hat{p}x = p \times x$ for all $p, x \in \mathbb{R}^3$. The velocity state is thus fully determined given the twist V_{0b}^b of the vehicle and the joint velocities $\dot{q}$.

In the case of $m < 6$ we define a selection matrix $H \in \mathbb{R}^{6\times m}$ such that the velocity vector of the vehicle is given by

$$V^b_{0b} = H\tilde{V}^b_{0b}, \tag{4}$$

where $\tilde{V}^b_{0b} \in \mathbb{R}^m$ determines the velocity state of the vehicle by selecting elements of V^b_{0b} that are not trivially zero. More generally we will write the allowed joint velocity as a vector $v_i \in \mathbb{R}^{n_i}$. The joint velocity is uniquely described by this vector and the joint twist can be expressed in terms of this vector as $T^{i,i}_j = X_i(Q)v_i$ with $X_i(Q) \in \mathbb{R}^{6\times n_i}$ a matrix describing the instantaneously allowed twists. If X is independent of the manipulator configuration we get $H = X$. In our case we have $v_i = \dot{q}_i$ for the Euclidean joints of the manipulator and the velocity vector $v_b = \tilde{V}^b_{0b}$ for the allowed vehicle velocities. The spacial velocity when $m < 6$ is then written by

$$V^0_{0i} = \begin{bmatrix} v^0_{0i} \\ \omega^0_{0i} \end{bmatrix} = V^0_{0b} + V^0_{bi} = \mathrm{Ad}_{g_{0b}}\left(H\tilde{V}^b_{0b} + J_i(q)\dot{q}\right). \tag{5}$$

2.2. Vehicle-Manipulator Dynamics

The previous section shows how the kinematics of the system can be naturally described in terms of the (global) state variables g_{0b}, q, V^b_{0b}, and $\dot{q}$. We now derive the dynamic equations for the system in terms of these state variables. We first assume the vehicle to be free-moving and then restrict the vehicle motion to be kinematically constrained.

To derive the dynamics of the complete mechanism (including the m-DoF between Ψ_0 and Ψ_b), we follow the generalized Lagrangian method introduced by Duindam et al. [27, 28]. This method gives the dynamic equations for a general mechanism described by a set $Q = \{Q_i\}$ of configuration states Q_i (not necessarily Euclidean), a vector v of velocity states $v_i \in \mathbb{R}^{n_i}$, and several mappings that describe the local Euclidean structure of the configuration states and their relation to the velocity states. More precisely, the neighborhood of every state $\bar{Q}_i$ is locally described by a set of Euclidean coordinates $\phi_i \in \mathbb{R}^{n_i}$ as $Q_i = \Phi_i(\bar{Q}_i, \phi_i)$ with $\Phi_i(\bar{Q}_i, 0) = \bar{Q}_i$. $\Phi_i(\bar{Q}_i, \phi_i)$ defines a local diffeomorphism between a neighborhood of $0 \in \mathbb{R}^{n_i}$ and a neighborhood of $\bar{Q}_i$.

The trick here is to first consider Q_i a parameter, even though it strictly speaking is a state variable. We then think of the local coordinate ϕ_i as a state variable. The global coordinates v are thought of as state variables in the normal way. The Lagrangian is then written in terms of v_i for velocity and $\Phi_i(\bar{Q}_i, \phi_i)$ for position and we differentiate with respect to the velocity variable v_i and the position variable ϕ_i, not $\bar{Q}_i$ which we for now consider a parameter. Recalling that $\Phi_i(\bar{Q}_i, 0) = \bar{Q}_i$, we see that evaluating the expressions at $\phi = 0$ allows us to consider Q_i a variable and we are done. The reason we can do this is that locally the variables ϕ describe the configuration state of the system in a neighborhood of any configuration $\bar{Q}_i$.

We start by deriving an expression for the kinetic co-energy of a mechanism, expressed in coordinates Q, v, but locally parameterized by the coordinate mappings for each joint. For joints that can be described by a matrix Lie group (actually for the group of $n \times n$ nonsingular real matrices $GL(n, \mathbb{R})$), this mapping can be given by the exponential map [30]. Let $A \in gl(n, \mathbb{R})$, where $gl(n, \mathbb{R})$ is the Lie Algebra of $GL(n, \mathbb{R})$. Then the exponential map $\exp(A)$ is given by

$$e^A = I + A + \frac{A^2}{2} \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{6}$$

where I (no subscript) is the identity matrix. This expression is valid for all subgroups of $SE(3)$ and $SE(3)$ itself by replacing A with the matrix representation of the Lie algebra associated with the Lie group. We denote the matrix representation of the corresponding Lie algebra by $\hat{\phi}$ and thus get

$$e^{\hat{\phi}} = I + \hat{\phi} + \frac{\hat{\phi}^2}{2} \cdots = \sum_{n=0}^{\infty} \frac{\hat{\phi}^n}{n!}. \tag{7}$$

Specific examples of $\hat{\phi}$ for different Lie groups are given in the following sections.

The dynamics are thus expressed in local coordinates ϕ for configuration and v for velocity, and we consider Q a parameter. After taking partial derivatives of the Lagrangian function, we evaluate the results at $\phi = 0$ (i.e., at configuration Q) to obtain the dynamics expressed in global coordinates Q and v as desired. We note that even though local coordinates ϕ appear in the derivations of the various equations, the final equations are all evaluated at $\phi = 0$ and hence these final equations do not depend on local coordinates. The global coordinates Q and v are the only dynamic state variables and the equations are valid globally, without the need for coordinate transitions between various areas of the configuration space, as is required in methods that use an atlas of local coordinate patches.

Note also that taking the partial derivatives of the Lagrangian and evaluating at $\phi = 0$ greatly simplifies (7) and the closed form expressions of the exponential map is not needed. We will use this observation to simplify the implementation and reduce the computational complexity of the algorithm. We will see several examples of how we can use this to simplify the expressions of the Coriolis matrices for different types of vehicles.

In general, the topology of a Lie group is not Euclidean. When deriving the dynamic equations for vehicles such as ships [29], AUVs [10], and spacecraft [3], this is normally dealt with by introducing a transformation matrix that relates the local and global velocity variables. However, forcing the dynamics into a vector representation in this way, without taking the topology of the configuration space into account, leads to singularities in the representation or other deficiencies. To preserve the topology of the configuration space we will use quasi-coordinates, i.e., velocity coordinates that are not simply the time-derivative of position coordinates, but given by a linear relation. Thus, there exist differentiable matrices S_i such that we can write $v_i = S_i(Q_i, \phi_i)\dot{\phi}_i$ for every Q_i. For Euclidean joints this relation is given simply by the identity map while for joints with a Lie group topology we can use the exponential map to derive this relation.

Given a mechanism with coordinates formulated in this generalized form, we can write its kinetic energy as $\mathcal{T}(Q, v) = \frac{1}{2} v^\mathsf{T} M(Q) v$ with $M(Q)$ the inertia matrix in coordinates Q and v the stacked velocities of the vehicle, represented by v_b, and the robot joints, represented by v_i, $i = 1 \dots n$. The dynamics of this system then satisfy

$$M(Q)\dot{v} + C(Q, v)v = \tau \tag{8}$$

with τ the vector of external and control wrenches (collocated with v), and $C(Q, v)$ the matrix describing Coriolis and centrifugal forces given by

$$\begin{aligned} C_{ij}(Q, v) := &\sum_{k,l} \left(\frac{\partial M_{ij}}{\partial \phi_k} S^{-1}_{kl} - \frac{1}{2} S^{-1}_{ki} \frac{\partial M_{jl}}{\partial \phi_k} \right) \Bigg|_{\phi=0} v_l \\ &+ \sum_{k,l,m,s} \left(S^{-1}_{mi} \left(\frac{\partial S_{mj}}{\partial \phi_s} - \frac{\partial S_{ms}}{\partial \phi_j} \right) S^{-1}_{sk} M_{kl} \right) \Bigg|_{\phi=0} v_l. \end{aligned} \tag{9}$$

More details and proofs can be found in references [27] and [28].

To apply this general result to systems of the form of Figure 1, we write $Q = \{g_{0b}, q\}$ as the set of configuration states where g_{0b} is the Lie group $SE(3)$ or one of its sub-groups, and $v = \begin{bmatrix} \tilde{V}^b_{0b} \\ \dot{q} \end{bmatrix}$ as the vector of velocity states. The local Euclidean structure for the state g_{0b} is given by exponential coordinates [30], while the state q is itself globally Euclidean. Mathematically, we can express configurations (g_{0b}, q) around a fixed state $(\bar{g}_{0b}, \bar{q})$ as

$$g_{0b} = \bar{g}_{0b} \exp\left(\sum_{j=1}^{6} b_j (\phi_b)_j \right), \qquad q_i = \bar{q}_i + \phi_i \quad \forall\, i \in \{1, \dots, n\}, \tag{10}$$

with b_j the standard basis elements of the Lie algebra $se(3)$ or one of its subalgebras. When $m < 6$ we set $b_i = 0$ for all the $n - m$ entries that are trivially zero, corresponding to Equation (4).

From expression (5) for the twist of each link in the mechanism, we can derive an expression for the total kinetic energy. Let $I_b \in \mathbb{R}^{m \times m}$ and $I_i \in \mathbb{R}^{6 \times 6}$ denote the constant positive-definite diagonal inertia tensor of the base and link i (expressed in Ψ_i), respectively. The kinetic energy $\mathcal{T}_i$ of link i then follows as

$$\begin{aligned} \mathcal{T}_i &= \frac{1}{2} \left(V^i_{0i} \right)^\mathsf{T} I_i V^i_{0i} \\ &= \frac{1}{2} \left(H \tilde{V}^b_{0b} + J_i(q)\dot{q} \right)^\mathsf{T} \mathrm{Ad}^\mathsf{T}_{g_{ib}} \, I_i \, \mathrm{Ad}_{g_{ib}} \left(H \tilde{V}^b_{0b} + J_i(q)\dot{q} \right) \\ &= \frac{1}{2} \left((\tilde{V}^b_{0b})^\mathsf{T} H^\mathsf{T} + \dot{q}^\mathsf{T} J_i(q)^\mathsf{T} \right) \mathrm{Ad}^\mathsf{T}_{g_{ib}} \, I_i \, \mathrm{Ad}_{g_{ib}} \left(H \tilde{V}^b_{0b} + J_i(q)\dot{q} \right) \\ &= \frac{1}{2} \begin{bmatrix} \left(\tilde{V}^b_{0b} \right)^\mathsf{T} & \dot{q}^\mathsf{T} \end{bmatrix} M_i(q) \begin{bmatrix} \tilde{V}^b_{0b} \\ \dot{q} \end{bmatrix} = \frac{1}{2} v^\mathsf{T} M_i(q) v \end{aligned} \tag{11}$$

with $M_b = \left[\begin{smallmatrix} I_b & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for the vehicle and

$$M_i(q) := \begin{bmatrix} H^\top \mathrm{Ad}_{g_{ib}}^\top I_i \mathrm{Ad}_{g_{ib}} H & H^\top \mathrm{Ad}_{g_{ib}}^\top I_i \mathrm{Ad}_{g_{ib}} J_i \\ J_i^\top \mathrm{Ad}_{g_{ib}}^\top I_i \mathrm{Ad}_{g_{ib}} H & J_i^\top \mathrm{Ad}_{g_{ib}}^\top I_i \mathrm{Ad}_{g_{ib}} J_i \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)} \tag{12}$$

for the links. Here, $H^\top$ is the transpose of H which works fine when dealing with the Lie groups treated here, so we will use this notation throughout this paper. The total kinetic energy of the mechanism is given by the sum of the kinetic energies of the mechanism links and the vehicle, that is,

$$\mathcal{T}(q, v) = \frac{1}{2} v^\top \underbrace{\left(\begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{n} M_i(q) \right)}_{\text{inertia matrix } M(q)} v \tag{13}$$

with $M(q)$ the inertia matrix of the total system. Note that neither $\mathcal{T}(q, v)$ nor $M(q)$ depend on the pose g_{0b} nor the choice of inertial reference frame Ψ_0.

We can write (8) in block-form as follows

$$\begin{bmatrix} M_{VV} & M_{qV}^\top \\ M_{qV} & M_{qq} \end{bmatrix} \begin{bmatrix} \dot{\tilde{V}}_{0b}^b \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} C_{VV} & C_{Vq} \\ C_{qV} & C_{qq} \end{bmatrix} \begin{bmatrix} \tilde{V}_{0b}^b \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \tau_V \\ \tau_q \end{bmatrix} \tag{14}$$

with τ_V a wrench of control and external forces acting on the vehicle, expressed in coordinates Ψ_b (such that it is collocated with $\tilde{V}_{0b}^b$). Here the subscript V refers to the first m entries and q the remaining $n - m$ entries. To compute the matrix $C(Q, v)$ for our system, we can use the observations that $M(q)$ is independent of g_{0b}, that $S(Q, \phi)$ is independent of q, and that $S(Q, 0) \equiv I$. Furthermore, the partial derivative of M with respect to ϕ_V is zero since M is independent of g_{0b}, and the second term of (9) is only non-zero for the C_{VV} block of $C(Q, v)$. This allows us to simplify $C(Q, v)$ slightly to

$$C_{ij}(Q, v) := \sum_{k=1}^{6+n} \left(\frac{\partial M_{ij}}{\partial \phi_k} - \frac{1}{2} \frac{\partial M_{jk}}{\partial \phi_i} \right) \Bigg|_{\phi=0} v_k + \sum_{k=1}^{6+n} \left(\frac{\partial S_{ij}}{\partial \phi_k} - \frac{\partial S_{ik}}{\partial \phi_j} \right) \Bigg|_{\phi=0} (M(q)v)_k. \tag{15}$$

Finally if gravitational forces are present we include these. Let the wrench associated with the gravitational force of link i with respect to coordinate frame Ψ_i be given by

$$F_g^i = \begin{bmatrix} f_g \\ \hat{r}_g^i f_g \end{bmatrix} = -m_i g \begin{bmatrix} R_{0i} e_z \\ \hat{r}_g^i R_{0i} e_z \end{bmatrix} \tag{16}$$

where $e_z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ and r_g^i is the center of mass of link i expressed in frame Ψ_i. In our case Ψ_i is chosen so that r_g^i is in the origin of Ψ_i so we have $r_g^i = 0$. The equivalent joint torque associated with link i is given by

$$\tau_g^i = J_i(q) \mathrm{Ad}_{g_{0i}}^\top (Q) F_g^i(Q) \tag{17}$$

where J_i is the geometric Jacobian and $\mathrm{Ad}_{g_{0i}} = \mathrm{Ad}_{g_{0b}} \mathrm{Ad}_{g_{bi}}$ is the transformation from the inertial frame to frame i. We note that both R_{0i} and $\mathrm{Ad}_{g_{0i}}$ depend on the vehicle configuration with respect to the inertial frame. The total effect of the gravity from all the links is then given by

$$n(Q) = \sum_{i=b}^{n} \tau_g^i \tag{18}$$

which enters (14) in the same way as τ.

2.3. Vehicles with Configuration Space $SE(3)$

The configuration space of a free-floating vehicle, such as an AUV or an aeroplane can be described by the matrix Lie group $SE(3)$. In this case we have the mapping [31]

$$V_{0b}^b = \left(I - \frac{1}{2}\,\mathrm{ad}_{\phi_V} + \frac{1}{6}\,\mathrm{ad}_{\phi_V}^2 - \dots\right)\dot{\phi}_V \tag{19}$$

with $\mathrm{ad}_p = \begin{bmatrix} \hat{p}_{4\dots6} & \hat{p}_{1\dots3} \\ 0 & \hat{p}_{4\dots6} \end{bmatrix} \in \mathbb{R}^{6\times6}$ for $p \in \mathbb{R}^6$ relating the local and global velocity variables, and $\tilde{V}_{0b}^b = V_{0b}^b$. The corresponding matrices S_i can be collected in one block-diagonal matrix S given by

$$S(Q,\phi) = \begin{bmatrix} \left(I - \frac{1}{2}\,\mathrm{ad}_{\phi_V} + \frac{1}{6}\,\mathrm{ad}_{\phi_V}^2 - \dots\right) & 0 \\ 0 & I \end{bmatrix} \in \mathbb{R}^{(6+n)\times(6+n)}. \tag{20}$$

This shows that the choice of coordinates (Q, v) has the required form. We note that when differentiating with respect to ϕ and substituting $\phi = 0$ this simplifies the expression substantially.

The precise computational details of the partial derivatives follow the same steps as in the classical approach [30]. C_{VV} depends on both the first and the second term in Equation (15). We have $i, j = 1\dots6$. Note that $\frac{\partial M_{ij}}{\partial \phi_k} = 0$ for $k < 7$ and $\frac{\partial S_{ij}}{\partial \phi_k} = 0$ for $i, j, k > 6$. This simplifies C_{VV} to

$$C_{ij}(Q,v) = \sum_{k=7}^{6+n} \left(\frac{\partial M_{ij}}{\partial \phi_k} - \underbrace{\frac{1}{2}\frac{\partial M_{jk}}{\partial \phi_i}}_{=0} \right)\Bigg|_{\phi=0} v_k + \sum_{k=1}^{6} \left(\frac{\partial S_{ij}}{\partial \phi_k} - \frac{\partial S_{ik}}{\partial \phi_j} \right)\Bigg|_{\phi=0} (M(q)v)_k. \tag{21}$$

Furthermore, if we write $S_b = (I - \frac{1}{2}\,\mathrm{ad}_{\phi_V} + \frac{1}{6}\,\mathrm{ad}_{\phi_V}^2 - \dots)$ we note that after differentiating and evaluating at $\phi = 0$ the matrices $\sum \frac{\partial S_{ij}}{\partial \phi_k}$ are equal to $-\frac{1}{2}\,\mathrm{ad}_{e_k}^{\mathsf{T}}$ where e_k is a 6-vector with 1 in the k^{th} entry and zeros elsewhere. Similarly, $\sum \frac{\partial S_{ik}}{\partial \phi_j}$ is equal to $\frac{1}{2}\,\mathrm{ad}_{e_k}^{\mathsf{T}}$. This is then multiplied by the k^{th} element of $M(q)v$

when differentiating with respect to ϕ_k. We then get

$$\begin{aligned} C_{VV}(Q,v) &= \sum_{k=1}^{6} \frac{\partial M_{VV}}{\partial q_k}\dot{q}_k - \frac{1}{2}\,\mathrm{ad}^{\mathsf{T}}_{(M(q)v)_V} - \frac{1}{2}\,\mathrm{ad}^{\mathsf{T}}_{(M(q)v)_V} \\ &= \sum_{k=1}^{6} \frac{\partial M_{VV}}{\partial q_k}\dot{q}_k - \mathrm{ad}^{\mathsf{T}}_{(M(q)v)_V} \end{aligned} \tag{22}$$

where $(M(q)v)_V$ is the vector of the first 6 entries (corresponding to V_{0b}^b) of the vector $M(q)v$.

$C_{Vq}(Q,v)$, i.e., $i = 1\dots6$ and $j = 7\dots(6+n)$, is found in a similar manner. First we note that $\frac{\partial M_{jk}}{\partial \phi_i} = 0$ for $i = 1\dots6$ and that $\frac{\partial S_{ij}}{\partial \phi_k} = 0$ and $\frac{\partial S_{ik}}{\partial \phi_j} = 0$ for $j = 7\dots(6+n)$, so only the first part is non-zero and we get

$$C_{Vq}(Q,v) = \sum_{k=1}^{6} \frac{\partial M_{Vq}}{\partial q_k}\dot{q}_k. \tag{23}$$

Finally, the terms C_{qV} and C_{qq} depend only on the first part of Equation (15) and can be written more explicitly as [13]

$$C_{qV} = \sum_{k=1}^{n} \frac{\partial M_{qV}}{\partial q_k}\dot{q}_k - \frac{1}{2}\frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix}\right), \tag{24}$$

$$C_{qq} = \sum_{k=1}^{n} \frac{\partial M_{qq}}{\partial q_k}\dot{q}_k - \frac{1}{2}\frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq} \end{bmatrix}\begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix}\right). \tag{25}$$

The C-matrix is thus given by

$$C(Q,v) = \sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} 2\,\mathrm{ad}^{\mathsf{T}}_{(M(q)v)_V} & 0 \\ \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix}\right) & \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq} \end{bmatrix}\begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix}\right) \end{bmatrix}. \tag{26}$$

2.4. Vehicles with Configuration Space SO(3)

The dynamics of a vehicle-manipulator system for a vehicle with configuration space $SO(3)$ are derived in the same way. The velocity state is thus fully determined by only three variables and we choose H so that

$$V_{0b}^b = H\tilde{V}_{0b}^b \tag{27}$$

with

$$H = \begin{bmatrix} 0_{3\times3} \\ I_{3\times3} \end{bmatrix}. \tag{28}$$

We then get

$$\tilde{V}_{0b}^b = \left(I - \frac{1}{2}\hat{\phi}_V + \frac{1}{6}\hat{\phi}_V^2 - \dots\right)\dot{\phi}_V. \tag{29}$$

The corresponding matrices S_i can be collected in one block-diagonal matrix S given by

$$S(Q,\phi) = \begin{bmatrix} \left(I - \frac{1}{2}\hat{\phi}_V + \frac{1}{6}\hat{\phi}_V^2 - \dots\right) & 0 \\ 0 & I \end{bmatrix} \in \mathbb{R}^{(3+n)\times(3+n)}. \tag{30}$$

We note that when differentiating with respect to ϕ and substituting $\phi = 0$, once again this simplifies the expression substantially. The precise computational details of the partial derivatives follow the same steps as for the $SE(3)$ case except for C_{VV}. Note that $\frac{\partial M_{ij}}{\partial \phi_k} = 0$ for $k < 4$ and $\frac{\partial S_{ij}}{\partial \phi_k} = 0$ for $i, j, k > 3$. When differentiating and evaluating at $\phi = 0$ the matrices $\sum \frac{\partial S_{ij}}{\partial \phi_k}$ are equal to $\frac{1}{2}\hat{e}_k$ where e_k is a 3-vector with 1 in the k^{th} entry and zeros elsewhere. Similarly, $\sum \frac{\partial S_{ik}}{\partial \phi_j}$ is equal to $-\frac{1}{2}\hat{e}_k$. We then get

$$C_{VV}(Q,v) = \sum_{k=1}^{6} \frac{\partial M_{VV}}{\partial q_k}\dot{q}_k + (\widehat{M(q)v})_{\tilde{V}} \tag{31}$$

where $(M(q)v)_{\tilde{V}}$ is the vector of the first three entries of the vector $M(q)v$ (corresponding to $\tilde{V}^b_{0b}$) and $\hat{p} \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix such that $\hat{p}x = p \times x$ for all $p, x \in \mathbb{R}^3$.

2.5. Summary

Table 2 shows the mapping from local to global velocity coordinates and the corresponding C-matrices for different Lie Groups.

3. AUV-Manipulator Systems

We start by presenting the state of the art dynamic equations of an AUV-manipulator system as it is normally presented in literature. It is well known that these are not valid globally due to the Euler angle singularity that arises when transforming from local to global velocity variables. Next, we show how to re-write the dynamics using the proposed framework in order to avoid the singularities. The dynamic equations have approximately the same complexity and are better suited for simulation and easier to implement. One drawback of the proposed approach is that the matrix $L = \dot{M} - 2C$ is not skew symmetric. This is a desired property in Lyapunov-based controller design but not in model-based controller design or simulation environments, for which computational speed, robustness, and ease are of higher importance.

3.1. State of the Art AUV Dynamics

A wide range of dynamical systems can be described by the Euler-Lagrange equations [32]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x})\right) - \frac{\partial \mathcal{L}}{\partial x}(x,\dot{x}) = \tau \tag{32}$$

Lie Group	S_{VV}	C
$SE(3)$	$I - \frac{1}{2}\,\mathrm{ad}_{\phi_V} + \frac{1}{6}\,\mathrm{ad}^2_{\phi_V} - \dots$	$\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} 2\,\mathrm{ad}^{\mathsf{T}}_{(M(q)v)_V} & 0 \\ A & B \end{bmatrix}$
$X(z)$	$I_{4\times 4}$	$\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix}$
$T(3)$,$SE(2)$	$I_{3\times 3}$	$\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix}$
$SO(3)$	$I - \frac{1}{2}\hat{\phi}_V + \frac{1}{6}\hat{\phi}^2_V - \dots$	$\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} -2(\widehat{M(q)v})_{\tilde{V}} & 0 \\ A & B \end{bmatrix}$
$T(2)$,$C(1)$	$I_{2\times 2}$	$\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix}$
$T(1)$,H,$SO(2)$	$I_{1\times 1}$	$\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix}$
	$A = \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} \tilde{V}^b_{0b} \\ \dot{q} \end{bmatrix}\right)$	$B = \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq} \end{bmatrix}\begin{bmatrix} \tilde{V}^b_{0b} \\ \dot{q} \end{bmatrix}\right)$

Table 2: The Coriolis matrix for different Lie subgroups of $SE(3)$.

where $x \in \mathbb{R}^n$ is a vector of generalized coordinates, $\tau \in \mathbb{R}^n$ are the generalized forces and

$$\mathcal{L}(x, \dot{x}) \,:\, \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} := \mathcal{T}(x, \dot{x}) - \mathcal{V}(x). \tag{33}$$

Here, $\mathcal{T}(x, \dot{x})$ is the kinetic and $\mathcal{V}(x)$ the potential energy function. We assume that the kinetic energy function is positive definite and in the form

$$\mathcal{T}(x, \dot{x}) := \frac{1}{2}\dot{x}^{\mathsf{T}} M(x)\dot{x}. \tag{34}$$

where $M(x)$ is the inertia matrix. For a kinetic energy function on this form we can recast the Euler-Lagrange equations (32) into the equivalent form

$$M_{RB}(x)\ddot{x} + C_{RB}(x, \dot{x})\dot{x} + n(x) = \tau \tag{35}$$

where $C_{RB}(x, \dot{x})$ is the Coriolis and centripetal matrix and $n(x)$ is the potential forces vector defined as

$$n(x) := \frac{\partial \mathcal{V}(x)}{\partial x}. \tag{36}$$

The Coriolis and centripetal matrix is normally obtained by the Christoffel symbols of the first kind as [33]

$$C_{RB}(x, \dot{x}) := \{c_{ij}\} = \left\{ \sum_{k=1}^{n} \Gamma_{ijk}\dot{x}_k \right\}, \tag{37}$$

$$\Gamma_{ijk} := \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial x_k} + \frac{\partial m_{ik}}{\partial x_j} - \frac{\partial m_{kj}}{\partial x_i}\right) \tag{38}$$

where $M(x) = \{m_{ij}\}$. When representing the dynamic equations using generalized coordinates we implicitly introduce non-inertial frames in which we represent the inertial properties of the rigid bodies. The Coriolis matrix arises as a result of these non-inertial frames. We note that there are several ways to define the Coriolis matrix so that $C_{ij}(x, \dot{x})\dot{x}_j = \Gamma_{ijk}\dot{x}_j\dot{x}_k$ is satisfied. Later, we will see that in deriving the dynamics using a different framework we get a different Coriolis matrix with different properties. Normally the terms where $i \neq j$ are identified with the Coriolis forces and $i = j$ with the centrifugal forces.

In addition, for floating or submerged vehicles we need to add the hydrodynamic forces and moments. The damping forces are collected in the damping matrix D and the restoring forces (weight and buoyancy) are normally included in $n(\eta)$. Furthermore, the motion of the AUV will generate a flow in the surrounding fluid. This is known as added mass. For completely submerged vehicles operating at low velocities the added mass is given by a constant matrix $M_A = M_A^\mathsf{T} > 0$. The corresponding Coriolis matrix is given by $C_A = -C_A^\mathsf{T}$ and is found in the same way as C_{RB} by replacing M_{RB} with M_A [34]. We also add environmental disturbances such as currents.

The dynamics of underwater vehicles are usually given as [29]

$$\dot{\eta} = J(\eta)\nu, \tag{39}$$

$$M\dot{\nu} + C(\nu)\nu + D(\nu)\nu + n(\eta) = \tau \tag{40}$$

where $\eta = \begin{bmatrix} x & y & z & \phi & \theta & \psi \end{bmatrix}^\mathsf{T}$ is the position and orientation of the vessel given in the inertial frame and $\nu = \begin{bmatrix} u & v & w & p & q & r \end{bmatrix}^\mathsf{T}$ is the linear and angular velocities given in the body frame. $D(\nu)\nu$ is the damping and friction matrix, $M = M_{RB} + M_A$ and $C(\nu) = C_{RB}(\nu) + C_A(\nu)$.

The velocity transformation matrix $J(\eta)$ in (39) transforms the velocities from the body frame to the inertial frame and is defined as

$$J(\eta) = \begin{bmatrix} R_{0b}(\Theta) & 0 \\ 0 & T_\Theta(\Theta) \end{bmatrix} \tag{41}$$

where $R_{0b}(\Theta)$ is the rotation matrix and depends only on the orientations of the vessel given by the Euler angles $\Theta = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^\mathsf{T}$, represented in the reference frame. $T_\Theta(\Theta)$ is given by (zyx-sequence)

$$T_\Theta(\Theta) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix}. \tag{42}$$

We note that $T_\Theta(\Theta)$, and thus also $J(\eta)$, are not defined for $\theta = \pm\frac{\pi}{2}$. This is the well known Euler angle singularity for the zyx-sequence. The inverse mappings $T_\Theta^{-1}(\Theta)$ and $J^{-1}(\eta)$ are defined for all $\theta \in \mathbb{R}$ but singular for $\theta = \pm\frac{\pi}{2}$.

This singularity can be removed from the operational space by deriving the kinematic equations using two Euler angle representations with different singularities and switching between these two representations. It can also be avoided using the unit quaternion representation, which does not have a singularity at the cost of introducing a fourth parameter to describe the orientation. The unit quaternion representation is computationally challenging when it comes to integration and normalization. Also, in computing the Euler angles from the quaternions the Euler angle singularity is present and precautions against computational errors close to this singularity must be taken.

We note that the representation $\nu = \begin{bmatrix} x & y & z & \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix}^\mathsf{T}$ where $Q = \begin{bmatrix} \eta & \epsilon_1 & \epsilon_2 & \epsilon_3 \end{bmatrix}^\mathsf{T}$ is the unit quaternion cannot be used in the Lagrangian approach since it is defined by 7 parameters. These parameters are hence not generalized coordinates.

We will assume that the ocean current ν_c is expressed in the inertial frame. Then the relative velocity in the body-fixed frame is given by

$$\nu_r = \nu - R_{0b}\nu_c. \tag{43}$$

The effects of the current are then included in the dynamics by using ν_r in the derivation of the Coriolis and centripetal matrices and the damping terms.

The relationship between the wrench acting on the vehicle τ and the control input of the thrusters u_V is highly non-linear. However, it is common to approximate this with a linear relation

$$\tau = Bu_V \tag{44}$$

where $B \in \mathbb{R}^{6\times p_u}$ is a known constant matrix, u_V is the p_u-dimensional vector of control inputs and p_u is the number of thrusters, rudders, sterns, etc.

We can rewrite the dynamics using general coordinates η, eliminating the body frame coordinates ν from the equations. We then get

$$\tilde{M}(\eta)\ddot{\eta} + \tilde{C}(\eta,\dot{\eta})\dot{\eta} + \tilde{D}(\eta,\dot{\eta})\dot{\eta} + \tilde{n}(\eta) = \tilde{\tau} \tag{45}$$

where

$$\tilde{M}(\eta) = J^{-\mathsf{T}}(\eta)MJ^{-1}(\eta), \tag{46}$$

$$\tilde{n}(\eta) = J^{-\mathsf{T}}(\eta)n(\eta), \tag{47}$$

$$\tilde{\tau} = J^{-\mathsf{T}}(\eta)\tau, \tag{48}$$

$$\tilde{D}(\eta,\dot{\eta}) = J^{-\mathsf{T}}(\eta)D(J^{-1}(\eta)\dot{\eta})J^{-1}(\eta), \tag{49}$$

$$\tilde{C}(\eta,\dot{\eta})\dot{\eta} = J^{-\mathsf{T}}(\eta)\left[C(J^{-1}(\eta)\eta) - MJ^{-1}(\eta)\dot{J}(\eta)\right]J^{-1}(\eta). \tag{50}$$

Note that the Equations (45-50) are only valid when $J^{-1}(\eta)$ is non-singular, i.e., for $\theta \neq \pm\frac{\pi}{2}$.

To formulate the Lagrange equations in a body-fixed coordinate frame directly we need to circumvent the fact the $\int_0^t \nu dt$ has no physical meaning. We do this by rewriting the Langrange equations using quasi-coordinates. Write $\nu_1 = \begin{bmatrix} u & v & w \end{bmatrix}^\mathsf{T}$ and $\nu_2 = \begin{bmatrix} p & q & r \end{bmatrix}^\mathsf{T}$ and similarly for τ. Then the dynamics can be written as [35]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{T}}{\partial \nu_1}\right) + \hat{\nu}_2 \frac{\partial \mathcal{T}}{\partial \nu_1} = \tau_1 \tag{51}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{T}}{\partial \nu_2}\right) + \hat{\nu}_2 \frac{\partial \mathcal{T}}{\partial \nu_2} + \hat{\nu}_1 \frac{\partial \mathcal{T}}{\partial \nu_1} = \tau_2. \tag{52}$$

We note that the dynamic equations are independent of the position vector η and the gravitational forces are thus not included in the dynamics. We thus need to augment the equations with (39) to get a complete description of the state space. Once again this introduces a singularity in the equations.

3.2. State of the Art AUV-Manipulator Dynamics

The dynamics of an AUV-manipulator system is given by [10]

$$\dot{\xi} = J(\xi)\zeta, \tag{53}$$

$$M(q)\dot{\zeta} + C(q,\zeta)\zeta + D(q,\zeta)\zeta + n(q, R_{0b}) = \tau \tag{54}$$

where $\xi = \begin{bmatrix} \eta^\mathsf{T} & q^\mathsf{T} \end{bmatrix}^\mathsf{T}$, $\zeta = \begin{bmatrix} \nu^\mathsf{T} & \dot{q}^\mathsf{T} \end{bmatrix}^\mathsf{T}$, $M(q) \in \mathbb{R}^{(6+n)\times(6+n)}$ is the inertia matrix including added mass, $C(q,\zeta) \in \mathbb{R}^{(6+n)\times(6+n)}$ is the Coriolis and centripetal matrix and $D(q,\zeta) \in \mathbb{R}^{(6+n)\times(6+n)}$ is the matrix representing the dissipative forces. τ is the vector of forces and moments working on the mechanism and is given by

$$\tau = \begin{bmatrix} \tau_V \\ \tau_q \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} u \tag{55}$$

where $u = \begin{bmatrix} u_V^\mathsf{T} & u_q^\mathsf{T} \end{bmatrix}^\mathsf{T}$ is the control input. The velocity transformation matrix is given by

$$J(\xi) = \begin{bmatrix} R_{0b}(\Theta) & 0 & 0 \\ 0 & T_\Theta(\Theta) & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{56}$$

3.3. The Proposed Approach

In this section we show how to derive the AUV-manipulator dynamics without the presence of singularities. The inertia matrix of the AUV is derived in two steps. First, M_{RB} is found from (13). Then the added mass $M_A = M_A^\mathsf{T} > 0$ is found from the hydrodynamic properties and we get $M = M_{RB} + M_A$. We can now use M instead of M_{RB} to derive the Coriolis and centripetal matrix [29] which gives us $C = C_{RB} + C_A$. As the configuration space of an AUV can be described by the matrix Lie group $SE(3)$ we get (following the mathematics of Equations (19-25)) the Coriolis matrix

$$C(Q,v) = \sum_{k=1}^{n} \frac{\partial M}{\partial q_k} \dot{q}_k - \frac{1}{2} \begin{bmatrix} 2\,\mathrm{ad}^{\mathsf{T}}_{(M(q)v)_V} & 0 \\ \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix} \right) & \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix} \right) \end{bmatrix}. \tag{57}$$

The dynamic equations can now be written as

$$M(Q)\dot{v} + C(Q,v)v + D(v)v + n(Q) = \tau. \tag{58}$$

Here, $v = \begin{bmatrix} (V_{0b}^b)^{\mathsf{T}} & \dot{q}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ where V_{0b}^b is the velocity state of the AUV and $\dot{q}$ the velocity state of the manipulator, and $Q = \{g_{0b}, q\}$ where $g_{0b} \in SE(3)$ determines the configuration space of the AUV (non-Euclidean) and q the configuration space of the manipulator (Euclidean). We note that the singularity in (53) is eliminated and the state space (Q, v) is valid globally. $D(v)$ and $n(Q)$ are found in the same way as for the conventional approach. Specifically, $n(Q)$ is found by (18). In the following we make some brief remarks on implementing the dynamic equations in a software environment.

3.3.1. Computing the Partial derivatives of $M(q_1, \dots, q_n)$

The partial derivatives of the inertia matrix with respect to $q_1, \dots, q_n$ are computed by

$$\frac{\partial M(q_1,\dots,q_n)}{\partial q_k} = \sum_{i=k}^{n} \left(\begin{bmatrix} I \\ J_i^{\mathsf{T}} \end{bmatrix} \left[\frac{\partial^{\mathsf{T}} \mathrm{Ad}_{g_{ib}}}{\partial q_k} I_i \,\mathrm{Ad}_{g_{ib}} + \mathrm{Ad}^{\mathsf{T}}_{g_{ib}} I_i \frac{\partial \,\mathrm{Ad}_{g_{ib}}}{\partial q_k} \right] \begin{bmatrix} I & J_i \end{bmatrix} \right)$$

$$+ \sum_{i=k+1}^{n} \begin{bmatrix} 0 & \mathrm{Ad}^{\mathsf{T}}_{g_{ib}} I_i \,\mathrm{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \\ \frac{\partial^{\mathsf{T}} J_i}{\partial q_k} \mathrm{Ad}^{\mathsf{T}}_{g_{ib}} I_i \,\mathrm{Ad}_{g_{ib}} & \frac{\partial^{\mathsf{T}} J_i}{\partial q_k} \mathrm{Ad}^{\mathsf{T}}_{g_{ib}} I_i \,\mathrm{Ad}_{g_{ib}} J_i + J_i^{\mathsf{T}} \mathrm{Ad}^{\mathsf{T}}_{g_{ib}} I_i \,\mathrm{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \end{bmatrix}. \tag{59}$$

3.3.2. Computing the Partial derivatives of $\mathrm{Ad}_{g_{ij}}$

The main computational burden is on the computation of the partial derivatives of M with respect to q for which we need the partial derivatives of the adjoint matrices, also with respect to q. To compute these one can use a relatively simple relation. If we express the velocity of joint k as $V_{(k-1)k}^{(k-1)} = X_k \dot{q}_k$ for constant X_k, then the following holds:

Proposition 3.1. *The partial derivatives of the adjoint matrix is given by*

$$\frac{\partial \,\mathrm{Ad}_{g_{ij}}}{\partial q_k} = \begin{cases} \mathrm{Ad}_{g_{i(k-1)}} \mathrm{ad}_{X_k} \mathrm{Ad}_{g_{(k-1)j}} & \text{for } i < k \le j, \\ -\mathrm{Ad}_{g_{i(k-1)}} \mathrm{ad}_{X_k} \mathrm{Ad}_{g_{(k-1)j}} & \text{for } j < k \le i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: To prove this, we start by writing out the spatial velocity of frame Ψ_k with respect to $\Psi_{(k-1)}$ when $i < k \le j$:

$$\hat{X}_k \dot{q}_k = \hat{V}^{(k-1)}_{(k-1)k} = \dot{g}_{(k-1)k} g^{-1}_{(k-1)k} = \frac{\partial g_{(k-1)k}}{\partial q_k} g_{k(k-1)} \dot{q}_k$$

where $\hat{X} := \begin{bmatrix} \hat{X}_\omega & X_v \\ 0 & 0 \end{bmatrix}$. If we compare the first and the last terms, we get

$$\frac{\partial R_{(k-1)k}}{\partial q_k} = \hat{X}_\omega R_{(k-1)k}, \tag{60}$$

$$\frac{\partial p_{(k-1)k}}{\partial q_k} = \hat{X}_\omega p_{(k-1)k} + X_v. \tag{61}$$

We can use this relation in the expression for the partial derivative of $\mathrm{Ad}_{g_{(k-1)k}}$:

$$\begin{aligned} \frac{\partial \mathrm{Ad}_{g_{(k-1)k}}}{\partial q} &= \begin{bmatrix} \frac{\partial R_{(k-1)k}}{\partial q_k} & \frac{\hat{p}_{(k-1)k}}{\partial q_k} R_{(k-1)k} + \hat{p}_{(k-1)k} \frac{\partial R_{(k-1)k}}{\partial q_k} \\ 0 & \frac{\partial R_{(k-1)k}}{\partial q_k} \end{bmatrix} \\ &= \begin{bmatrix} \hat{X}_\omega & \hat{X}_v \\ 0 & \hat{X}_\omega \end{bmatrix} \begin{bmatrix} R_{(k-1)k} & \hat{p}_{(k-1)k} R_{(k-1)k} \\ 0 & R_{(k-1)k} \end{bmatrix} \\ &= \mathrm{ad}_{X_k} \mathrm{Ad}_{g_{(k-1)k}}. \end{aligned} \tag{62}$$

It is now straight forward to show that

$$\begin{aligned} \frac{\partial \mathrm{Ad}_{g_{ij}}}{\partial q_k} &= \mathrm{Ad}_{g_{i(k-1)}} \frac{\partial \mathrm{Ad}_{g_{(k-1)k}}}{\partial q_k} \mathrm{Ad}_{g_{kj}} \\ &= \mathrm{Ad}_{g_{i(k-1)}} \mathrm{ad}_{X_k} \mathrm{Ad}_{g_{(k-1)k}} \mathrm{Ad}_{g_{kj}} \\ &= \mathrm{Ad}_{g_{i(k-1)}} \mathrm{ad}_{X_k} \mathrm{Ad}_{g_{(k-1)j}}. \end{aligned} \tag{63}$$

The proof is similar for $j < k \le i$. The details are found in Appendix A.

3.3.3. Computing the Jacobian and its Partial Derivatives

The Jacobian J_i of link i is given by

$$J_i(q) = \begin{bmatrix} X_1 & \mathrm{Ad}_{g_{b1}} X_2 & \mathrm{Ad}_{g_{b2}} X_3 & \cdots & \mathrm{Ad}_{g_{b(i-1)}} X_i & 0 & \cdots & 0 \end{bmatrix}. \tag{64}$$

When the partial derivatives of the adjoint map are found we can also use these to find the partial derivatives of the Jacobian, i.e.,

$$\frac{\partial J_i}{\partial q_k} = \begin{bmatrix} 0_{(k+1)\times 6} & \frac{\partial \mathrm{Ad}_{g_{bk}}}{\partial q_k} X_{k+1} & \frac{\partial \mathrm{Ad}_{g_{b(k+1)}}}{\partial q_k} X_{k+2} & \cdots & \frac{\partial \mathrm{Ad}_{g_{b(i-1)}}}{\partial q_k} X_5 & 0_{(6-i)\times 6} \end{bmatrix} \tag{65}$$

For the special case when the twist of each joint cannot be represented as a constant vector the computation is somewhat more involved. The proposed framework does, however, allow for joints with non-constant twists. This is shown in Appendix B.

3.3.4. Implementation

We first define the vector

$$(M(q)v)_V = \begin{bmatrix} (M(q)v)_1 \\ (M(q)v)_2 \\ \vdots \\ (M(q)v)_m \end{bmatrix} = \begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix}. \qquad (66)$$

This gives the adjoint part of the second part of (57) as

$$\mathrm{ad}_{(M(q)v)_V} = \qquad (67)$$

$$\begin{bmatrix} 0 & -(M(q)v)_6 & (M(q)v)_5 & 0 & -(M(q)v)_3 & (M(q)v)_2 \\ (M(q)v)_6 & 0 & -(M(q)v)_4 & (M(q)v)_3 & 0 & -(M(q)v)_1 \\ -(M(q)v)_5 & (M(q)v)_4 & 0 & -(M(q)v)_2 & (M(q)v)_1 & 0 \\ 0 & 0 & 0 & 0 & -(M(q)v)_6 & (M(q)v)_5 \\ 0 & 0 & 0 & (M(q)v)_6 & 0 & -(M(q)v)_4 \\ 0 & 0 & 0 & -(M(q)v)_5 & (M(q)v)_4 & 0 \end{bmatrix}.$$

The lower part of the matrix in the second term in (57) is calculated in the following way

$$\frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix}\right) \qquad (68)$$

$$= \begin{bmatrix} \frac{\partial (M(q)v)_1}{\partial q_1} & \frac{\partial (M(q)v)_2}{\partial q_1} & \cdots & \frac{\partial (M(q)v)_6}{\partial q_1} \\ \frac{\partial (M(q)v)_1}{\partial q_2} & \frac{\partial (M(q)v)_2}{\partial q_2} & \cdots & \frac{\partial (M(q)v)_6}{\partial q_2} \\ \vdots & & \ddots & \vdots \\ \frac{\partial (M(q)v)_1}{\partial q_n} & \frac{\partial (M(q)v)_2}{\partial q_n} & \cdots & \frac{\partial (M(q)v)_6}{\partial q_n} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{6+n} \frac{\partial M_{1i}(q)}{\partial q_1} v_i & \sum_{i=1}^{6+n} \frac{\partial M_{2i}(q)}{\partial q_1} v_i & \cdots & \sum_{i=1}^{6+n} \frac{\partial M_{6i}(q)}{\partial q_1} v_i \\ \sum_{i=1}^{6+n} \frac{\partial M_{1i}(q)}{\partial q_2} v_i & \sum_{i=1}^{6+n} \frac{\partial M_{2i}(q)}{\partial q_2} v_i & \cdots & \sum_{i=1}^{6+n} \frac{\partial M_{6i}(q)}{\partial q_2} v_i \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^{6+n} \frac{\partial M_{1i}(q)}{\partial q_n} v_i & \sum_{i=1}^{6+n} \frac{\partial M_{2i}(q)}{\partial q_n} v_i & \cdots & \sum_{i=1}^{6+n} \frac{\partial M_{6i}(q)}{\partial q_n} v_i \end{bmatrix}$$

$$\frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix}\right) \qquad (69)$$

$$= \begin{bmatrix} \sum_{i=1}^{6+n} \frac{\partial M_{(m+1)i}(q)}{\partial q_1} v_i & \sum_{i=1}^{6+n} \frac{\partial M_{(m+2)i}(q)}{\partial q_1} v_i & \cdots & \sum_{i=1}^{6+n} \frac{\partial M_{(m+n)i}(q)}{\partial q_1} v_i \\ \sum_{i=1}^{6+n} \frac{\partial M_{(m+1)i}(q)}{\partial q_2} v_i & \sum_{i=1}^{6+n} \frac{\partial M_{(m+2)i}(q)}{\partial q_2} v_i & \cdots & \sum_{i=1}^{6+n} \frac{\partial M_{(m+n)i}(q)}{\partial q_2} v_i \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^{6+n} \frac{\partial M_{(m+1)i}(q)}{\partial q_n} v_i & \sum_{i=1}^{6+n} \frac{\partial M_{(m+2)i}(q)}{\partial q_n} v_i & \cdots & \sum_{i=1}^{6+n} \frac{\partial M_{(m+n)i}(q)}{\partial q_n} v_i \end{bmatrix}$$

and is thus also given by the partial derivative of the inertia matrix. We thus only need to compute the partial derivative $\frac{\partial M(q)}{\partial q_i}$ once and use the result in the both in the first and second part of (57). This approach can be used to obtain the dynamic equations for an arbitrary n-link mechanism mounted on an AUV.

4. Spacecraft-Manipulator Systems

Spacecraft-manipulator systems are different from conventional earth-based manipulators in that they are placed in a free fall environment and that the base is not fixed (free-floating). In general there are three different cases that must be considered [2]. Firstly, if we have reaction jets available and use these to keep the spacecraft stationary we obtain a fixed spacecraft model which very much resembles the conventional fixed-based model. Secondly, if no actuation is used for the spacecraft we have a free-floating spacecraft with reduced fuel consumption at the expense of dynamic coupling between the spacecraft and the manipulator and a reduced workspace model. Finally, if the attitude, but not the position, of the spacecraft is actively controlled, we have a constrained spacecraft. We note that for free-floating spacecraft the center of mass (CM) of the spacecraft-manipulator system does not accelerate. However, when reaction jets or momentum wheels are used for control or other external forces are present, the center of mass is not constant in the orbit-fixed reference frame. The main challenge in modeling spacecraft-manipulator systems is that the base-fixed coordinate frame cannot simply be fixed in the orbit-fixed frame. There are two main approaches to deal with a floating base; the virtual manipulator approach [36] or the barycentric vector approach [37].

4.1. State of the Art Spacecraft Dynamics

The attitude of a spacecraft is normally described by the Euler parameters, or unit quaternion. This is motivated by their properties as a nonsingular representation. We note that this is not the minimal representation, nor generalized coordinates, and thus not suited for the Lagrangian approach. Also, when transforming back to Euler angles from the unit quaternion representation a singularity is present for $\theta = \pm\frac{\pi}{2}$.

Any positive rotation ψ about a fixed unit vector n can be represented by the four-tuple

$$Q = \begin{bmatrix} \eta \\ \epsilon \end{bmatrix}, \tag{70}$$

where $\eta \in \mathbb{R}$ is known as the scalar part and $\epsilon \in \mathbb{R}^3$ as the vector part. $Q(\psi, n)$ is written in terms of ψ and n by

$$\eta = \cos(\frac{\psi}{2}), \qquad \epsilon = \sin(\frac{\psi}{2})n. \tag{71}$$

The kinematic differential equations can now be given by

$$\dot{\eta} = -\frac{1}{2}\epsilon^{\mathsf{T}}\omega_{0b}^{0} \tag{72}$$

$$\dot{\epsilon} = \frac{1}{2}(\eta I_b + \hat{\epsilon})\omega_{0b}^{0} \tag{73}$$

where ω_{0b}^{0} is the angular velocity of the body frame with respect to the orbit frame and I_b is the spacecraft inertia matrix. The attitude dynamics are given by [3]

$$I_b\dot{\omega}_{0b}^{0} + \hat{\omega}_{0b}^{0} I_b \omega_{0b}^{0} = \tau. \tag{74}$$

4.2. State of the Art Spacecraft-Manipulator Dynamics

The equations of motion of a spacecraft-manipulator system can be written as [1]

$$M(Q)\dot{v} + C(Q, v)v = \tau. \tag{75}$$

Here, $v = \begin{bmatrix}\dot{r}_0^\mathsf{T} & (\omega_{0b}^0)^\mathsf{T} & \dot{q}^\mathsf{T}\end{bmatrix}^\mathsf{T}$ where r_0 is the position of the center of mass of the vehicle, ω_{0b}^0 the angular velocity of the vehicle and q is the joint position of the manipulator.

Alternatively we can use the center of mass of the whole system to represent the translational motion. Then $v = \begin{bmatrix}\dot{r}_{cm}^\mathsf{T} & (\omega_{0b}^0)^\mathsf{T} & \dot{q}^\mathsf{T}\end{bmatrix}^\mathsf{T}$ where $\dot{r}_{cm}$ is the linear velocity of the center of mass of the vehicle-manipulator system. This is decoupled from the angular velocity ω_{0b}^0 and the inertia matrix of a free-flying spacecraft-manipulator system can be written as [2]

$$M = \begin{bmatrix} mI & 0 & 0 \\ 0 & M_{\omega\omega} & M_{q\omega}^\mathsf{T} \\ 0 & M_{q\omega} & M_{qq} \end{bmatrix} \tag{76}$$

where m is the total mass of the system. The Euler angle rates $\dot{\Theta}_{0b}$ relate to ω_{0b}^0 by

$$\dot{\Theta}_{0b} = T_{\Theta_{0b}}(\Theta_{0b})\omega_{0b}^0. \tag{77}$$

Again $T_{\Theta_{0b}}(\Theta_{0b})$ is singular at isolated points. The control torques are given by $\tau = \begin{bmatrix}\tau_v^\mathsf{T} & \tau_\omega^\mathsf{T} & \tau_q^\mathsf{T}\end{bmatrix}^\mathsf{T}$ where τ_v is the spacecraft forces generated by thrusters, τ_ω is the spacecraft moments generated by thrusters, momentum gyros or reaction wheels, and τ_q is the manipulator torques.

Other models are also available depending on the actuators available to control the spacecraft. In the case where $\tau_v, \tau_w \neq 0$ (free-flying space robots) the center of mass of the system is not constant, but described by the variable r_{cm} of Equation (75) if we let $v = \begin{bmatrix}\dot{r}_{cm}^\mathsf{T} & (\omega_{0b}^0)^\mathsf{T} & \dot{q}^\mathsf{T}\end{bmatrix}^\mathsf{T}$. If no external forces act on the system and the spacecraft is not actuated with thrusters, the center of mass does not accelerate, i.e., the system linear momentum is constant and $\dot{r}_{cm} = 0$. This can be used to simplify the equations to an n-dimensional system with inertia matrix $M_r = M_{qq} - M_{q\omega}M_{\omega\omega}^{-1}M_{q\omega}^\mathsf{T}$ and we get the reduced system by eliminating ω [2, 37]

$$M_r(Q)\ddot{q} + C_r(Q, v)\dot{q} = \tau_q. \tag{78}$$

The attitude of the spacecraft is then found from

$$\omega = -M_{\omega\omega}^{-1}M_{q\omega}^\mathsf{T}\dot{q}. \tag{79}$$

The dynamic coupling between the manipulator and the spacecraft complicates the modeling and control of such systems. One way to deal with this is to derive a fixed-based manipulator with the same kinematic and dynamic properties as the free-floating spacecraft-manipulator system. The dynamically equivalent manipulator (DEM) [6, 7] is a fixed-base manipulator with the base fixed in the center of mass of the space manipulator. Here, space manipulator refers to both the satellite and the manipulator. When no external forces are present, the center of mass does not move and the end-effector of this manipulator is thus given in the inertial frame. It can be shown that a given sequence of actuator torques acting on the DEM will produce the same joint trajectories for the space manipulator as for the DEM.

The dynamic equations of the free-floating space manipulator can be derived from from Lagrange's equations. We assume that all the joints are stiff and a free fall environment. The Lagrangian of the space manipulator is then given by the kinetic energy only, i.e.,

$$\mathcal{T} := \sum_{i=b}^{n+1} \left[\frac{1}{2} \dot{\rho}_i^\top m_i \dot{\rho}_i + \frac{1}{2} \omega_i^\top R_{0i} I_i R_{0i}^\top \omega_i \right] \tag{80}$$

for both the spacecraft and the links, which is different from Equation (12) in that the inertia matrix depends on the configuration of both the spacecraft and the joints. m_i is the total mass of link i and ρ_i is the distance from the center of mass of the system to the center of mass of link i.

Similarly, we can define a fixed-based manipulator with a spherical first joint and kinetic energy

$$\mathcal{T}' := \sum_{i=1}^{n+1} \left[\frac{1}{2} v_i^\top m_i' v_i + \frac{1}{2} (\omega_i')^\top R_{0i}' I_i' (R_{0i}')^\top \omega_i' \right] \tag{81}$$

where v_i is the velocity of link i with respect to the base. It can be shown that the kinematic and dynamic parameters of the space manipulator can be mapped to the DEM by [6, 7]

$$\begin{aligned}
m_i' &= m_i \frac{\left(\sum_{k=1}^{n+1} m_k\right)^2}{\sum_{k=1}^{i-1} m_k \sum_{k=1}^{i} m_k}, \ i = 2 \ldots n+1, \\
I_i' &= I_i, \ i = 1 \ldots n+1, \\
W_1 &= \frac{R_1 m_1}{\sum_{k=1}^{n+1} m_k}, \\
W_i &= R_i \left(\frac{\sum_{k=1}^{i} m_k}{\sum_{k=1}^{n+1} m_k} \right) + L_i \left(\frac{\sum_{k=1}^{i-1} m_k}{\sum_{k=1}^{n+1} m_k} \right), \ i = 2 \ldots n+1, \\
l_{c1} &= 0, \\
l_{ci} &= L_i \left(\frac{\sum_{k=1}^{i-1} m_k}{\sum_{k=1}^{n+1} m_k} \right), \ i = 2 \ldots n+1,
\end{aligned} \tag{82}$$

where the vector W_i connecting joint i with joint $i+1$ of the DEM is given by R_i and L_i of the space manipulator where R_i is the vector connecting the center of mass of link i and joint $i+1$ and L_i is the vector connecting joint i with the center of mass of link i. l_{ci} is the vector connecting joint i and the center of mass of joint i in the DEM. We refer to Liang et al. [6] and Parlaktuna and Ozkan [7] for details.

4.3. The Proposed Approach - $SE(3)$

As for the AUV, the configuration space of a spacecraft can be described by the matrix Lie group $SE(3)$ with respect to an orbit-fixed frame. The dynamic equations can be written as

$$\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \\ M_{qV} & M_{qq} \end{bmatrix} \begin{bmatrix} \dot{V}_{0b}^{b} \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} C_{VV} & C_{Vq} \\ C_{qV} & C_{qq} \end{bmatrix} \begin{bmatrix} V_{0b}^{b} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \tau_V \\ \tau_q \end{bmatrix} \tag{83}$$

where

$$C(Q,v) = \sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k - \frac{1}{2}\begin{bmatrix} 2\,\mathrm{ad}_{(M(q)v)_V}^{\mathsf{T}} & 0 \\ \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{VV} & M_{qV}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} V_{0b}^{b} \\ \dot{q} \end{bmatrix}\right) & \frac{\partial^{\mathsf{T}}}{\partial q}\left(\begin{bmatrix} M_{qV} & M_{qq} \end{bmatrix}\begin{bmatrix} V_{0b}^{b} \\ \dot{q} \end{bmatrix}\right) \end{bmatrix}. \tag{84}$$

This can be used both for actuated and unactuated spacecraft.

For free-floating spacecraft we have $\tau_V = 0$ and we can simplify the dynamics substantially by re-writing the mass matrix as

$$M_r = M_{qq} - M_{qV} M_{VV} M_{qV}^{\mathsf{T}}. \tag{85}$$

The Coriolis matrix is then found by

$$C_r(Q,v) = \sum_{k=1}^{n} \frac{\partial M_r}{\partial q_k}\dot{q}_k - \frac{1}{2}\frac{\partial^{\mathsf{T}}}{\partial q}(M_r v) \tag{86}$$

with M_r given as in (85) and the dynamics are described by

$$M_r \ddot{q} + C_r^{\mathsf{T}} \dot{q} = \tau_q. \tag{87}$$

When $\ddot{q}$ and $\dot{q}$ are known, the base velocity vector can be found by

$$M_{VV}\dot{V}_{0b}^{b} + C_{VV}V_{0b}^{b} = -(M_{qV}^{\mathsf{T}}\ddot{q} + C_{Vq}\dot{q}). \tag{88}$$

This can be done either by projecting g_{0b} onto the allowed configuration space $SE(3)$ [38] or by using structure-preserving integration methods [39]. As these equations are based on the singularity-free dynamics (83) these are also singularity-free with the state variables $Q = \{g \in SE(3), q \in \mathbb{R}^n\}$ and $v = \begin{bmatrix} (V_{0b}^{b})^{\mathsf{T}} & \dot{q}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{6+n}$.

4.4. The Proposed Approach: The Dynamically Equivalent Manipulator - $SO(3)$

In this section we reformulate the dynamic equations of a space manipulator and its dynamically equivalent manipulator using the proposed framework. This removes the singularities in the representation, but is otherwise similar. Assume no spacecraft actuation, i.e., $\dot{r}_{cm} = 0$. Then the kinetic energy of link i of the space manipulator is given by

$$
\begin{aligned}
\mathcal{T}_i &= \frac{1}{2} \left(V_{0i}^i\right)^\mathsf{T} I_i V_{0i}^i \\
&= \frac{1}{2} \left((\tilde{V}_{0b}^b)^\mathsf{T} H^\mathsf{T} + \dot{q}^\mathsf{T} J_i(q)^\mathsf{T} \right) \mathrm{Ad}_{g_{ib}}^\mathsf{T} \, I_i \, \mathrm{Ad}_{g_{ib}} \left(H \tilde{V}_{0b}^b + J_i(q)\dot{q} \right) \\
&= \frac{1}{2} \left((\omega_{0b}^0)^\mathsf{T} H^\mathsf{T} + \dot{q}^\mathsf{T} J_i(q)^\mathsf{T} \right) \mathrm{Ad}_{g_{ib}}^\mathsf{T} \, I_i \, \mathrm{Ad}_{g_{ib}} \left(H \omega_{0b}^0 + J_i(q)\dot{q} \right) \\
&= \frac{1}{2} \begin{bmatrix} (\omega_{0b}^0)^\mathsf{T} & \dot{q}^\mathsf{T} \end{bmatrix} M_i(q) \begin{bmatrix} \omega_{0b}^0 \\ \dot{q} \end{bmatrix} = \frac{1}{2} v^\mathsf{T} M_i(q) v
\end{aligned}
\tag{89}
$$

where

$$
M_i(q) := \begin{bmatrix} H^\mathsf{T} \, \mathrm{Ad}_{g_{ib}}^\mathsf{T} \, I_i \, \mathrm{Ad}_{g_{ib}} \, H & H^\mathsf{T} \, \mathrm{Ad}_{g_{ib}}^\mathsf{T} \, I_i \, \mathrm{Ad}_{g_{ib}} \, J_i \\ J_i^\mathsf{T} \, \mathrm{Ad}_{g_{ib}}^\mathsf{T} \, I_i \, \mathrm{Ad}_{g_{ib}} \, H & J_i^\mathsf{T} \, \mathrm{Ad}_{g_{ib}}^\mathsf{T} \, I_i \, \mathrm{Ad}_{g_{ib}} \, J_i \end{bmatrix}
\tag{90}
$$

and the inertia matrix is given by substituting this into (13) and H given as in (28). The configuration space is then given by $Q = \{R_{0b}, q\}$.

Similarly, we can define a fixed-based manipulator with a spherical first joint, also with configuration space $SO(3)$. The corresponding inertia matrices are then given by

$$
M_i'(q) := \begin{bmatrix} H^\mathsf{T} \, \mathrm{Ad}_{g_{ib}'}^\mathsf{T} \, I_i' \, \mathrm{Ad}_{g_{ib}'} \, H & H^\mathsf{T} \, \mathrm{Ad}_{g_{ib}'}^\mathsf{T} \, I_i' \, \mathrm{Ad}_{g_{ib}'} \, J_i' \\ (J_i')^\mathsf{T} \, \mathrm{Ad}_{g_{ib}'}^\mathsf{T} \, I_i' \, \mathrm{Ad}_{g_{ib}'} \, H & (J_i')^\mathsf{T} \, \mathrm{Ad}_{g_{ib}'}^\mathsf{T} \, I_i' \, \mathrm{Ad}_{g_{ib}'} \, J_i' \end{bmatrix}
\tag{91}
$$

where I_i' and the kinematic relations used to compute R_{0i}' and J_i' are found from (82). Thus, we have $\tilde{V}_{0b}^b = \tilde{V}_{0b}'^b$ as required. The spacecraft inertia matrix is given by

$$
I_b = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}
\tag{92}
$$

which also represents the inertial properties of the spherical base link. The Coriolis matrix then becomes (following the mathematics of (27-31))

$$
\begin{aligned}
C'(Q, v) = &\sum_{k=1}^{n} \frac{\partial M'}{\partial q_k} \dot{q}_k \\
&- \frac{1}{2} \begin{bmatrix} -2(\widehat{M'(q)v})_{\tilde{V}} & 0 \\ \frac{\partial^\mathsf{T}}{\partial q} \left(\begin{bmatrix} M_{VV}' & (M')_{qV}^\mathsf{T} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix} \right) & \frac{\partial^\mathsf{T}}{\partial q} \left(\begin{bmatrix} M_{qV}' & (M')_{qq}^\mathsf{T} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix} \right) \end{bmatrix}
\end{aligned}
\tag{93}
$$

where $(M'(q)v)_{\tilde{V}}$ is the vector of the first three entries of the vector $M'(q)v$ (corresponding to $\tilde{V}_{0b}^b = \omega_{0b}^0$). The specific computations of the inertia and Coriolis matrices are performed in the same way as for the AUV (see Section 3.3) except from the partial derivatives of the inertia matrices which now depend on the selection matrix H. This is shown in Section 4.4.1.

The dynamic equations can now be written as

$$M'(q)\dot{v} + C'(Q, v)v = \tau. \tag{94}$$

Here, $v = \begin{bmatrix}(\omega^0_{0b})^\mathsf{T} & \dot{q}^\mathsf{T}\end{bmatrix}^\mathsf{T}$ where ω^0_{0b} is the velocity state of the passive spherical base joint of the DEM (and thus also the spacecraft) and $\dot{q}$ the velocity state of the manipulator of the DEM (and the space manipulator), and $Q = \{R_{0b}, q\}$ where $R_{0b} \in SO(3)$ determines the configuration of the spherical joint/spacecraft and q the configuration of the manipulators of the DEM and space manipulator. We note that the singularity that normally arises when using the Euler angles is eliminated and the state space (Q, v) is valid globally.

Most importantly, we can now use this fixed-base DEM for simulation and control of the space manipulator. Similar to the conventional approach, the DEM described by (94) have the same kinetic and dynamic properties as the space manipulator and if the same actuator torques $\tau(t) = \tau'(t)$, $\forall t$ are applied on both the DEM and the space manipulator, this will produce the same joint trajectory $q(t) = q'(t)$ for $\forall t \in [t_0, \infty]$ if $q(t_0) = q'(t_0)$.

4.4.1. Computing the Partial derivatives of $M(q_1, \ldots, q_n)$

The partial derivatives of the inertia matrix with respect to $q_1, \ldots, q_n$ are computed by

$$\frac{\partial M(q_1, \ldots, q_n)}{\partial q_k} = \sum_{i=k}^{n} \left(\begin{bmatrix} H^\mathsf{T} \\ J_i^\mathsf{T} \end{bmatrix} \left[\frac{\partial^\mathsf{T} \operatorname{Ad}_{g_{ib}}}{\partial q_k} I_i \operatorname{Ad}_{g_{ib}} + \operatorname{Ad}^\mathsf{T}_{g_{ib}} I_i \frac{\partial \operatorname{Ad}_{g_{ib}}}{\partial q_k} \right] \begin{bmatrix} H & J_i \end{bmatrix} \right)$$
$$+ \sum_{i=k+1}^{n} \begin{bmatrix} 0_{m\times m} & H^\mathsf{T} \operatorname{Ad}^\mathsf{T}_{g_{ib}} I_i \operatorname{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \\ \frac{\partial^\mathsf{T} J_i}{\partial q_k} \operatorname{Ad}^\mathsf{T}_{g_{ib}} I_i \operatorname{Ad}_{g_{ib}} H & \frac{\partial^\mathsf{T} J_i}{\partial q_k} \operatorname{Ad}^\mathsf{T}_{g_{ib}} I_i \operatorname{Ad}_{g_{ib}} J_i + J_i^\mathsf{T} \operatorname{Ad}^\mathsf{T}_{g_{ib}} I_i \operatorname{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \end{bmatrix} \tag{95}$$

which only differs from (59) in that the identity matrix I is substituted by H and H^T in the first part and we multiply by H and H^T to get the right dimensions in the second part.

5. Ground Vehicle-Manipulator Systems

We now consider a ground vehicle with no non-holonomic constraints. The configuration space can be described by the matrix Lie group $SE(2)$. The velocity state is thus fully determined by only three variables and we choose H so that

$$V^b_{0b} = H\tilde{V}^b_{0b} \tag{96}$$

with

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\mathsf{T}. \tag{97}$$

For Euclidean joints Equation (19) simplifies to

$$\tilde{V}_{0b}^{b} = \dot{\phi}_V. \tag{98}$$

S is thus given by the identity matrix, the partial derivatives of S vanish and we get

$$C_{VV}(Q, v) = \sum_{k=1}^{6} \frac{\partial M_{VV}}{\partial q_k} \dot{q}_k. \tag{99}$$

The inertia matrix

$$I = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_z \end{bmatrix} \tag{100}$$

then determines the dynamic equations.

If non-holonomic constraints are present, such as for wheeled mechanisms, we get the selection matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} \tag{101}$$

and velocity state $\tilde{V}_{0b}^{b} = \left[\begin{smallmatrix} v_x \\ \omega_z \end{smallmatrix}\right]$. The dynamics are then found by substituting H and $\tilde{V}_{0b}^{b}$ into the formalism presented in Section 2.

6. A Simple Example

Consider the general structure of the equations for a mechanism with one joint with joint variable q_1 mounted on a vehicle with configuration space $SE(3)$. We can write its inertia matrix as follows

$$M(q_1) = \begin{bmatrix} I_b + \mathrm{Ad}_{g_{1b}}^{\mathsf{T}} I_1 \mathrm{Ad}_{g_{1b}} & \mathrm{Ad}_{g_{1b}}^{\mathsf{T}} I_1 \mathrm{Ad}_{g_{1b}} X_1 \\ X_1^{\mathsf{T}} \mathrm{Ad}_{g_{1b}}^{\mathsf{T}} I_1 \mathrm{Ad}_{g_{1b}} & X_1^{\mathsf{T}} \mathrm{Ad}_{g_{1b}}^{\mathsf{T}} I_1 \mathrm{Ad}_{g_{1b}} X_1 \end{bmatrix}. \tag{102}$$

Its partial derivative with respect to q is a single matrix

$$\frac{\partial M(q_1)}{\partial q_1} = \begin{bmatrix} I \\ X_1^{\mathsf{T}} \end{bmatrix} \left[\tfrac{\partial^{\mathsf{T}} \mathrm{Ad}_{g_{1b}}}{\partial q_1} I_1 \mathrm{Ad}_{g_{1b}} + \mathrm{Ad}_{g_{1b}}^{\mathsf{T}} I_1 \tfrac{\partial \mathrm{Ad}_{g_{1b}}}{\partial q_1} \right] \begin{bmatrix} I & X_1 \end{bmatrix} \tag{103}$$

with

$$\frac{\partial g_{1b}}{\partial q_1} = -g_{1b} \hat{X}_1 g_{bb} = -g_{1b} \hat{X}_1. \tag{104}$$

Note that the Jacobian matrix is constant and hence no partial derivatives are taken.

Consider as an example the robot in Figure 2 with a single prismatic joint. We can write the Jacobian as $J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$ and the inertia matrix

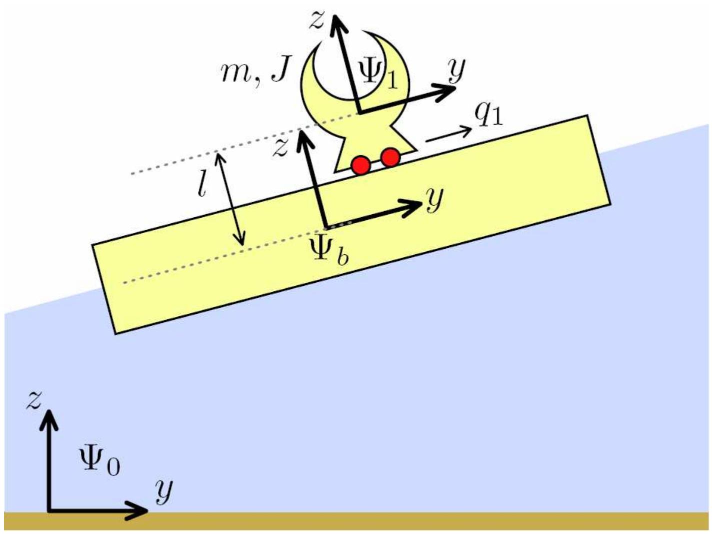

Figure 2: One-link robot with a prismatic joint attached to a non-inertial base with configuration space $SE(3)$.

as

$$M(q) = \begin{bmatrix} M_b & 0 & 0 & 0 & ml & -mq_1 & 0 \\ 0 & M_b & 0 & -ml & 0 & 0 & m \\ 0 & 0 & M_b & mq_1 & 0 & 0 & 0 \\ 0 & -ml & mq_1 & J_{t,x} + ml^2 + mq_1^2 & 0 & 0 & -ml \\ ml & 0 & 0 & 0 & J_{t,y} + ml^2 & -mlq_1 & 0 \\ -mq_1 & 0 & 0 & 0 & -mlq_1 & J_{t,z} + mq_1^2 & 0 \\ 0 & m & 0 & -ml & 0 & 0 & m \end{bmatrix} \tag{105}$$

where $M_b = m_b + m$ and $J_{t,x} = J_{b,x} + J_x$, etc. Assume we are interested in the dynamics of the prismatic joint. This is given by the last row of the inertia and Coriolis matrix. The Coriolis matrix is given by (26) where the first part is zero and the second part gives

$$C(q, V_{0b}^b, \dot{q}) = \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ \frac{m}{2}\omega_z & 0 & -\frac{m}{2}\omega_x & -\frac{m}{2}v_z - mq_1\omega_x & \frac{m}{2}l\omega_z & \frac{m}{2}(v_x + l\omega_y) - mq_1\omega_z & 0 \end{bmatrix}. \tag{106}$$

The last row here is given by multiplying the $\frac{\partial M(q_1)}{\partial q_1} \in \mathbb{R}^{7\times 7}$ with the vector $v = \begin{bmatrix} (V_{0b}^b)^\top & \dot{q}_1 \end{bmatrix}^\top$. Using these expressions, we can write the dynamics of the prismatic joint due to the motion of the vehicle as

$$
\begin{aligned}
\begin{bmatrix} M_{qV} & M_{qq} \end{bmatrix} \begin{bmatrix} \dot{V}_{0b}^b \\ \ddot{q} \end{bmatrix} + \begin{bmatrix} C_{qV} & C_{qq}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} V_{0b}^b \\ \dot{q} \end{bmatrix} &= \tau \\
m\ddot{q}_1 + m\dot{v}_y - ml\dot{\omega}_x + \frac{m}{2}\omega_z v_x - \frac{m}{2}\omega_x v_z & \\
-\frac{m}{2}v_z\omega_x - mq_1\omega_x^2 - \frac{m}{2}(v_x + l\omega_y)\omega_z - mq_1\omega_z^2 &= \tau \qquad (107) \\
\ddot{q}_1 + \dot{v}_y - l\dot{\omega}_x + (v_x + l\omega_y)\omega_z - v_z\omega_x - q_1\omega_x^2 - q_1\omega_z^2 &= \frac{\tau}{m}. \qquad (108)
\end{aligned}
$$

Similarly, if we consider a single rigid body in $SE(3)$ the inertia matrix becomes (dropping the subscript b)

$$
M = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & J_x & 0 & 0 \\ 0 & 0 & 0 & 0 & J_y & 0 \\ 0 & 0 & 0 & 0 & 0 & J_z \end{bmatrix} \tag{109}
$$

and when computing the Coriolis matrix we note that the first part of (26) is zero and the second part is given by $\mathrm{ad}_{(Mv)}^{\mathsf{T}}$ and the Coriolis matrix is thus given by

$$
C(q) = \begin{bmatrix} 0 & -J_z\omega_z & J_y\omega_y & 0 & 0 & 0 \\ J_z\omega_z & 0 & -J_x\omega_x & 0 & 0 & 0 \\ -J_y\omega_y & J_x\omega_x & 0 & 0 & 0 & 0 \\ 0 & -mv_z & mv_y & 0 & -J_z\omega_z & J_y\omega_y \\ mv_z & 0 & -mv_x & J_z\omega_z & 0 & -J_x\omega_x \\ -mv_y & mv_x & 0 & -J_y\omega_y & J_x\omega_x & 0 \end{bmatrix} \tag{110}
$$

which we recognize as Kirchhoff's equations. Kirchhoff's equations are, however, valid for systems with only kinetic energy.

There are many ways for computing the Coriolis matrix for rigid bodies. One commonly found formulation is ship modeling is

$$
C(q) = -\begin{bmatrix} 0 & \widehat{M_{11}\nu_1} + \widehat{M_{12}\nu_2} \\ \widehat{M_{11}\nu_1} + \widehat{M_{12}\nu_2} & \widehat{M_{21}\nu_1} + \widehat{M_{22}\nu_2} \end{bmatrix} \tag{111}
$$

and the dynamics are given by (39) and (40). The expression in (111) can also be reformulated to the form of (110). We note that using this approach we end up with the transformation in (39) which singularity prone.

7. Conclusions

In this paper the dynamic equations of vehicle-manipulator systems are derived based on Lagrange's equations. The main contribution is to close the gap between manipulator dynamics which are normally derived based on the Lagrangian approach and vehicle dynamics which are normally derived using other approaches in order to avoid singularities. The proposed framework allows us to derive the dynamics of vehicles with a Lie group topology using a minimal, singularity-free representation based on Lagrange's equations which naturally extends to include also the manipulator dynamics. The globally valid vehicle-manipulator dynamics are thus derived for the first time using the proposed framework. Several examples of how to derive the dynamics for different vehicles, such as spacecraft, AUVs, and ground vehicles are shown to illustrate the simple analytical form of the final equations.

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Appendix A. Partial Derivatives of Ad_g - By Direct Computation

The partial derivative of $\mathrm{Ad}_{g_{ij}}$ with respect to q_k when $i < k \leq j$ can be written as

$$
\begin{aligned}
&\frac{\partial \mathrm{Ad}_{g_{ij}}}{\partial q_k} = \mathrm{Ad}_{g_{i(k-1)}} \frac{\partial \mathrm{Ad}_{g_{(k-1)k}}}{\partial q_k} \mathrm{Ad}_{g_{kj}} \\
&= \begin{bmatrix} R_{i(k-1)} & \hat{p}_{i(k-1)} R_{i(k-1)} \\ 0 & R_{i(k-1)} \end{bmatrix} \begin{bmatrix} \frac{\partial R_{(k-1)k}}{\partial q_k} & \frac{\hat{p}_{(k-1)k}}{\partial q_k} R_{(k-1)k} + \hat{p}_{(k-1)k} \frac{\partial R_{(k-1)k}}{\partial q_k} \\ 0 & \frac{\partial R_{(k-1)k}}{\partial q_k} \end{bmatrix} \begin{bmatrix} R_{kj} & \hat{p}_{kj} R_{kj} \\ 0 & R_{kj} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} & \begin{bmatrix} R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} \hat{p}_{kj} R_{kj} + R_{i(k-1)} \frac{\hat{p}_{(k-1)k}}{\partial q_k} R_{(k-1)j} + \\ R_{i(k-1)} \hat{p}_{(k-1)k} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} + \hat{p}_{i(k-1)} R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} \end{bmatrix} \\ 0 & R_{i(k-1)} \frac{\partial R_{(k-1)k}}{\partial q_k} R_{kj} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} R_{kj} & \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} \hat{p}_{kj} R_{kj} + R_{i(k-1)} ((\widehat{\hat{X}_\omega p_{(k-1)k}}) + \hat{X}_v) R_{(k-1)j} + \\ R_{i(k-1)} \hat{p}_{(k-1)k} \hat{X}_\omega R_{(k-1)k} R_{kj} + \hat{p}_{i(k-1)} R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} R_{kj} \end{bmatrix} \\ 0 & R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} R_{kj} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} & \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)k} \hat{p}_{kj} R_{kj} + R_{i(k-1)} (\widehat{\hat{X}_\omega p_{(k-1)k}}) R_{(k-1)j} + \\ R_{i(k-1)} \hat{X}_v R_{(k-1)j} + R_{i(k-1)} \hat{p}_{(k-1)k} \hat{X}_\omega R_{(k-1)j} + \hat{p}_{i(k-1)} R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} \end{bmatrix} \\ 0 & R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} & R_{i(k-1)} \hat{X}_v R_{(k-1)j} + R_{i(k-1)} \hat{X}_\omega \hat{p}_{(k-1)j} R_{(k-1)j} + \hat{p}_{i(k-1)} R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} \\ 0 & R_{i(k-1)} \hat{X}_\omega R_{(k-1)j} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} \hat{X}_\omega & R_{i(k-1)} \hat{X}_v + \hat{p}_{i(k-1)} R_{i(k-1)} \hat{X}_\omega \\ 0 & R_{i(k-1)} \hat{X}_\omega \end{bmatrix} \begin{bmatrix} R_{(k-1)j} & \hat{p}_{(k-1)j} R_{(k-1)j} \\ 0 & R_{(k-1)j} \end{bmatrix} \\
&= \begin{bmatrix} R_{i(k-1)} & \hat{p}_{i(k-1)} R_{i(k-1)} \\ 0 & R_{i(k-1)} \end{bmatrix} \begin{bmatrix} \hat{X}_\omega & \hat{X}_v \\ 0 & \hat{X}_\omega \end{bmatrix} \begin{bmatrix} R_{(k-1)j} & \hat{p}_{(k-1)j} R_{(k-1)j} \\ 0 & R_{(k-1)j} \end{bmatrix} \\
&= \mathrm{Ad}_{g_{i(k-1)}} \mathrm{ad}_{X_k} \mathrm{Ad}_{g_{(k-1)j}}
\end{aligned} \tag{A.1}
$$

where we have used that

$$\hat{a}\hat{b} = (\widehat{\hat{a}b}) + \hat{b}\hat{a}, \tag{A.2}$$

and

$$\hat{p}_{(k-1)j} = (\widehat{R_{(k-1)k} p_{kj}}) + \hat{p}_{(k-1)k}. \tag{A.3}$$

The proof when $j < k \leq i$ follows the same approach.

Appendix B. Partial Derivatives of the Mass Matrix for Joints with Non-Constant Twist

For a non-constant twist X_k, we get the following expression for the partial derivatives of the inertia matrix

$$
\begin{aligned}
\frac{\partial M(q_1, \ldots, q_n)}{\partial q_k} &= \sum_{i=k}^{n} \left(\begin{bmatrix} H^\intercal \\ J_i^\intercal \end{bmatrix} \left[\frac{\partial^\intercal \mathrm{Ad}_{g_{ib}}}{\partial q_k} I_i \mathrm{Ad}_{g_{ib}} + \mathrm{Ad}_{g_{ib}}^\intercal I_i \frac{\partial \mathrm{Ad}_{g_{ib}}}{\partial q_k} \right] \begin{bmatrix} H & J_i \end{bmatrix} \right) \\
&+ \sum_{i=k}^{n} \left(\begin{bmatrix} 0_{m \times m} & H^\intercal \mathrm{Ad}_{g_{ib}}^\intercal I_i \mathrm{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \\ \frac{\partial^\intercal J_i}{\partial q_k} \mathrm{Ad}_{g_{ib}}^\intercal I_i \mathrm{Ad}_{g_{ib}} H & \frac{\partial^\intercal J_i}{\partial q_k} \mathrm{Ad}_{g_{ib}}^\intercal I_i \mathrm{Ad}_{g_{ib}} J_i + J_i^\intercal \mathrm{Ad}_{g_{ib}}^\intercal I_i \mathrm{Ad}_{g_{ib}} \frac{\partial J_i}{\partial q_k} \end{bmatrix} \right)
\end{aligned} \tag{B.1}
$$

where the only difference from the constant twist expression (95) is that the summing starts from k and not $k+1$ in the last term and that the partial derivatives of the Jacobian are given by

$$\frac{\partial J_i}{\partial q_k} = \begin{bmatrix} 0 & \underbrace{\mathrm{Ad}_{g_{b(k-1)}} \frac{\partial}{\partial q_k} X_k(q_k)}_{\text{For non-constant twists only}} & \frac{\partial}{\partial q_k}(\mathrm{Ad}_{g_{bk}}) X_{k+1} \cdots & \frac{\partial}{\partial q_k}(\mathrm{Ad}_{g_{b(i-1)}}) X_i(q_i) & 0 \end{bmatrix} \tag{B.2}$$

We still have that $q_k = \bar{q}_k + \phi$ and thus for a constant $\bar{q}_k$ we get $\dot{q}_k = \dot{\phi}_k$ so that the transformation from local to global coordinates for the manipulator is still given by $\dot{q} = S(q, \phi)\dot{\phi}$ with $S(q, \phi) = I$. Thus the expression for the Coriolis matrix does not change.

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