Stabilization of Planar Switched Linear Systems Using Polar Coordinates*

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ABSTRACT

Analysis of stability and stabilizability of switched linear systems is a well-researched topic. This article pursues a polar coordinate approach which offers a convenient framework to analyze second-order continuous time switched linear systems. We elaborate on the analytic utility of polar coordinates and present necessary and sufficient conditions under which a stabilizing switched control law can be constructed. Implications of polar coordinate analysis for switched linear systems include sensitivity analysis of switching control laws and the design of oscillators.

Categories and Subject Descriptors

B.1.2 [**Control Structure Performance Analysis and Design Aids**]: Formal Models

General Terms

Theory

Keywords

Switched linear systems, stabilizability

1. INTRODUCTION

By a switched linear system, we refer to a system $\dot{x} = A_q x$, $x \in \mathbb{R}^n$, $A \in \{A_1, \ldots, A_k\} \subset \mathbb{R}^{n \times n}$, where a switching signal, $q = \{1, 2, \ldots, k\}$, alters the continuous dynamics to effect control over the continuous system state.

In practical applications, when presented with a collection of possible linear dynamical systems, one is often faced with the task of designing a switching controller that will stabilize a trajectory of the system to equilibrium. This is the problem of stabilizability, which has been explored widely in the literature [2, 5, 8, 10, 11, 13, 14, 21, 22, 23, 24].

In this paper, we define a switched linear systems as a hybrid system in which the continuous dynamics are linear and second-order, and the domains are finite unions of conic subsets of $\mathbb{R}^2$. Transitions between discrete states are forced when the continuous state reaches one of the linear guards. We study the problems of stability and stabilizability of the equilibria of such systems with a polar coordinate transformation of the continuous state variables. The use of polar coordinates allows us to present, for a restricted set of switched linear systems, necessary and sufficient conditions for global exponential stability of the equilibrium. We further leverage polar coordinates to present necessary and sufficient conditions for a collection of unstable linear systems to be stabilized by a particular form of switching. We illustrate the application of our approach with an example motivated by Branicky [1].

1.1 Related Works

The issues addressed in this paper have been explored before; the contribution of this paper stems from the application of the polar coordinate transformation to an understanding of the problems of stability and stabilizability of switched linear systems. While uncommon in linear system analysis, the use of the polar coordinate transformation is not without precedent. Huang et. al. [9] leveraged the polar coordinate transformation to derive conditions for stability of equilibria in switched linear systems under arbitrary switching signals. In the process, they developed foundational techniques for analyzing stability of equilibria in switched linear systems. In this paper, we make explicit the ability of the polar coordinate transformation of the continuous dynamics to determine the net change in the radial component, r, with respect to a given change in angular component, θ. We extend the results of Huang et. al. and derive conditions under which a switched linear system with both stable and unstable subsystem equilibria will have a stable equilibrium. In addition, we address the problem of stabilizability, which is unaddressed in [9].

There is an abundance of work on the problems of stability and stabilizability of switched linear systems in the Hybrid Systems literature. In the late 1990's, Liberzon and Morse [12] presented a succinct description of three core

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problems: 1) finding conditions to guarantee stability for arbitrary switching, 2) identifying which switching signals stabilize a given system, and 3) constructing a stabilizing switching law for a given system. Other authors (e.g. [5, 14, 16]) separate the problem into, roughly, 1) analyzing stability for arbitrary or particular switching signals, and 2) analyzing the existence of, and constructing, stabilizing switching laws, which we refer to as the problems of 1) stability, and 2) stabilizability.

Stability under arbitrary switching, the problem considered by Huang et. al. [9], is useful in many contexts, when switching signals are not known a priori, and are possibly reactive to environmental stimuli [4]. In this context, a number of interesting algebraic conditions have been derived, summarized in [16].

When stability under arbitrary switching cannot be guaranteed (e.g. all subsystems are unstable), it is still possible that there exist switching control laws that will stabilize the system. The problem of stabilizability of stable and unstable linear subsystems has been known to be a challenging problem, and has yielded very slowly to results. It wasn't until recently that some necessary and sufficient conditions for stabilizability were introduced [15]. The majority of authors studying stabilizability of switched linear systems apply Lyapunov theory, requiring common or multiple Lyapunov functions [1, 3, 4, 5, 7, 11, 15, 17, 21, 23]. A review of these approaches can be found in the review paper by Shorten et. al. [19] or by Decarlo et. al. [5]. The most recent results on the problems of stability and stabilizability of switched linear systems can be found in review papers [16, 20]. Other work has been invested in efficient methods of finding Lyapunov functions, which apply Linear Matrix Inequalities [3, 19], which may be solved efficiently.

Part of the difficulty of studying switched systems is that, even when all subsystems are stable, particular switching signals can destabilize the system. Such pernicious effects of switching are summarized well by Decarlo et. al. [5], building off of earlier examples by Branicky [1]. Such examples illustrate that the stabilizing or destabilizing effects of switching are not intuitive and that they have some dependence on the construction of the switching surfaces, or equivalently, the domains and guards of the hybrid system.

Here, we depart from the major vein of the literature and seek conditions for stability and stabilizability of switched linear systems without applying Lyapunov theory. Instead, we focus on conditions derived directly from an analysis of system trajectories in polar coordinates to deal more simply with the effects of switching.

1.2 Paper Organization

We apply a polar coordinate transformation to the continuous dynamics of a switched linear system with state-dependent switching to approach stability analysis and the problem of stabilizability. In Section 2, we introduce model of switched linear systems as a special, restricted class of hybrid systems. We proceed with a polar coordinate transformation and leverage this transformation to explore a few types of trajectories of switched linear systems which may converge to the origin. We culminate this section with a derivation of a number related to system trajectories, called the stability exponent, which we use to generate conditions for stability of the equilibrium in switched linear systems. We further utilize the stability exponent in Section 3 to design the domains, guards, and edges of a particular, restricted hybrid systems model of a switched linear system to stabilize the equilibrium. Finally, in Section 4, we present an application of the stability exponent to a study of the sensitivity of switched linear systems to perturbations in switching surfaces. An example motivated by Branicky [1] illustrates the application of the material presented.

2. PRELIMINARIES

In this section, we define switched linear systems with state-dependent switching in the context of hybrid systems. Then, we explore some important properties of the polar coordinate transformation for the analysis of switched linear systems.

2.1 Hybrid System Definition

This paper focuses on hybrid systems with second-order linear dynamics, and with state-dependent switching laws. These systems are described as follows, and will be referred to throughout the paper as $\mathcal{H}$:

$$\mathcal{H} = (\mathcal{X}, \mathcal{Q}, f, \text{Init}, \mathcal{R}) \tag{1}$$

where

$$\begin{aligned} \mathcal{X} &= \mathbb{R}_+ \times [0, 2\pi) \\ \mathcal{Q} &= \{1, 2, \ldots, k\} \subset \mathbb{N} \\ f &: \mathcal{X} \times \mathcal{Q} \to \mathbb{R}^2 \\ \text{Init} &\subset \{(x, q) \in \mathcal{X} \times \mathcal{Q}\} \\ \mathcal{R} &: \mathcal{X} \times \mathcal{Q} \to 2^{\mathcal{X} \times \mathcal{Q}} \end{aligned}$$

The continuous state is $x \in \mathcal{X} \subset \mathbb{R}^2$. In Section 2.2, we will apply a polar coordinate transformation, so the continuous state may equivalently be represented as $(r, \theta) \in \mathbb{R}_+ \times [0, 2\pi)$, with the understanding that $\theta = 0$ is equivalent to $\theta = 2\pi$. The discrete state is represented by $q \in \mathcal{Q}$. There are $k \in \mathbb{N}$ discrete states of the system. The continuous dynamics of the system are represented by the map f which is of the form $\dot{x} = A_q x$ (which we refer to as a linear subsystem of $\mathcal{H}$), where the discrete state q indexes the particular 2x2 matrix which governs the continuous linear dynamics. It is assumed that $A_q \in \mathcal{A} = \{A_1, A_2, \ldots, A_k\} \subset \mathbb{R}^{2\times 2}$.

The switching laws are determined by the map $\mathcal{R}$, the reset relation. For this model, the reset relation takes on a restricted form, and is described in terms of domains, edges, and guards. The reset relation preserves continuity of the continuous state but allows switching of the discrete state. The domains are constructed with finite unions of non-empty convex conic sets. Consider the partitioning of the plane into N sets as follows:

$$d_j = \left\{(r, \theta) \mid \theta \in \left[\overline{\theta}_{j-1}, \overline{\theta}_j\right] \cup \left[\pi + \overline{\theta}_{j-1}, \pi + \overline{\theta}_j\right]\right\}$$

for $j = 1, 2, \ldots, N$, and $\overline{\theta}_j \in [0, \pi)$, $\overline{\theta}_N = \overline{\theta}_0 + \pi$, and $\overline{\theta}_0 < \overline{\theta}_1 < \ldots < \overline{\theta}_N$. These sets look like those in Figure 1; they are each the union of two convex conic sets. We refer to these sets as fundamental domain sets of the hybrid system $\mathcal{H}$.

The domain, $\mathcal{D}_q$, associated with discrete state $q \in \mathcal{Q}$, is defined as some finite union of sets d_j as follows:

$$\mathcal{D}_q = \bigcup_{i \in i_q} d_i$$

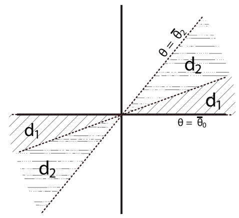


Figure 1: Illustration of sets d_j that comprise discrete domains

where $i_q \subset \{1, 2, \ldots, N\} \subset \mathbb{N}$ lists the indices of the fundamental domain sets which comprise the domain of discrete state q. We require that $\bigcup_{j \in \{1,2,\ldots,k\}} \mathcal{D}_j = \mathbb{R}^2$, so a trajectory of the hybrid system will exist for any initial condition on the continuous state. The initialization set is Init $:= \{(x, q) \mid x \in \mathcal{D}_q\}$.

The edges between discrete states are directional and define possible discrete state transitions. The reset maps maintain continuity of the continuous state trajectory of the system and are deterministic, so the edges are configured to allow transitions only between discrete states with adjacent domains. For these systems, an edge may exist between discrete states q_i and q_j if $\mathcal{D}_i \bigcap \mathcal{D}_j \neq \{0\}$, the set containing only the origin. In the case that $k > 1$, each discrete state must have at least one inbound edge and one outbound edge so there always exists a trajectory of the system $\mathcal{H}$ for any initial condition $(x_0, q_0) \in$ Init.

Linear switching surfaces are described as guards of the form $\{(r, \theta) \mid \theta = \overline{\theta}\} \in \mathcal{D}_i \bigcap \mathcal{D}_j$, for $i \neq j$, and force discrete state transitions when reached. Thus, discrete state transitions occur only on the boundaries of the convex conic sets d_i, $i = 1, \ldots, N$, which are lines through the origin. We require that the system be deterministic, so no two outbound edges from a given discrete state may share a guard.

Note that a controller generating the discrete state switching signals would be a state-feedback controller. Similar state-dependent switching has been thoroughly investigated in the literature [1, 6, 8, 11, 15, 17, 18, 21, 22, 24]. In this definition, there are a fixed, finite set of switching surfaces. This limitation is motivated by real-world switched systems, in which there are limits on the switching capabilities of the system. Since physical systems cannot switch infinitely fast and infinitely often, there is an upper limit on the number of switches that are possible. Also, note that there are no continuous control inputs to the system. Control of these systems is achieved only by the design of switching laws.

Finally, the possibility of sliding modes complicates matters. Following other works, such as [1, 15], and to simplify the development, we avoid sliding modes, and require that only a finite number of discrete state switches may occur in finite time. Fortunately, one additional restriction on the form of the reset maps eliminates all possible sliding modes. Since this condition is more easily discussed after the polar coordinate transformation, it will be introduced in Section 2.3.

2.2 The Polar Coordinate Transformation

While used infrequently in linear systems theory, there is a precedent to the use of polar coordinates. The core development of this section mirrors that of Huang et. al. [9]. We repeat the derivation of the polar coordinate representation of the dynamics of $\mathcal{H}$ for clarity, due to differences in notation and to differences in the eventual application of the material.

For hybrid systems with linear continuous dynamics, the polar transformation is convenient, due to radial symmetry of the linear subsystems: for $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and $\alpha \in \mathbb{R}$, $\dot{x} = A(\alpha x) = \alpha A x$. That is, the vector field of the continuous dynamics at any point on a line through the origin is a scaled copy of the vector field at any other point on the line.

Consider $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, for $x_i(t) \in \mathbb{R}$, $i = 1, 2$. For ease of notation, let x denote $x(t)$. Define the polar transformation map $T(r, \theta) : (\mathbb{R}_+ \times [0, 2\pi)) \to \mathbb{R}^2$ as

$$\begin{aligned} x_1 &= r \cos\theta \\ x_2 &= r \sin\theta \end{aligned}$$

where $\mathbb{R}_+$ is the set of nonnegative real numbers, and $[0, 2\pi) \subset \mathbb{R}$. Thus, applying the transformation to the dynamics of x, $\dot{x} = A_q x$ yields the dynamics for r and θ as follows:

$$\begin{bmatrix} \dot{r}\cos\theta - r\sin\theta\dot{\theta} \\ \dot{r}\sin\theta + r\cos\theta\dot{\theta} \end{bmatrix} = r A_q \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{2}$$

Let $\omega(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}^T$ and $\gamma(\theta) = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix}^T$. Note that $\omega(\theta)$ is orthogonal to $\gamma(\theta)$ for all θ, and that each vector is unit length. Multiplying Equation (2) on the left by $\omega(\theta)^T$ yields:

$$\frac{1}{r}\dot{r} = \omega(\theta)^T A_q \omega(\theta) \tag{3}$$

Equation (3) yields the proportional dynamics of the state variable r, where the proportional dynamics of a continuous state $x \in \mathbb{R}^n$ are given by $\frac{\dot{x}(t)}{||x(t)||_2}$.

Multiplying Equation (2) on the left by $\gamma(\theta)^T$ yields:

$$\dot{\theta} = \gamma(\theta)^T A_q \omega(\theta) \tag{4}$$

Note that $\dot{\theta}$ is a function only of θ. Also, note that the proportional dynamics of r depend only on θ, or equivalently, the proportional dynamics of r are constant along any given line through the origin. This is a direct consequence of the fact that $A(\alpha x) = \alpha A x$: along a given line through the origin, the vector field points in the same direction, but the magnitude of the vector field scales with the distance from the origin.

For a given discrete state q, let $\alpha_q(\theta) := \frac{1}{r}\frac{\dot{r}}{\dot{\theta}} \equiv \frac{\omega(\theta)^T A_q \omega(\theta)}{\gamma(\theta)^T A_q \omega(\theta)}$, which is well-defined only when $\gamma(\theta)^T A_q \omega(\theta) \neq 0$. Note that $\gamma(\theta)^T A_q \omega(\theta) = 0$ if and only if $\omega(\theta)$ is an eigenvector of A_q.

Now, note that the proportional dynamics of $r(t)$ are implicitly related to time, through $\theta(t)$, which has dynamics independent from $r(t)$. We seek an expression for $r(\theta)$, which will enable stability analysis for trajectories of the system without explicitly considering time. Consider a trajectory of $\mathcal{H}$ with initial condition $(r_0, \overline{\theta}_0, q_0) \in$ Init, assuming that $\dot{\theta}(\overline{\theta}_0) \neq 0$, so $\alpha_{q_0}(\overline{\theta}_0)$ is well-defined. The discrete state transitions are labeled so the transitions occur in sequential order. That is, the system is initialized in q_0, and proceeds to q_1, q_2, and eventually to q_R. The discrete state transition between q_i and q_{i+1}, $i = 0, \ldots, R-1$, is driven by the guard $\theta = \overline{\theta}_{i+1}$. After some time, the continuous state reaches $\theta = \overline{\theta}_{R+1}$, in the domain of discrete state q_R.

Proceeding informally, we assume that $\dot{\theta} \neq 0$ for all time and that $\frac{dr/dt}{d\theta/dt} = \frac{dr}{d\theta}$. Thus, we have $\frac{1}{r}\frac{dr}{d\theta} = \alpha_q(\theta)$, for given θ and q. We then multiply both sides by $d\theta$ and integrate, yielding $\int_{r(\overline{\theta}_0)}^{r(\overline{\theta}_{R+1})} \frac{dr(\theta)}{r(\theta)} = \sum_{j=0}^{R} \int_{\overline{\theta}_j}^{\overline{\theta}_{j+1}} \alpha_{q_j}(\theta) d\theta$. Evaluating the integral on the left-hand side and solving for $r(\overline{\theta}_{R+1})$ yields

$$r(\overline{\theta}_{R+1}) = r(\overline{\theta}_0) \exp\left(\Gamma(\overline{\theta}_0, \overline{\theta}_{R+1}, \mathbf{q})\right) \quad (5)$$

where $\Gamma(\overline{\theta}_0, \overline{\theta}_{R+1}, \mathbf{q}) = \sum_{j=0}^{R} \int_{\overline{\theta}_j}^{\overline{\theta}_{j+1}} \alpha_{q_j}(\theta) d\theta$ is called the stability exponent of a given trajectory of $\mathcal{H}$ with corresponding ordered sequence of discrete states $\mathbf{q} = (q_0, \ldots, q_R)$. This equation is central to the study of stability of equilibria of switched linear systems, and is very similar in form to Equation (8) presented by Huang et. al. [9].

Note that for the stability exponent to be well-defined, we require that $\theta = \overline{\theta}_{R+1}$ is reachable from the initial condition $(r_0, \overline{\theta}_0, q_0)$ through the sequence of discrete states $\mathbf{q}$. Figure 2 illustrates a situation in which a given $\Gamma(\overline{\theta}_0, \overline{\theta}_{R+1}, \mathbf{q})$ is not well-defined, when the continuous state approaches an eigenvector of one of the linear subsystems.

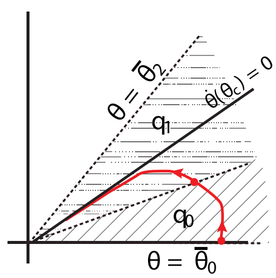


Figure 2: Initial condition: $(r_0, \overline{\theta}_0, q_0)$. **As** $t \to \infty$, $\theta \to \theta_c$, **as** $\gamma(\theta_c)A_1\omega(\theta_c) = 0$. $\Gamma(\overline{\theta}_0, \overline{\theta}_2, (q_0, q_1))$ **is not well-defined, as** $\theta > \theta_c$ **is not reachable.**

Proposition 1. *Given a initial angle* θ_i*, final angle* θ_f*, and a corresponding sequence of discrete states* $\boldsymbol{q}$*,* $r(\theta_f) < r(\theta_i) \Leftrightarrow \Gamma(\theta_i, \theta_f, \boldsymbol{q}) < 0$*. Thus, the sign of the stability exponent is enough information to determine convergence or divergence of trajectories with respect to the origin over a given interval in* θ *and corresponding sequence of discrete states* $\boldsymbol{q}$*.*

Proof. Assume there is some trajectory of $\mathcal{H}$ initialized at θ_i, which reaches θ_f in finite time, and has corresponding sequence of discrete states $\mathbf{q}$. By Equation (5), $r(\overline{\theta}_{R+1}) = r(\overline{\theta}_0)\exp(\Gamma(\theta_i, \theta_f, \mathbf{q}))$. Thus, $r(\overline{\theta}_{R+1}) < r(\overline{\theta}_0)$ if and only if $\exp(\Gamma(\theta_i, \theta_f, \mathbf{q})) < 1$, or equivalently, $\Gamma(\theta_i, \theta_f, \mathbf{q}) < 0$. □

2.3 Describing the behavior of $\mathcal{H}$

With the polar coordinate transform introduced, we now present four additional conditions on $\mathcal{H}$ to avoid sliding modes and to limit the behavior of trajectories of the system to three simple cases. To avoid sliding modes, we require that, after a discrete state switch, the continuous state proceeds to the interior of a fundamental domain set. For this, we require two conditions. The first requires the dynamics of θ to be non-zero on switching surfaces. The second requires that the sign of the dynamics of θ cannot change sign between two adjacent fundamental domains sets. For some fundamental domain set d_j, with boundaries $\theta = \overline{\theta}_j$ and $\theta = \overline{\theta}_{j+1}$,

$$d_j \bigcap \mathcal{D}_u = d_j \Rightarrow \gamma(\overline{\theta}_i)^T A_u \omega(\overline{\theta}_i) \neq 0 \text{ for } i = (j, j+1) \quad (6)$$

$$d_j \bigcap \mathcal{D}_u \bigcap \mathcal{D}_v = \{(r,\theta) \mid \theta = \overline{\theta}\} \Rightarrow$$
$$sgn(\gamma(\overline{\theta})^T A_u \omega(\overline{\theta})) \neq sgn(\gamma(\overline{\theta}) A_v \omega(\overline{\theta})) \quad (7)$$

where $sgn(\cdot)$ is the signum function, which returns -1 for a negative argument, 0 for an argument of 0, and 1 for a positive argument.

Next, we require that the interiors of the domains of any three discrete states intersect only at the origin, or equivalently, that at most two discrete states may incorporate a given fundamental domain set, d_j, in their domains.

$$u, v, w \in \mathcal{Q}, u \neq v \neq w \Rightarrow \mathcal{D}_u^0 \bigcap \mathcal{D}_v^0 \bigcap \mathcal{D}_w^0 = \{0\} \quad (8)$$

In addition, we require that discrete states with domains that overlap in the interior of a fundamental domain set have continuous dynamics that circulate in opposing directions ($\dot{\theta} > 0$ is counter-clockwise, and $\dot{\theta} < 0$ is clockwise):

$$d_j \bigcap \mathcal{D}_u \bigcap \mathcal{D}_v = d_j \Rightarrow$$
$$sgn(\gamma(\overline{\theta}_i)^T A_u \omega(\overline{\theta}_i)) \neq sgn(\gamma(\overline{\theta}_i)^T A_v \omega(\overline{\theta}_i)) \quad (9)$$

for $i = (j, j+1)$.

Theorem 2. *For* $\mathcal{H}$ *with additional assumptions 6,7,8,9,* $\exists T > 0$ *such that, after* $t = T$*, the trajectory may be*

- *approaching an eigenvector. Assume that, at* $t = T$*, the discrete state* $q = u$ *and* $(r, \theta) \in d_j \subset \mathcal{D}_u$*, then* $q = u\ \forall t > T$*. Further,* $\exists \theta_c$ *such that* $\{(r,\theta) \mid \theta = \theta_c\} \in d_j^0$ *(the interior of* d_j*) and* $\theta \to \theta_c$ *as* $t \to \infty$*. Any eigenvector of a subsystem that a trajectory of* $\mathcal{H}$ *may approach is called an eigenvector of* $\mathcal{H}$*.*

- *cyclic. Assume that, at* $t = T$*,* $\exists \tau > 0$ *such that* $\forall t > T$*,* $\forall m \in \mathbb{N}$*,* $\theta(t) = \theta(t + m\tau)$*, and for all* $\vartheta \in [0, 2\pi)$*,* $\exists \tau' < \tau$ *such that* $\theta(t + \tau') = \vartheta$*.*

- *conic. Assume that at* $t = T$*,* $\theta(T) = \overline{\theta}_j$*, on the boundary of* d_j*.* $\exists i \neq j$ *such that* $\overline{\theta}_i$ *is on the boundary of* d_i *and for all* $t > T$*,* $\theta(t) \in [\overline{\theta}_i, \overline{\theta}_j]$*. In addition,* $\exists \tau, \tau' > 0$ *such that for all* $t > T$*, for all* $m \in \mathbb{N}$*,* $\theta(T + m\tau) = \overline{\theta}_j$ *and* $\theta(T + m\tau + \tau') = \overline{\theta}_i$*.*

Figure 3 illustrates an example conic trajectory. A cyclic trajectory is analogous to trajectories of planar linear time-invariant systems with complex eigenvalues.

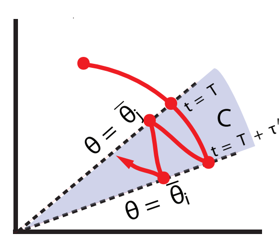


Figure 3: A sample "conic" trajectory: The direction of circulation of the continuous state switches only at the boundaries, $\theta = \overline{\theta}_i$ **and** $\theta = \overline{\theta}_j$**.**

PROOF. First, for $j \in \{1, \ldots, N\}$, consider an initial condition at $t = T$, $(r_0, \theta_0) \in d_j^0$, and q_0 where $d_j = \{(r, \theta) \mid \theta \in \vartheta_j = [\overline{\theta}_j, \overline{\theta}_{j+1}]\} \subset \mathcal{D}_{q_0}$, where $\overline{\theta}_j < \overline{\theta}_{j+1}$. Suppose $\exists \theta_c \in \vartheta_j^0$ such that $\gamma(\theta_c)^T A_{q_0} \omega(\theta_c) = 0$, implying that $\omega(\theta_c)$ is an eigenvector of A_{q_0}. If $\theta_0 = \theta_c$, then $\theta(t) \to \theta_c$ as $t \to \infty$, since $\theta(t) = \theta_c$ for all time. So, the trajectory approaches an eigenvector, and no discrete-state switches occur. Suppose instead, w.l.o.g. that $\theta_0 < \theta_c$ and that $\gamma(\theta_0)^T A_{q_0} \omega(\theta_0) > 0$. Since $\gamma(\theta)^T A_{q_0} \omega(\theta)$ is a continuous function of θ, and trajectories of linear systems are unique, $\nexists t > T$ such that $\theta = \theta_c$. Because $\forall t > T$, $\dot{\theta}(t) > 0$, as $t \to \infty$, $\theta(t) \to \theta_c$, and therefore the trajectory approaches an eigenvector of A_{q_0}. Since the line $\theta = \theta_c$ is in the interior of d_j, and switches of $\mathcal{H}$ only occur on the boundaries, no discrete state switches occur for all time. Next, note that there may be two eigenvectors of A_{q_0}, so there may be $\theta_{c_1}, \theta_{c_2} \in \vartheta_j^0$ such that for $i = 1, 2$, $\gamma(\theta_{c_i})^T A_{q_0} \omega(\theta_{c_i}) = 0$. The remaining case for trajectories starting in the interior of d_j is if we assume, w.l.o.g., that $\theta_{c_2} \geq \theta_{c_1} > \theta_0$ and that $\gamma(\theta_0)^T A_{q_0} \omega(\theta_0) < 0$. Now, $\forall \theta \in [\theta_0, \overline{\theta}_j]$, $\gamma(\theta)^T A_{q_0} \omega(\theta) < 0$, so in finite time, $\theta = \overline{\theta}_j$, and a discrete-state switch is triggered by a guard.

Assume that the discrete state switch occurs at time $t = \tau$. Now, we may apply Conditions (6) and (7) to ensure that $\exists \epsilon > 0$ such that $\forall 0 < \delta < \epsilon$, $\theta(\tau + \delta) \in \vartheta_j^0$ (if $\dot{\theta}$ changed sign) or $\theta(\tau + \delta) \in \vartheta_{j-1}^0$ (otherwise). Once in the interior of a fundamental domain set, by the argument above, the continuous trajectory either approaches an eigenvector, or reaches a boundary of either d_j or d_{j-1} in finite time.

Now, consider an initial condition at time $t = T$ such that $\theta_0 = \overline{\theta}_j$. For the remainder of the proof, assume that, $\forall t > T$, $\nexists \theta_c$ such that $\theta \to \theta_c$ as $t \to \infty$. Then, in some finite time τ, assume w.l.o.g. that $\theta(T + \tau) = \overline{\theta}_{j+1}$. Note that the time τ (called the traversal time) is constant for all initial r_0 on the line $\theta = \overline{\theta}_j$.

Then, assume that after time $t = T$, every discrete state transition preserves the sign of $\dot{\theta}$, and that $\theta(T) = \overline{\theta}_j$, on the boundary of d_j, and the discrete state is q_0. Thus, for $c > 0$ and $c' < 0$ $\forall t > T$, either $\dot{\theta}(t) > c$ or $\dot{\theta}(t) < c'$. Then, in some finite time $\tau > 0$, $\theta(T + \tau) = \theta(T)$. In this finite time, the discrete state follows some sequence $\mathbf{q} = (q_0, q_1, \ldots, q_N)$. By Conditions (8) and (9), we guarantee that at time $T + \tau$, the discrete state is once again q_0, after the continuous state has traversed $\vartheta \in [0, 2\pi)$. Further, since the traversal time is independent of r, in time τ the discrete state follows sequence $\mathbf{q}$ and the continuous state returns once again to the line $\theta = \overline{\theta}_j$. Thus, $\exists \tau > 0$ such that $\forall t > T$, $\forall m \in \mathbb{N}$ $\theta(t) = \theta(t + m\tau)$, so the trajectory is cyclic.

Finally, assume $\nexists t > 0$ such that every discrete state transition preserves the sign of $\dot{\theta}(t)$. Assume that at some $t = T > 0$, $\theta(T) = \overline{\theta}_j$, at the boundary of d_j. Assume $\exists \epsilon, \epsilon' > 0$ such that $\forall 0 < \delta < \epsilon$ and $\forall 0 < \delta' < \epsilon'$, $\theta(T - \delta) \in \vartheta_j^0$ and $\theta(T + \delta') \in \vartheta_j^0$. Thus, the dynamics of θ change sign at ϵ. Since the dynamics of θ must change sign again, $\exists \overline{\theta}_i$ at the boundary of d_i such that $\exists \tau' > 0$ such that $\theta(T + \tau') = \overline{\theta}_i$ and $\exists \epsilon, \epsilon' > 0$ such that $\forall \delta, \delta'$ such that $0 < \delta < \epsilon$ and $0 < \delta' < \epsilon'$, $\theta(T + \tau' - \delta) \in \vartheta_i^0$ and $\theta(T + \tau' + \delta') \in \vartheta_i^0$. Assume that $\exists \nu > 0$ such that $\theta(T + \tau' + \nu) = \overline{\theta}_j$. Let $\tau' + \nu = \tau$. Then, by Conditions (8) and (9), $\forall m \in \mathbb{N}$, $\theta(T + m\tau) = \overline{\theta}_j$. Further, $\forall t > T$, $\theta(t) \in [\overline{\theta}_i, \overline{\theta}_j]$, and $\theta(T + m\tau + \tau') = \overline{\theta}_i$. Thus, any trajectory that does not approach an eigenvector and is not cyclic is conic. $\square$

2.4 Stability Criteria

While the majority of stability and stabilizability analysis focuses on Lyapunov methods, we take an alternative approach, and utilize the stability exponent. For what follows, we refer to systems $\mathcal{H}$ with cyclic trajectories as having a cyclic switching law, while systems with conic trajectories are said to have a conic switching law.

Given a system with a conic switching law, we know that for some initial conditions, and for some $T > 0$, $\forall t > T$, $\theta(t) \in \mathcal{C} = [\overline{\theta}_i, \overline{\theta}_j] \subset [0, 2\pi]$. Given Conditions (8) and (9), a trajectory initialized at $\overline{\theta}_i$ will proceed counter-clockwise from $\overline{\theta}_i$ to $\overline{\theta}_j$ following discrete state sequence $\mathbf{q}$, then proceed clockwise from $\overline{\theta}_j$ to $\overline{\theta}_i$ following discrete state sequence $\mathbf{q}'$, and repeat for all time. Since $\forall t > 0$, $\dot{\theta}(t) \neq 0$, we may consider the fundamental stability exponent of this class of conic trajectories:

$$\Gamma_{\mathcal{C}} = \Gamma(\overline{\theta}_i, \overline{\theta}_j, \mathbf{q}) + \Gamma(\overline{\theta}_j, \overline{\theta}_i, \mathbf{q}')$$

In the case that $\mathcal{H}$ has a cyclic switching law, we may guarantee that for some $T > 0$, the continuous state returns to the line $\theta(T) = \overline{\theta}$ with period τ for all $t > T$, and will follow a fixed sequence of discrete states, $\mathbf{q}$, within each period. The fundamental stability exponent of a cyclic system is

$$sgn(\dot{\theta}) \Gamma(\overline{\theta}_i, \overline{\theta}_i + 2\pi, \mathbf{q})$$

for any $\overline{\theta}_i \in [0, 2\pi)$, and for the corresponding sequence of discrete states, $\mathbf{q}$, where $sgn(\dot{\theta})$ indicates the direction of circulation of the cyclic trajectory.

Utilizing the definitions of fundamental stability exponents, it is now possible to introduce stability theorems. Consider first the case where the system has at least one conic switching law region $\mathcal{C}$ or trajectory which approaches an eigenvector.

THEOREM 3. *Given a system $\mathcal{H}$ with Conditions (6), (7), (8), and (9), with $p \in \mathbb{N}$ distinct conic switching law regions $\mathcal{C}_i$. The origin of $\mathcal{H}$ is globally exponentially stable if and only if $\forall i = 1, \ldots, p$, $\Gamma_{\mathcal{C}_i} < 0$ and every eigenvector of $\mathcal{H}$ has a corresponding real valued negative eigenvalue.*

PROOF. We show this first in the forward direction. By Theorem 2, for systems $\mathcal{H}$ that do not have a cyclic switching law, for any initial condition $(r_0, \theta_0, q_0) \in \text{Init}$, trajectories will either approach an eigenvector or enter one of the p positively invariant conic switching law regions $\mathcal{C}_i$ in finite time. Assume the trajectory approaches an eigenvector of discrete state q, in some set $d_j = \{(r, \theta) \mid \theta \in \vartheta_j = [\overline{\theta}_j, \overline{\theta}_{j+1}]\}$. By our assumption, $\exists \theta_c \in \vartheta_j^0$ such that $A_q \omega(\theta_c) = \lambda \omega(\theta_c)$, where $\lambda < 0$. Consider $\frac{1}{r}\dot{r}(\theta) = \omega(\theta)^T A_q \omega(\theta)$. Note that $\omega(\theta_c)$ is orthogonal to $\gamma(\theta_c)$, and make a basis for $\mathbb{R}^2$. Thus, $\exists \beta_1, \beta_2 \in \mathbb{R}$ such that $\beta_1^2 + \beta_2^2 = 1$ and $\omega(\theta) = \beta_1 \omega(\theta_c) + \beta_2 \gamma(\theta_c)$. Therefore, $\omega(\theta)^T A_q \omega(\theta) = \beta_1^2 \lambda + \beta_2^2$. As $\theta \to \theta_c$, $\beta_1 \to 1$ and $\beta_2 \to 0$, so $\exists T > 0$, $c < 0$ such that $\forall t > T$, $\frac{\dot{r}}{r} \leq c$, implying that $r(t) \leq r(T)e^{ct}$ for all $t > T$, so the trajectory is exponentially stable.

Next, assume that the trajectory enters one of the positively invariant conic switching law regions $\mathcal{C}_i = [\overline{\theta}_u, \overline{\theta}_v]$ at some time $T > 0$. Thus, $\theta(T) = \overline{\theta}_u$. Then, assume that $\Gamma_{\mathcal{C}_i} = c < 0$. By Equation (5), and the definition of a conic trajectory, $r(T + \tau) = r(T)e^c$. Note that $r(T + k\tau) = r(T + (k-1)\tau)e^c$ for all integers $m > 1$, so $r(T + m\tau) = r(T)e^{mc}$.

As $t \to \infty$, $m \to \infty$, and $r(T + m\tau) \to 0$, so the trajectory is exponentially stable.

Now, we prove the necessity of the stated conditions. Assume that the origin is globally exponentially stable. Assume that a given trajectory approaches an eigenvector of A_q, $\omega(\theta_c)$. We note that, if $A_q\omega(\theta_c) = \lambda\omega(\theta_c)$ where $\lambda > 0$, any trajectory initialized at $\theta = \theta_c$ in discrete state q will diverge exponentially, a contradiction. Next, assume that at time $t = T$, a given trajectory hits the boundary of a conic switching region, $\mathcal{C}_i$. Assume that $\Gamma(\mathcal{C}_i) = c \geq 0$. Then, by Equation (5), $r(T+\tau) = r(T)e^c$. Again, note that $r(T + m\tau) = r(T)e^{mc}$ for all $m \in \mathbb{N}$. Note that, as $t \to \infty$, and therefore $m \to \infty$, $r(T + m\tau) \nrightarrow 0$, contradicting the assumption that the origin of the system is exponentially stable. □

Next, we propose conditions for stability of the origin for cyclic systems $\mathcal{H}$.

THEOREM 4. *Given a cyclic system $\mathcal{H}$ with Conditions (6), (7), (8), and (9). Then, the origin of $\mathcal{H}$ is globally exponentially stable if and only if for any $\theta = \theta_i$, and corresponding sequence of discrete states $\mathbf{q}$,*

$$sgn(\dot{\theta})\Gamma(\theta_i, \theta_i + 2\pi, \mathbf{q}) < 0$$

PROOF. By our assumption that any trajectory of the system is cyclic, we know $\exists T > 0, \tau > 0$ such that for all $t > T$, $\theta(t) = \theta(t + \tau)$.

Next, assume that, for any θ_i such that $\theta(T) = \theta_i$ and $\theta(T + \tau) = \theta_i$, $\Gamma(\theta_i, \theta_i + 2\pi, \mathbf{q}) = c < 0$. By Equation (5), we note that $r(T + \tau) = r(T)e^c$, and for all $m \in \mathbb{N}$, $r(T + m\tau) = r(T)e^{mc}$. As $t \to \infty$, $m \to \infty$, and $r(T + m\tau) \to 0$. Further, since the continuous state trajectory is continuous, $\exists\beta > 0$ such that $\forall t \in [T, T + \tau]$, $r(t) < \beta r(T)$. Also, $\forall t \in [mT, mT + \tau]$, $r(t) < \beta r(mT)$. Therefore, the trajectory converges to the origin exponentially. Then, we assume that the origin of $\mathcal{H}$ is globally exponentially stable. Since the system has a cyclic switching law, after some time $T > 0$, we can guarantee that $\forall t > T$, $\theta(t + \tau) = \theta(t)$. For a given θ_i, we can guarantee that the discrete state follows sequence $\mathbf{q}$. To show necessity, we derive a contradiction. Assume that, for some θ_i, $\Gamma(\theta_i, \theta_i + 2\pi, \mathbf{q}) = c \geq 0$. This implies that $r(T+m\tau) = r(T)e^{mc}$. As $t \to \infty$, $m \to \infty$, and $r(T + m\tau) \nrightarrow 0$, so the origin is not globally exponentially stable, which is a contradiction. □

3. STABILIZING SWITCHING LAWS

In this section, we apply the stability results of the previous section to the problem of stabilizability. For hybrid systems $\mathcal{H}$ with the addition of Conditions (6), (7), (8), and (9), the problem of stabilizability is equivalent to deciding, for a given collection of discrete states $\mathcal{Q}$ with corresponding linear dynamics $\dot{x} = A_i x$ with $A_i \in \mathcal{A} = \{A_1, A_2, \ldots, A_k\}$, if the domains and reset map can be designed, in such a way that the origin of $\mathcal{H}$ is globally exponentially stable. Trivially, if any of the linear subsystems, say A_j, has an exponentially stable equilibrium, then we may let $\mathcal{D}_j = \mathbb{R}^2$, making trajectories of $\mathcal{H}$ equivalent to trajectories of $\dot{x} = A_j x$, and the system is globally exponentially stable. So, we assume that all linear subsystems have an unstable equilibrium.

We approach the stabilizability problem for cyclic switching laws constructively, and offer a principled approach to constructing the domains and reset map of $\mathcal{H}$. We show that, if the designed system is not well-defined or is unstable, then the given collection of linear subsystems is not stabilizable by the by a system $\mathcal{H}$ with Conditions (6), (7), (8), (9), and a cyclic switching law.

3.1 Constructive Cyclic Stabilizability Solution

We assume, in this section, that there are no eigenvectors that are common to all subsystems A_i. If this is the case, no switching law can be created that would not include an unstable eigenvector line, implying that there does not exist a set of domains and a reset map such that the system $\mathcal{H}$ is stable.

With the end-goal of generating a cyclic switching law, we require that, for all θ, the trajectory circulates either clockwise ($\dot{\theta} < 0$) or counter-clockwise ($\dot{\theta} > 0$). Thus, at every $\theta \in [0, \pi)$, we partition $\mathcal{Q}$ into

$$\begin{aligned} Q_c(\theta) &= \{q \mid \gamma(\theta)^T A_q \omega(\theta) < 0\} \\ Q_{cc}(\theta) &= \{q \mid \gamma(\theta)^T A_q \omega(\theta) > 0\} \\ Q_0(\theta) &= \{q \mid \gamma(\theta)^T A_q \omega(\theta) = 0\} \end{aligned}$$

The former two sets represent the set of discrete states with continuous dynamics which circulate clockwise and counterclockwise, respectively. The latter set represents the set of discrete states, q, such that $\omega(\theta)$ is an eigenvector of A_q. Note that, $\forall\theta$ $Q_c(\theta) \bigcup Q_{cc}(\theta) \bigcup Q_0(\theta) = \mathcal{Q}$. With this partitioning, we introduce the following two piecewise-constant functions:

$$\begin{aligned} q_c^*(\theta) &= \arg\max_{q \in \mathcal{Q}_c(\theta)} \{\alpha_q(\theta)\} \\ q_{cc}^*(\theta) &= \arg\min_{q \in \mathcal{Q}_{cc}(\theta)} \{\alpha_q(\theta)\} \end{aligned} \quad (10)$$

Note that $\alpha_q(\theta) = \alpha_q(\theta + \pi)$, as $A(\alpha)x = \alpha Ax$ for $A \in \mathbb{R}^{2\times2}$ and $x \in \mathbb{R}^2$. This selection law enables direct construction of the fundamental domain sets, d_j, $j = 1, \ldots, N$ that have the form described in Equation (1), for two systems $\mathcal{H}_c$ and $\mathcal{H}_{cc}$. When a single discrete state, q_i is chosen for an interval $\theta \in [\theta_0, \theta_f)$, what results is a fundamental domain set $d_j = \{(r, \theta) \mid \theta \in [\theta_0, \theta_f) \bigcup [\theta_0 + \pi, \theta_f + \pi)\}$ for some j, which contributes to the set $\mathcal{D}_i$. However, to design a well-defined system $\mathcal{H}$, we must verify is that this selection rule will create non-empty fundamental domain sets.

LEMMA 5. *Given a set of fundamental domain sets d_j, $j = 1, \ldots, N$, designed by point-wise selection of discrete states at each θ according to decision rule (10). Then, for $j = 1, \ldots, N$, d_j is non-empty.*

PROOF. We seek to analyze properties of a collection of functions $\alpha_q(\theta) = \frac{\omega(\theta)^T A_q \omega(\theta)}{\gamma(\theta)^T A_q \omega(\theta)}$ for $q = 1, \ldots, k$, which are well-defined when $\gamma(\theta)^T A_q \omega(\theta) \neq 0$. Rather than consider the trigonometric functions $\omega(\theta)$ and $\gamma(\theta)$ over the interval $[0, \pi/2)$, we consider instead $\hat{\omega}(\xi) = \begin{bmatrix} 1 - \xi & \xi \end{bmatrix}^T$ and $\hat{\gamma}(\xi) = \begin{bmatrix} -\xi & 1 - \xi \end{bmatrix}$ for $\xi \in [0, 1]$. Note that, if we scale $\hat{\omega}(\xi)$ and $\hat{\gamma}(\xi)$ by a ξ-dependent factor $\beta(\xi) = \frac{1}{\sqrt{\xi^2 + (1-\xi)^2}}$, each vector becomes unit length for all ξ. Thus, $\omega(\theta) = \beta(\xi)\hat{\omega}(\xi)$, and $\gamma(\theta) = \beta(\xi)\hat{\gamma}(\xi)$, for $\sin\theta = \frac{\xi}{\sqrt{\xi^2 + (1-\xi)^2}}$. This is illustrated in Figure 4. Thus, $\alpha_q(\xi) = \frac{\hat{\omega}(\xi)^T A_q \hat{\omega}(\xi)}{\hat{\gamma}(\xi)^T A_q \hat{\omega}(\xi)}$.

Note that $\alpha_q(\xi)$ is a (not necessarily proper) rational function, where the numerator and the denominator are both quadratic in ξ:

$$\frac{a_{11} + \xi(-2a_{11} + a_{12} + a_{21}) + \xi^2(a_{11} + a_{22} - a_{12} - a_{21})}{a_{21} + \xi(-a_{11} - a_{21} + a_{22}) + \xi^2(a_{11} - a_{12} + a_{21} - a_{22})} = \\ = \frac{N_q(\xi)}{D_q(\xi)}$$

where $A_q = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

We seek to find a $q \in Q_c$ or Q_{cc} to maximize $\alpha_q(\xi)$ for given ξ, which may be determined by pairwise comparisons between $\alpha_i(\theta)$ and $\alpha_j(\theta)$ for $i \neq j$. Pairwise comparison may be achieved by checking the sign of $\alpha_i(\theta) - \alpha_j(\theta)$. The difference between two rational functions is also a rational function: $\frac{N_1(\xi)}{D_1(\xi)} - \frac{N_2(\xi)}{D_2(\xi)} = \frac{N_1(\xi)D_2(\xi) - N_2(\xi)D_1(\xi)}{D_1(\xi)D_2(\xi)}$. In this case, the numerator and denominator of this rational function are at most fourth-order polynomials in ξ. This rational function may change sign only around poles and zeros. Since there are a finite number of poles and zeros of a rational function, and there are a finite number of discrete states, there are a finite number of points of discontinuity in the decision rule of Equation (10). Since a continuous interval exists between any finite set of distinct points on a line, the fundamental domain sets designed by Equation (10) will be non-empty conic sets. □

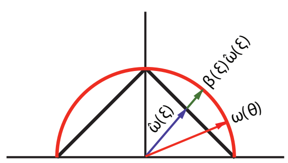


Figure 4: Simplifying re-parameterization for proof of Lemma 5

Thus, the decision rule described in Equation (10) will design the domains for two hybrid systems, $\mathcal{H}_c$ and $\mathcal{H}_{cc}$, which automatically satisfy Conditions (7), (8), and (9). However, it is not guaranteed that $\bigcup_{i=1}^{k} \mathcal{D}_i = \mathbb{R}^2$ for either $\mathcal{H}_c$ or $\mathcal{H}_{cc}$.

THEOREM 6. *Either $\mathcal{H}_c$ or $\mathcal{H}_{cc}$ is well-defined and exponentially stable if and only if the collection of linear subsystems described by the set $\mathcal{A} = \{A_1, \dots, A_n\}$ are stabilizable with a cyclic switching law.*

PROOF. Suppose that $q_c^*(\theta)$ is not defined for some $\theta \in [0, \pi)$. This happens if and only if $\bigcup_{i=1}^{k} \mathcal{D}_i \neq \mathbb{R}^2$, meaning that the hybrid system $\mathcal{H}_c$ is not well-defined. $q_c^*(\theta)$ is not defined at a given θ if and only if $\exists \theta \forall q \in \{1, \dots, k\}$ such that $\gamma(\theta)^T A_q \omega(\theta) \geq 0$. By the definition of cyclic switching laws, this implies that the system is not stabilizable by a clockwise circulating cyclic switching law.

Suppose instead that $q_c^*(\theta)$ is defined for all $\theta \in [0, \pi)$. Thus, Condition (6) is automatically satisfied, and $\mathcal{H}_c$ is cyclic. Assume that the equilibrium of $\mathcal{H}_c$ is globally exponentially stable. The obvious implication is that the collection of linear subsystems $\mathcal{A}$ is stabilizable by a clockwise cyclic switching law. If, instead, the equilibrium of $\mathcal{H}_c$ is not globally exponentially stable, then by Theorem 4, $\Gamma(2\pi, 0, \mathbf{q}) \geq 0$ for the appropriate sequence of discrete states $\mathbf{q}$, which is determined by $q_c^*(\theta)$. Consider an alternative sequence of discrete states $\mathbf{q}' = \{q_0', q_1', \dots, q_N'\} \neq \mathbf{q}$. By the pointwise maximization rule of Equation (10), we have

$$\begin{aligned} \Gamma(2\pi, 0, \mathbf{q}) &= \sum_{i=1}^{N} \int_{\theta_{i-1}}^{\theta_i} \alpha_{q_i}(\theta) d\theta \\ &\leq \sum_{i=1}^{N} \int_{\theta_{i-1}}^{\theta_i} \alpha_{q_i'}(\theta) d\theta \\ &\leq \Gamma(2\pi, 0, \mathbf{q}') \end{aligned}$$

Thus, the stability exponent is minimized over all possible cyclic systems generated from $\mathcal{A}$. By Theorem 4, a cyclic trajectory is exponentially stable if and only if the corresponding stability exponent is negative, so there does not exist a clockwise cyclic switching law to stabilize the system if $\mathcal{H}_c$ is not exponentially stable. The same argument applies for counter-clockwise cyclic switching laws. □

4. SENSITIVITY TO PERTURBATIONS IN SWITCHING LAW

Previously, we determined that the sign of the stability exponent indicates whether or not a given trajectory converges to an equilibrium. The magnitude of the stability exponent provides useful information, too. While yielding some insight about the rate of convergence of the trajectory, the stability exponent permits analysis of the sensitivity of the stability of the equilibrium with respect to perturbations in the switching law. While sensitivity analysis can be performed using standard methods [8, 18, 23], this approach offers an interesting alternative.

Consider a given hybrid system $\mathcal{H}$ of the form of Equation (1), with Conditions (6), (7), (8), and (9), which has guards at lines $\theta = \overline{\theta}_i$ for $i = 1, \dots, N$. By Theorem 4, if $\mathcal{H}$ has a cyclic switching law and the equilibrium is globally exponentially stable, the fundamental stability exponent for a cyclic trajectory will be negative. If the system $\mathcal{H}$ has a conic switching law, by Theorem 3, the fundamental stability exponent of any positively invariant set $\mathcal{C}_i$ will be negative. However, we would like to know how sensitive this stability is to perturbations in the switching law. Switches, in general, don't toggle instantaneously. If some configuration relies on very fast and/or precise switching to stabilize a system, then that configuration is rather tenuous. We seek to understand conditions when any small change in the domains and reset maps may effect a large change in the observed behavior of the system.

Consider rotating the guards and domains of $\mathcal{H}$ by some constant angle ϕ_0 about the origin. For cyclic systems, the fundamental stability exponent is

$$\Gamma(\overline{\theta}_0 + \phi_0, \overline{\theta}_0 + \phi_0 + 2\pi, \mathbf{q}) = \sum_{i=1}^{N} \int_{\overline{\theta}_{i-1} + \phi_0}^{\overline{\theta}_i + \phi_0} \alpha_i(\theta) d\theta \quad (11)$$

We refer to the perturbation of all guards and domains by the same angle as a zero-order delay perturbation. We assume that ϕ_0 is limited so that the modified system, $\mathcal{H}'$, has no eigenvector, so the resulting system still has a cyclic switching law. If this is the case, Equation (11) describes a continuous function of ϕ_0 over the domain $[0, \pi)$ (or some subset of the domain). The extent of ϕ_0 on the interval $[0, \pi)$ such that Equation (11) is negative indicates the sensitivity of the stability of the equilibrium to zero-order delay perturbations.

We introduce an example to illustrate the application of the stability exponent. Consider a slight perturbation to a canonical example which was explored by Branicky [1].

$$A_1 = \begin{bmatrix} 0.1 & 1 \\ -10 & 0.1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.1 & 10 \\ -1 & 0.1 \end{bmatrix}$$

Discrete state q_1 induces continuous dynamics $\dot{x} = A_1 x$, and discrete state q_2 induces continuous dynamics $\dot{x} = A_2 x$. While both $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ are unstable, it is well known that there is one configuration of switching surfaces that is clearly exponentially stable, where the domain of q_1 is the union of quadrants 1 and 3 and the domain of q_2 is the union of quadrants 2 and 4. Guards $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ force a transition between discrete states q_1 and q_2. Guards $\theta = \pi$ and $\theta = 0$ force a transition between discrete states q_2 and q_1. However, if the domains of the discrete states are interchanged, the equilibrium of the hybrid system becomes unstable. This exchange of domains is equivalent to rotating the switching surfaces by $\phi_0 = \frac{\pi}{2}$. Note that, for a rotation of $\phi_0 = \frac{\pi}{4}$, the system is unstable, though it diverges slowly. This is illustrated in Figure 5.

We can plot the stability exponent of a cyclic system for various ϕ_0 to determine the sensitivity of the nominal stable system to zero-order delay perturbations. This is illustrated in Figure 6. Note that a stability exponent of 0 implies that the trajectory is periodic. This condition offers insight into the problem of designing non-linear oscillators from given linear systems. Given a nominal system $\mathcal{H}$, the trajectories may be designed to be periodic simply by rotating domains and guards to zero the stability exponent.

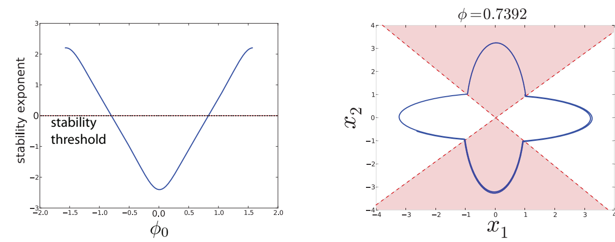
The stability exponent may also be leveraged to consider first-order delay perturbations, where the angles between guards change, along with the boundaries of the fundamental domain sets. A first-order perturbation of the guard $\overline{\theta}_0$ and associated boundary of fundamental domain set d_1 generates the fundamental stability exponent:

$$\Gamma(\overline{\theta}_0 + \phi_1, \overline{\theta}_0 + \phi_1 + 2\pi, \mathbf{q}) =$$
$$= \int_{\overline{\theta}_0+\phi_1}^{\overline{\theta}_1} \alpha_1(\theta) d\theta + \sum_{i=2}^{N-1} \int_{\overline{\theta}_{i-1}}^{\overline{\theta}_i} \alpha_i(\theta) d\theta +$$
$$+ \int_{\overline{\theta}_{N-1}}^{\overline{\theta}_N + \phi_1} \alpha_{N-1}(\theta) d\theta$$

The sequence of discrete states corresponding to the cyclic trajectory, $\mathbf{q}$, is assumed to be preserved. Thus, the limits on ϕ_1 are $\phi_1 \in [0, \overline{\theta}_1 - \overline{\theta}_0)$. Figure 7 illustrates the analysis of a first-order perturbation of our example. Note that similar analysis may be pursued for conic switching laws, using the fundamental stability exponent for a given conic switching law region.

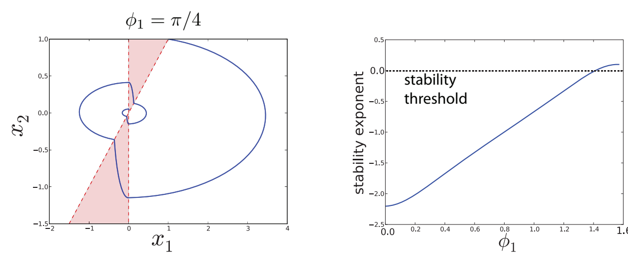
5. CONCLUSIONS

In this paper, we illustrated the use of polar coordinate transformations for analyzing hybrid systems with second-order linear continuous dynamics, and linear discrete-state switching surfaces. Necessary and sufficient conditions for stability and stabilizability are introduced for this particular class of hybrid systems. The approach is illustrated on a simple example and is further applied to sensitivity analysis of the stability of an equilibrium to perturbations in the switching law.



(a) (b)

Figure 6: (a): Stability exponent for Branicky's example for various zero-order perturbations. Note that stability is maintained with respect to zero-order perturbations almost up to $[-\frac{\pi}{4}, \frac{\pi}{4}]$. (b): Example trajectory when stability exponent is almost 0, a marginally stable solution.



(a) (b)

Figure 7: (a): First-order perturbation example. Stability preserved under relatively large perturbation. (b): Stability exponent for first-order perturbation. Note the stability exponent graph lies mostly beneath the stability threshold line. This stability of the equilibrium of this system is not very sensitive to first-order perturbations.

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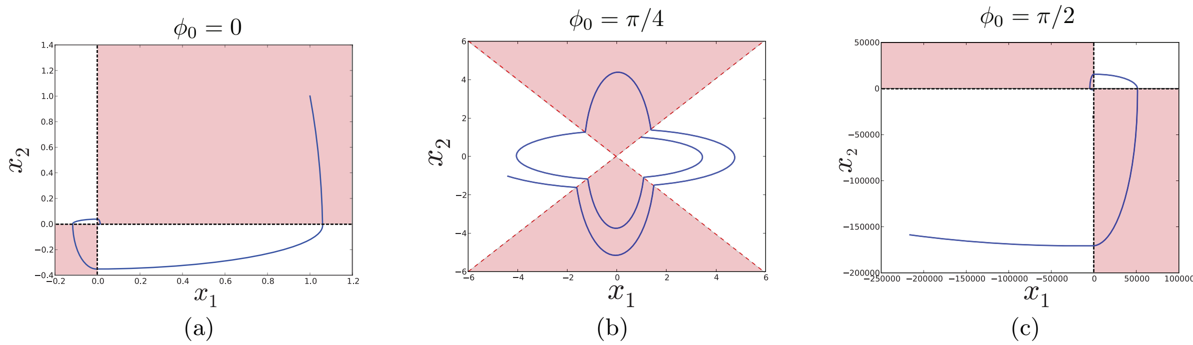

Figure 5: Example trajectories of Branicky's system for different switching laws. (a): Nominal stable system, $\dot{x} = A_1 x$ in first and third quadrants, $\dot{x} = A_2 x$ elsewhere. (b): Intermediate system. Zero-order perturbation $\phi_0 = \frac{\pi}{4}$. Unstable, but diverges slowly. (c): Unstable. $\dot{x} = A_1 x$ in second and fourth quadrants.

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