

# Compressive Phase Retrieval via Lifting

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Phase retrieval has been a longstanding problem in optics and x-ray crystallography since the 1970s [3, 2]. Early methods to recover the phase signal using Fourier transform mostly relied on additional information about the signal, such as band limitation, nonzero support, real-valuedness, and nonnegativity. Common drawbacks of these iterative methods are that they may not converge to the global solution, and the rate of convergence is often slow. More recently, phase retrieval has been framed as a low-rank matrix completion problem in [1]. Given an *overdetermined* system, a lifting technique was used to approximate the linear model constraint as a semidefinite program (SDP). The authors also derived an upper-bound for the sampling rate that guarantees exact recovery in the noise-free case and stable recovery in the noisy case.

The main contribution of this work is a convex formulation of the *underdetermined* compressive phase retrieval problem. Using the lifting technique, the NP-hard problem is relaxed as a semidefinite program. We derive bounds for guaranteed recovery of the true signal and compare the performance of our algorithm with traditional compressive sensing and phase retrieval algorithms. The results extend the type of applications that compressive sensing can be applied to, namely, applications where only magnitudes can be observed. A potential application of interest with this property is x-ray diffraction.

## 1 Compressive Phase Retrieval via Lifting

In a linear model  $\mathbf{y} = A\mathbf{x}$ , we assume that only the squared magnitude of the output is observed:

$$b_i = |y_i|^2 = |\langle \mathbf{x}, \mathbf{a}_i \rangle|^2 = |\mathbf{a}_i^H \mathbf{x}|^2, \quad i = 1, \dots, N, \quad (1)$$

where  $A^H = [\mathbf{a}_1, \dots, \mathbf{a}_N] \in \mathbb{C}^{n \times N}$ ,  $\mathbf{y}^T = [y_1, \dots, y_N] \in \mathbb{C}^{1 \times N}$ , and  $\mathbf{a}^H$  denotes the Hermitian transpose of  $\mathbf{a}$ . We assume  $\{b_i, \mathbf{a}_i\}_{i=1}^N$  are known and seek  $\mathbf{x}$ . When  $N \geq n$ , this combinatorial problem is referred to as the phase retrieval problem. For compressive phase retrieval (CPR), *i.e.*,  $N < n$ , additional assumptions are needed to find a unique solution up to a global phase [4].

Motivated by compressive sensing, we seek the sparsest solution of CPR satisfying (1). We have shown in [5] that the intractable problem can be relaxed to a semidefinite program (SDP)

$$\min_{X \succeq 0} \text{Tr}(X) + \lambda \|X\|_1, \quad \text{subj. to} \quad b_i = \text{Tr}(\mathbf{a}_i \mathbf{a}_i^H X), \quad i = 1, \dots, N, \quad (2)$$

where  $\lambda > 0$  is a design parameter. Finally, the estimate of  $\mathbf{x}$  can be found by computing the rank-1 decomposition of  $X$  via SVD. We refer to the solution as Compressive Phase Retrieval via Lifting (CPRL).

## 2 Theoretical Results

In order to state some theoretical properties, we need a generalization of the restricted isometry property (RIP) and mutual coherence. Firstly, define the linear operator  $B$  of  $X$  as  $B : X \in \mathbb{C}^{n \times n} \mapsto \{\text{Tr}(\mathbf{a}_i \mathbf{a}_i^H X)\}_{1 \leq i \leq N} \in \mathbb{R}^N$ . We now say that a linear operator  $B$  is  $(\epsilon, k)$ -RIP if for all  $X \neq 0$  s.t.  $\|X\|_0 \leq k$  we have  $\left| \frac{\|B(X)\|_2^2}{\|X\|_2^2} - 1 \right| < \epsilon$ . Secondly, let  $B(\cdot)$  be the matrix satisfying  $\mathbf{b} = B(\cdot)X(\cdot)$  with  $X(\cdot)$  the vectorized version of  $X$ . Now, for a matrix  $A$ , define the *mutual coherence* as  $\mu(A) = \max_{1 \leq k, j \leq n, k \neq j} \frac{|\mathbf{a}_k^H \mathbf{a}_j|}{\|\mathbf{a}_k\|_2 \|\mathbf{a}_j\|_2}$  and let  $\tilde{\mathbf{x}}$  be the sparsest solution to (1). We then have that the solution of (2),  $\tilde{X}$ , is equal to  $\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H$  if it has rank 1 and  $B$  is  $(\epsilon, 2\|\tilde{X}\|_0)$ -RIP with  $\epsilon < 1$ , or if  $\|\tilde{X}\|_0 < 0.5(1 + 1/\mu(B(\cdot)))$ . The proof is given in [5].

## 3 Experiments

Since (2) is an SDP, it can be solved by standard convex optimization software. However, it is well known that the standard toolboxes based on interior-point methods suffer when the dimension of  $X$  is large. We therefore propose a greedy approximate algorithm tailored to solving (2). Under mild conditions, the algorithm can accurately solve large problems of the form (2). The algorithm is summarized as follows:

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**Algorithm 1:** Greedy Compressive Phase Retrieval via Lifting (GCPRL)
 

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Set support set  $\mathcal{I} = \emptyset$  and let  $\gamma > 0, \epsilon > 0$ .  
**repeat**  
   **for**  $k = 1, \dots, N$ , **do**  
     Set  $\mathcal{I}_k = \mathcal{I} \cup \{k\}$  and solve  $X_k \doteq \arg \min_{X \succeq 0} \text{Tr}(X^{(\mathcal{I}_k)}) + \gamma \sum_{i=1}^N (b_i - \text{Tr}(\mathbf{a}_i^{(\mathcal{I}_k)} \mathbf{a}_i^{(\mathcal{I}_k)H} X^{(\mathcal{I}_k)}))^2$ .  
     Let  $W_k$  denote the corresponding objective value.  
   Let  $p$  be such that  $W_p \leq W_k, k = 1, \dots, N$ . Set  $\mathcal{I} = \mathcal{I} \cup \{p\}$  and  $X = X_p$ .  
**until**  $W_p < \epsilon$ ;

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To demonstrate the effectiveness of GCPRL and as an illustration of the theoretical bounds, let us consider a numerical example. Let the true  $\mathbf{x}_0 \in \mathbb{C}^n$  be a  $k$ -sparse signal, let the nonzero elements be randomly chosen and their values randomly distributed on the complex unit circle. Let  $A \in \mathbb{C}^{N \times n}$  be generated by sampling from a complex unit Gaussian distribution.

If we fix  $n/N = 2$ , that is, twice as many unknowns as measurements, and apply GCPRL for different values of  $n, N$  and  $k$ , we obtain the computational times visualized in the left plot of Figure 1. In all simulations  $\gamma = 10$  and  $\epsilon = 10^{-3}$  are used in GCPRL. The true sparsity pattern is always recovered. Since GCPRL can be executed in parallel, the execution time can be further divided by the number of cores used (the average run time in Figure 1 is computed on a standard laptop running Matlab, 2 cores, and using CVX to solve the low-dimensional SDP of GCPRL). Hence, the algorithm is several magnitudes faster than the standard interior-point methods used in CVX.

We also use this particular example to show the theoretical bounds of successful recovery. The middle plot of Figure 1 shows the quantity  $0.5(1 + 1/\mu(B))$ . We conclude that if the solution to (2) has rank 1,  $25 \leq n, N \leq 125$  and only one nonzero component, then it is also the sparse solution to (1). As shown in the right plot of Figure 1, the empirical 95% success curve, the theoretical bound is also very conservative.

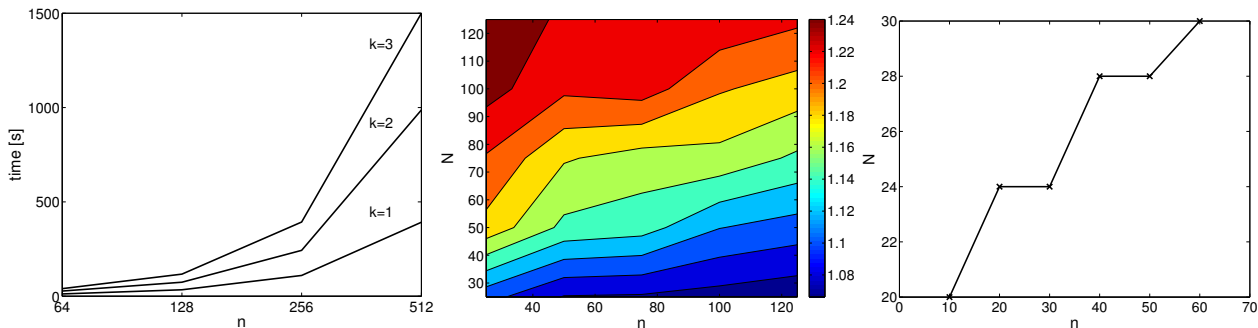


Figure 1: **Left:** Average run time of GCPRL in Matlab CVX environment. **Middle:** The quantity  $0.5(1 + 1/\mu(B))$ . **Right:** The 95% success rate of solving (2) for  $k = 2$  via CVX.

## References

- [1] E. J. Candès, Y. Eldar, T. Strohmer, and V. Voroninski. Phase retrieval via matrix completion. Technical Report arXiv:1109.0573, Stanford Univeristy, September 2011.
- [2] R. Gonsalves. Phase retrieval from modulus data. *Journal of Optical Society of America*, 66(9):961–964, 1976.
- [3] D. Kohler and L. Mandel. Source reconstruction from the modulus of the correlation function: a practical approach to the phase problem of optical coherence theory. *Journal of the Optical Society of America*, 63(2):126–134, 1973.
- [4] M. Moravec, J. Romberg, and R. Baraniuk. Compressive phase retrieval. In *SPIE International Symposium on Optical Science and Technology*, 2007.
- [5] H. Ohlsson, A. Y. Yang, R. Dong, and S. Sastry. Compressive Phase Retrieval From Squared Output Measurements Via Semidefinite Programming. Technical Report arXiv:1111.6323, University of California, Berkeley, November 2011.

**Topic:** visual processing and pattern recognition

**Preference:** oral