52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Pricing Design for Robustness in Linear Quadratic Games

Daniel J. Calderone, Lillian J. Ratliff, and S. Shankar Sastry

Abstract— **We show that designing prices to induce a socially optimal Nash equilibrium that is robust to small parameter perturbations in a linear-quadratic game can be framed as a convex program. In addition, we use a similar analysis to develop convex conditions that guarantee that the induced equilibrium is stable with respect to small deviations in players' feedback strategies.**

I. INTRODUCTION

Engineering problems involving decision making agents with competing interests are appearing more frequently in the literature as technology is integrated into infrastructure [1]–[4]. These types of problems can be modeled as dynamic games where competitive agents are strategic players seeking to optimize their own utility functions. In such scenarios, selfish agents often converge to Nash equilibria that are inefficient from a societal view point. As a result, it is desirable for a social planner to modify players' utility functions with pricing, taxation, or rewards so that the resulting equilibrium coincides with the socially optimal strategies.

In the case of linear quadratic (LQ) games, Ratliff, et.al. [5] showed that designing prices for coordination can be framed as a convex program. In this work, we show that a convex objective and a set of convex constraints can be added to this original program to design prices that induce an equilibrium that is robust to small perturbations in the game parameters. In general, perturbations in the parameters of an LQ game will cause the Nash equilibrium to shift and we seek to design prices to minimize this shift as much as possible.

Designing control strategies which are robust to noise and parameter perturbations has been studied extensively in the literature (see, [6] and references therein). In addition, there is literature studying the stability of Nash equilibria (see, e.g. [7]–[9]). This work draws on these techniques and relies on a first-order perturbation analysis similar to that done in [10]. The program developed also serves to design prices that guarantee the stability of the induced equilibrium with respect to small deviations in the players' feedback strategies. This analysis could be considered an extension of the work presented in [7] to LQ games with more than two players.

The rest of the paper is organized as follows. In Section II, we review the pricing design problem for LQ games presented in [5]. In Section III, we develop the theory for designing prices that are robust to parameter perturbations. We define a robustness objective and constraints that are incorporated into the optimization problem outlined in Section II. In Section IV, we present numerical examples. In Section V, we show that similar theory can be used to guarantee that the Nash equilibrium is robust to perturbations in the players' strategies. In Section VI, we summarize the results and make remarks on future work.

The authors are with the Department of Electrical Engineering and Computer Sciences at the University of California, Berkeley, Berkeley CA, 94720. {danjc, ratliffl, sastry} at eecs.berkeley.edu

The research presented in this paper is supported by National Science Foundation (NSF) Graduate Research Fellowships, NSF CPS:Large:ActionWebs award number 0931843, TRUST (Team for Research in Ubiquitous Secure Technology) which receives support from the NSF (CCF-0424422), and AFOSR MURI CHASE award number FA9550-10-1-0567.

II. REVIEW OF PRICING DESIGN FEASIBILITY PROBLEM

Consider a linear quadratic dynamic game with p players denoted as members of the index set $\mathcal{I} := \{1, \dots, p\}$. Let $x_i \in \mathbb{R}^{n_i}$ denote the state of the ith player (with dimension n_i) and let $u_i \in \mathbb{R}^{m_i}$ denote the control input of the ith player (with dimension m_i). Furthermore, let $n := \sum_{i\in\mathcal{I}} n_i$, $m := \sum_{i\in\mathcal{I}} m_i$, and

$$x := \begin{bmatrix} x_1^T & \dots & x_p^T \end{bmatrix}^T, \qquad u := \begin{bmatrix} u_1^T & \dots & u_p^T \end{bmatrix}^T. \quad (1)$$

The states are dynamically coupled and evolve under the following dynamics:

$$\dot{x}(t) = Ax(t) + \textstyle\sum_{i\in\mathcal{I}} B_i u_i(t) \quad (2)$$

for some matrix $A \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m_i}$ for $i \in \mathcal{I}$. We assume that $(A, \begin{bmatrix} B_1 \cdots B_p \end{bmatrix})$ is stabilizable. The ith player incurs the quadratic cost J_i given by

$$J_i := \textstyle\int_0^\infty x(t)^T Q_i x(t) + u(t)^T R_i u(t)\ dt \quad (3)$$

where $Q_i \in \mathbb{R}^{n\times n}$, $Q_i \succeq 0$, and $R_i \in \mathbb{R}^{m\times m}$ is a symmetric matrix to be designed. We restrict player i's strategy space to be the set of causal, memoryless state feedback controls (denoted by Γ_i). A Nash equilibrium is a set of strategies $\{u_i^* \in \Gamma_i\}_{i\in\mathcal{I}}$ such that

$$J_i(u_i^*, u_{-i}^*) \le J_i(u_i, u_{-i}^*) \quad \forall u_i \in \Gamma_i,\ \forall i \quad (4)$$

where u_{-i}^* denotes the set of actions taken by players other than player i, i.e. $-i := \{1, \dots, i-1, i+1, \dots, p\}$. When the players play a Nash equilibrium, no single one can achieve a lower cost by unilaterally changing his or her strategy.

As in [5], the goal of the linear quadratic pricing problem is to design a set of matrices, $\{R_i\}_{i\in\mathcal{I}}$, such that players are induced to use feedback gains, $\{K_i\}_{i\in\mathcal{I}}$; that is, $\{u_i^* = -K_i x\}_{i\in\mathcal{I}}$ becomes a Nash equilibrium for the game defined by the dynamics given in (2) and the costs given in (3). In general, $\{K_i\}_{i\in\mathcal{I}}$ would be the set of optimal feedback controls for some team optimization problem. Consistent with this, we make the assumption that $\{K_i\}_{i\in\mathcal{I}}$ stabilize the system. For notation purposes, we partition each R_i into blocks, $R_i^{jk} \in \mathbb{R}^{m_j \times m_k}$. Note that since R_i is symmetric, it must be that $R_i^{jk} = (R_i^{kj})^T$ for all $j, k \in \mathcal{I}$.

For each individual player, applying the definition of a Nash equilibrium, i.e. plugging the other players' controls into the dynamics and into the individual player's cost, and then writing down the optimality conditions for that player's individual LQR problem gives a set of convex constraints in $\{R_i\}_{i\in\mathcal{I}}$ and $\{P_i\}_{i\in\mathcal{I}}$. These constraints define the set of prices that induce the desired set of feedback strategies. This is the main result of [5] and is reproduced below.

Theorem 1 (Pricing Design Feasibility Problem)**.** *If there exists $\{R_i\}_{i\in\mathcal{I}}$ and $\{P_i\}_{i\in\mathcal{I}}$ such that the convex feasibility problem below is feasible for all $i \in \mathcal{I}$, then $\{u_i^* = -K_i x\}_{i\in\mathcal{I}}$ is a Nash equilibrium of the linear quadratic game defined by (2) and (3).*

$$P_i \succeq 0 \tag{5}$$

$$R_i^{ii} \succ 0 \tag{6}$$

$$\begin{bmatrix} \tilde{Q}_i & \tilde{N}_i \\ (\tilde{N}_i)^T & R_i^{ii} \end{bmatrix} \succeq 0 \tag{7}$$

$$(\tilde{A}_i)^T P_i + P_i(\tilde{A}_i) + \tilde{Q}_i - (K_i)^T R_i^{ii} K_i = 0 \tag{8}$$

$$(B_i^T P_i + (\tilde{N}_i)^T) = R_i^{ii} K_i \tag{9}$$

where

$$\tilde{A}_i := A - \textstyle\sum_{j\neq i} B_j K_j \tag{10}$$

$$\tilde{Q}_i := Q_i + \textstyle\sum_{j\neq i}\sum_{k\neq i} (K_j)^T R_i^{jk} K_k$$

$$\tilde{N}_i := -\textstyle\sum_{j\neq i} (K_j)^T R_i^{ji}. \tag{11}$$

We note that, in general, when such prices $\{R_i\}_{i\in\mathcal{I}}$ exist, they are not unique affording us the opportunity to add objective functions of the values of $\{R_i\}_{i\in\mathcal{I}}$ and $\{P_i\}_{i\in\mathcal{I}}$ to the convex program defined above in order to choose prices with desirable properties. In [5], we propose several objective functions the most interesting of which is a convex objective to make the sum of the individuals' costs after pricing as close as possible to some societal cost. This is referred to as forcing the prices to be *revenue neutral.*

In the current work, we present a new set of convex constraints and objective terms that when added to the program in Theorem 1 seek to minimize the effect of perturbations in the game parameters on the Nash equilibrium of the game and also make the equilibrium stable to small perturbations in the players' strategies. We first discuss pricing design for robustness to parameter perturbations.

III. Designing Prices for Robustness to Parameter Perturbations

A. Perturbations to the Solution Sets of Coupled Riccati Equations

To simplify notation, we define three vector spaces,

$$\mathcal{Z} := \mathbb{R}^{nxn} \times \Pi_{i\in\mathcal{I}}\mathbb{R}^{n\times m_i} \times \Pi_{i\in\mathcal{I}}\mathbb{R}^{n\times n} \times \Pi_{i\in\mathcal{I}}\mathbb{R}^{n\times n} \tag{12}$$

that contains the parameters of p coupled Riccati equations,

$$\mathcal{P} := \Pi_{i\in\mathcal{I}}\mathbb{R}^{n\times n} \tag{13}$$

that contains the cost-to-go matrices for each of the p players in the game, and

$$\mathcal{K} := \Pi_{i\in\mathcal{I}}\mathbb{R}^{m_i\times n} \tag{14}$$

that contains the gain matrices for each of the players. Let $Z := [A\ B_1 \dots B_p\ Q_1 \dots Q_p\ R_1 \dots R_p]$ denote an element of $\mathcal{Z}$, $P := [P_1 \dots P_p]$ denote an element of $\mathcal{P}$, and $K := \begin{bmatrix} K_1 \dots K_p \end{bmatrix}$ denote an element of $\mathcal{K}$. For a subset $\mathcal{I}_a \subseteq \mathcal{I}$, we will use $\mathcal{P}_a$ to denote the subspace of $\mathcal{P}$ corresponding to the players in index set $\mathcal{I}_a$ and P_a to denote an element of that subspace. Similarly for $\mathcal{K}_a$ and K_a.

We use the standard arrow notation $\vec{\cdot}$ to denote the vectorized form of each of these objects stacked column-wise in the usual way. We now define a Riccati operator for each player, $F_i : \mathcal{Z} \times \mathcal{P} \times \mathcal{K} \to \mathbb{R}^{n\times n}$, as

$$\begin{aligned} F_i(Z,P,K) &:= (A - \sum_{j\neq i} B_j K_j)^T P_i + P_i(A - \sum_{j\neq i} B_j K_j) \\ &+ Q_i - (P_i B_i - \sum_{j\neq i} K_j^T R_i^{ji})(R_i^{ii})^{-1}(B_i^T P_i - \sum_{j\neq i} R_i^{ij} K_j) \\ &+ \sum_{j\neq i}\sum_{k\neq i} K_j^T R_i^{jk} K_k \qquad \text{for} \quad i \in \mathcal{I} \end{aligned} \tag{15}$$

and a gain operator for each player, $G_i : \mathcal{Z} \times \mathcal{P} \times \mathcal{K} \to \mathbb{R}^{m_i\times n}$, as

$$G_i(Z,P,K) := R_i^{ii} K_i + \sum_{j\neq i} R_i^{ij} K_j - B_i^T P_i \quad \text{for} \quad i \in \mathcal{I}. \tag{16}$$

We also define several combined operators for the entire system, $F(\cdot) := \begin{bmatrix} F_1(\cdot) \dots F_p(\cdot) \end{bmatrix}$, $G(\cdot) := \begin{bmatrix} G_1(\cdot) \dots G_p(\cdot) \end{bmatrix}$, and $L(\cdot) := \begin{bmatrix} F(\cdot) & G(\cdot) \end{bmatrix}$. By definition, the Nash equilibrium of a linear quadratic game for a given set of parameters, Z, is implicitly defined as the solution, (P, K), to the equation

$$L(Z,P,K) = \mathbf{0}. \tag{17}$$

In general, finding solutions to (17) for a given parameter set Z is a difficult problem and a topic of ongoing research. However, in the case of the pricing design problem, we are guaranteed a solution since we are given K and we design $\{R_i\}_{i\in\mathcal{I}}$ (and thus $\{P_i\}_{i\in\mathcal{I}}$) such that (17) is satisfied. Thus in this context, it is reasonable to ask how does the Nash equilibrium shift with perturbations in the problem parameters and how can we design prices to mitigate this shift. In order to do this, we will invoke the implicit function theorem, and to this end, we enumerate the partial derivatives of the operators defined above.

We denote the matrix form of the partial derivative of F_i with respect to some subset $X \subseteq \mathcal{Z} \times \mathcal{P} \times \mathcal{K}$ as $F_{i,X}$. Similarly, we denote the partial derivatives of G_i, F, G, and L with respect to X as $G_{i,X}$, F_X, G_X, and L_X respectively. The partial derivatives of F_i and G_i with respect to various

$$
F_{i,X} = \begin{cases} I_n \otimes P_i^T + (P_i^T \otimes I_n)\Pi_p & ; X = A \\ -(P_i \otimes K_j^T)\Pi_p - K_j^T \otimes P_i & ; X = B_j, \forall j \\ I_n \otimes I_n & ; X = Q_i, \\ K_i^T \otimes K_i^T & ; X = R_i^{ii} \\ K_k^T \otimes K_j^T & ; X = R_i^{jk}, j,k \neq i \\ K_i^T \otimes K_j^T & ; X = R_i^{ji}, j \neq i \\ K_j^T \otimes K_i^T & ; X = R_i^{ij}, j \neq i \\ I_n \otimes A_{CL}^T + A_{CL}^T \otimes I_n & ; X = P_i \\ I_n \otimes \left(-P_i B_j + K_i^T R_i^{ij} + \sum_{k\neq i} K_k^T R_i^{kj}\right) & \\ + \left(-P_i B_j + K_i^T R_i^{ij} + \sum_{k\neq i} K_k^T R_i^{kj}\right) \otimes I_n \Pi_p & ; X = K_j, j \neq i \\ \mathbf{0} & ; \text{otherwise} \end{cases}
\quad
G_{i,X} = \begin{cases} -(P_i^T \otimes I_{m_i})\Pi_p & ; X = B_i \\ K_i^T \otimes I_{m_i} & ; X = R_i^{ii} \\ K_j^T \otimes I_{m_i} & ; X = R_i^{ij}, j \neq i \\ -I_n \otimes B_i^T & ; X = P_i \\ I_n \otimes R_i^{ii} & ; X = K_i \\ I_n \otimes R_i^{ij} & ; X = K_j, j \neq i \\ \mathbf{0} & ; \text{otherwise} \end{cases}
\tag{18}
$$

subspaces are given in Equation (18). We also have that

$$
F_P := \begin{bmatrix} F_{1,P_1} & \dots & F_{1,P_p} \\ \vdots & \ddots & \vdots \\ F_{p,P_1} & \dots & F_{p,P_p} \end{bmatrix} \; G_P := \begin{bmatrix} G_{1,P_1} & \dots & G_{1,P_p} \\ \vdots & \ddots & \vdots \\ G_{p,P_1} & \dots & G_{p,P_p} \end{bmatrix}
\tag{19}
$$

and similarly for F_K and G_K. We also define L_{PK} and L_Z as

$$
L_{PK} := \begin{bmatrix} L_P & L_K \end{bmatrix} = \begin{bmatrix} F_P & F_K \\ G_P & G_K \end{bmatrix} \; , \; L_Z := \begin{bmatrix} F_Z \\ G_Z \end{bmatrix}. \tag{20}
$$

Note that F_P and G_P are block diagonal since $F_{i,P_j} = \mathbf{0}$ and $G_{i,P_j} = \mathbf{0}$ for $j \neq i$. (Π_p is defined as the permutation matrix of appropriate dimensions such that $(\vec{X^T}) = \Pi_p \vec{X}$ for $X \in \mathbb{R}^{n\times m}$.)

Given an initial set of parameters, Z, and the resulting equilibrium, (P, K), a small perturbation in the parameters, $Z + \partial Z$, will lead to a perturbation in the equilibrium, $(P + \partial P, K + \partial K)$. Up to first order, $(\partial P, \partial K)$ can be calculated using the implicit function theorem as

$$
\begin{bmatrix} \partial \vec{P} \\ \partial \vec{K} \end{bmatrix} = -\begin{bmatrix} F_P & F_K \\ G_P & G_K \end{bmatrix}^{-1} \begin{bmatrix} F_Z \\ G_Z \end{bmatrix} \partial \vec{Z} \tag{21}
$$

assuming that L_{PK} is invertible. (See Theorem C.40 in [11] for details.) To design prices to induce feedback strategies, K, and make them robust to perturbations, we need to ensure that there exists a bound $\mathcal{H}$ such that $\left|\partial \vec{K}\right| < \mathcal{H} \left|\partial \vec{Z}\right|$ and to minimize $\mathcal{H}$ as much as possible. (Enforcing a bound on $\left|\partial \vec{K}\right|$ will also ensure that $\left|\partial \vec{P}\right|$ is bounded though we will not focus on minimizing $\left|\partial \vec{P}\right|$.)

To this end, we review the matrix inversion lemma.

Lemma 1. *For a block matrix*

$$
L := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{22}
$$

where A is square and invertible, L is invertible if and only if $D - CA^{-1}B$ is invertible.

Assuming that A_{CL} is stable, we have from Equation (18) that F_P is invertible. Thus Lemma 1 applies to L_{PK}. In addition, we can calculate

$$
\partial \vec{K} = -\Big[\underbrace{G_K - G_P F_P^{-1} F_K}_{:=\, H_{PK}}\Big]^{-1} \Big[\underbrace{G_Z - G_P F_P^{-1} F_Z}_{:=\, H_{PZ}}\Big] \partial \vec{Z}. \tag{23}
$$

(See Section 9.1.5 of [12] for details). Thus to control the size of ∂K, we need to ensure that H_{PK} is invertible and to minimize $\left|H_{PK}^{-1}\right|$ and $|H_{PZ}|$. We first consider how to bound $\left|H_{PK}^{-1}\right|$. (We use $|\cdot|$ to represent the 2-norm of a vector or the induced 2-norm of a matrix depending on context.)

B. Convex Constraints for Bounding $\left|H_{PK}^{-1}\right|$

Lemma 2. *Let A be a square block matrix of the form*

$$
A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \tag{24}
$$

where A_{ii} are also square. If

$$
\sigma_{min}(A_{ii}) > nt + \sum_{j\neq i} |A_{ij}| \qquad \forall i \tag{25}
$$

for $t > 0$, where σ_{min} is the minimum singular value, then

$$
\sigma_{min}(A) > t. \tag{26}
$$

Consequently, A is invertible and $\left|A^{-1}\right| < \frac{1}{t}$.
This could be considered a block matrix extension to the Gerschgorin Circle Theorem (Theorem 3.2.1 in [13]).

Proof: First note that for A square

$$
\sigma_{\min}(A) = \min_{|x|=1} |Ax|. \tag{27}
$$

We prove the contrapositive. Suppose not (26). It follows that there exists a non-zero vector x of unit norm such that

$$
|Ax| \leq t. \tag{28}
$$

We decompose x into components that agree with the dimensions of the blocks of A

$$
x := \left[x_1^T \dots x_n^T\right]^T. \tag{29}
$$

Let i be an index such that

$$|x_i| \geq |x_j| \qquad \forall j \neq i. \tag{30}$$

Calculating Ax and considering the i-th piece gives

$$t \geq |(Ax)_i| \geq |A_{ii}x_i| - \left|\sum_{j\neq i} -A_{ij}x_j\right| \tag{31}$$

$$\geq \sigma_{\text{min}}(A_{ii})\,|x_i| - \sum_{j\neq i}|A_{ij}|\,|x_j| \tag{32}$$

$$\geq \Big(\sigma_{\text{min}}(A_{ii}) - \sum_{j\neq i}|A_{ij}|\Big)\,|x_i|\,. \tag{33}$$

Noting that $|x_i| \geq \frac{1}{n}$ gives (25).

Since $\sigma_{\text{min}}(A) > t > 0$, A is obviously invertible and since

$$\left|A^{-1}\right| = \sigma_{\text{max}}(A^{-1}) = \frac{1}{\sigma_{\text{min}}(A)}, \tag{34}$$

we have that $\left|A^{-1}\right| < \frac{1}{t}$. ■

Since F_P and G_P are block diagonal, calculating H_{PK} from Equations (18) gives that the i,jth block of H_{PK} is given by

$$\Big[H_{PK}\Big]_{i,j} = G_{i,K_j} - G_{i,P_i}\left(F_{i,P_i}\right)^{-1}F_{i,K_j}. \tag{35}$$

We note that $F_{i,K_i} = \mathbf{0}$, and thus $[H_{PK}]_{ii} = G_{i,K_i}$. Thus we can apply Lemma 2 to get the following theorem.

Theorem 2. *If*

$$\sigma_{min}\left(G_{i,K_i}\right) \geq nt + \sum_{j\neq i}\left|G_{i,K_j} - G_{i,P_i}\left(F_{i,P_i}\right)^{-1}F_{i,K_j}\right| \tag{36}$$

for all i, then H_{PK} is invertible and $\left|H_{PK}^{-1}\right| < \frac{1}{t}$.

Proof: The proof is a straight forward application of Lemma 2 to Equation (35). ■

Since G_{i,K_i} is positive definite and $\sigma_{\text{min}}(\cdot)$ is concave on the set of positive definite matrices, the LHS of (36) is concave. Since G_{i,P_i} and F_{i,P_i} are constant and G_{i,K_j} and F_{i,K_j} are affine in the optimization variables, the RHS of (36) is convex in the original optimization variables and t. Thus the conditions in (36) form a set of convex constraints that can be added to our optimization program to control $\left|H_{PK}^{-1}\right|$.

C. Convex Objective for Minimizing $|H_{PZ}|$

We note from (18) that G_P and F_P are constant given the problem parameters, and F_Z and G_Z are affine in the optimization variables. Thus $|H_{PZ}|$ is a convex function and can be added to our program as an objective.

D. Convex Program for Robust Pricing Design

In order to reduce $\left|H_{PK}^{-1}\right|\,|H_{PZ}|$, we can add the following objective and constraints to our original program.

$$\min_{t,\{R_i,P_i\}_{i\in\mathcal{I}}} \quad \frac{1}{t}\,|H_{PZ}| \qquad \text{s.t.} \quad (36) \quad \text{satisfied}, \quad t > 0. \tag{37}$$

To obtain a convex objective function, we substitute the objective in (37) for the convex objective

$$\min_{t,\{R_i,P_i\}_{i\in\mathcal{I}}} \quad \frac{1}{t}\,|H_{PZ}|^2 \qquad \text{s.t.} \quad (36) \quad \text{satisfied}, \quad t > 0. \tag{38}$$

We do not make a rigorous argument for doing this. The objective in (38) is only an upper bound on the objective in (37) when $|H_{PZ}| \geq 1$; however, the two objectives behave similarly for $|H_{PZ}| > 0$ and $t > 0$.

The addition of (38) to our original convex program ensures that H_{PK} is invertible and helps to reduce $\left|H_{PK}^{-1}\right|\,|H_{PZ}|$. We can also add a weighting matrix, $W_Z \succeq 0$, to the objective function in (38), i.e.

$$\frac{1}{t}\,|H_{PZ}W_Z|^2\,, \tag{39}$$

to account for any prior we have on the size of likely parameter perturbations.

IV. Numerical Examples

In this section, we construct several numerical examples to demonstrate the robustness objective and constraints. We solve the convex program using code written in MATLAB that employs YALMIP [14]. For the examples that follow, we consider the dynamical system defined by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-3 & 0.5 & 0.25\\ 0.25 & -1 & 0.5\\ 0.5 & 0.5 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + I_{3\times 3}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} \tag{40}$$

where $I_{3\times 3}$ is the identity matrix in $\mathbb{R}^{3\times 3}$. Each player has a nominal cost defined as follows: for each $i \in \{1,2,3\}$, $Q_i = Q$ where

$$[Q]_{jk} := \begin{cases} 0.1, & \text{if } k = j\\ 0, & \text{otherwise}\end{cases} \tag{41}$$

and $R_i = R$ where

$$[R]_{jk} := \begin{cases} 1, & \text{if } k = j\\ 0.1, & \text{if } k \neq i,\ j \neq i\\ 0, & \text{otherwise.}\end{cases} \tag{42}$$

We determine the desired equilibrium, $\{K_i\}_{i\in\mathcal{I}}$, by solving the centralized LQR problem with cost given by the sum of the players' nominal costs.

A. Perturbations in the Dynamics

In this first example, we consider perturbations in the system dynamics given in Equation (40). We calculate pricing matrices with various combinations of the robustness constraints and the robustness and revenue neutral objectives. For the robustness objective, we use a weighting matrix, W_Z, (as shown in Equation (39)) given by

$$W_Z = \begin{bmatrix}W_A & \mathbf{0}\\ \mathbf{0} & W_B\end{bmatrix} \qquad \begin{aligned} W_A &= \text{diag}(c_1\text{vec}(|A|))\\ W_B &= I_{n^2\times n^2}.\end{aligned} \tag{43}$$

For this example, we chose $c_1 = 1.5$.

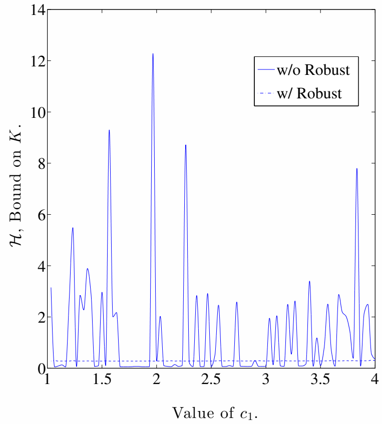

Fig. 1. Bound on perturbations in K ($\mathcal{H}$) resulting from varying c_1 in the prior W_A. The results depicted are from solving the pricing optimization problem with the revenue neutral objective.

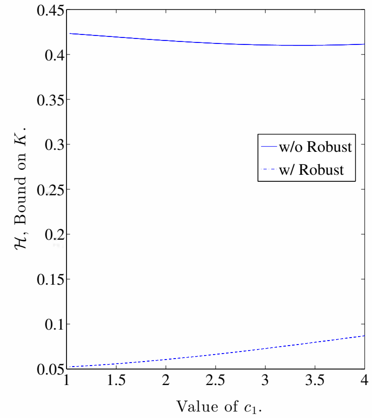

Fig. 2. Bound on perturbations in K ($\mathcal{H}$) resulting from varying c_1 in the prior W_A. The results depicted are from solving the pricing optimization problem without the revenue neutral objective.

$\mathcal{H}$[a]	w/ Robust[b]	w/o Robust
w/ Revenue[c]	0.214	3.828
w/o Revenue	0.053	0.4219

[a] $\mathcal{H} := \left|H_{PK}^{-1}\right| |H_{PZ} W_Z|$
[b] Robustness objective and constraints
[c] Revenue neutral objective

TABLE I

PERTURBATIONS IN K UNDER DIFFERENT PRICING OPTIMIZATION PROBLEM SOLUTIONS. THE TABLE INCLUDES THE VALUES OF $\mathcal{H}$ FOR THE DIFFERENT COMBINATIONS OF OBJECTIVES AND CONSTRAINTS.

We define $\mathcal{H} := \left|H_{PK}^{-1}\right| |H_{PZ} W_Z|$. In Table I, we report $\mathcal{H}$ under the various combinations of constraints and objectives. It is clear that the robustness objective and constraints greatly improve (by at least an order of magnitude) the bound on the perturbations in the feedback gains.

B. Varying Weighting Matrix W_A

We vary the constant c_1 to show the effect of changing the ratio between expected perturbations in the system matrix A and expected perturbations in B and plot $\mathcal{H}$ for different values of c_1. The results in Figure 1 are with the revenue neutral objective and the results in Figure 2 are without the revenue neutral objective. From the figures, one can see that the robustness objective and constraints improve the robustness bound in both cases, but in particular from Figure 1, we see that without the robustness objective and constraints, the revenue neutral objective can lead to prices with a very poor robustness bound.

V. DESIGNING PRICES FOR ROBUSTNESS TO PERTURBATIONS IN PLAYERS' FEEDBACK STRATEGIES

In addition to allowing us to design prices against the possibility of perturbations in the problem parameters, we can extend this analysis to design prices for robustness to perturbations in various players feedback controls. If a subset of players plays a set of feedback strategies different from their Nash strategies and then the other players play Nash in response, the set of players' strategies will shift away from the Nash equilibrium. However, unlike in the case of parameter perturbations after one such iteration, the game will no longer be at a Nash equilibrium because any subset of the players who deviated originally could re-optimize and improve their strategies. Thus a deviation in a subset of players' feedback strategies could lead to a back and forth iterative process where at each step some subset of players is playing Nash with respect to each other while the other players' controls remain fixed. We can extend the above analysis to cover this case of perturbations in $\{K_i\}_{i\in\mathcal{I}}$ by treating the controls of the players that played Nash at the previous step to be problem parameters for the players playing Nash at the current step. We consider a simplified case where the players are divided into two groups denoted by index sets $\mathcal{I}_a$ and $\mathcal{I}_b$ where $\mathcal{I}_a \cup \mathcal{I}_b = \mathcal{I}$ and $\mathcal{I}_a \cap \mathcal{I}_b = \emptyset$. The first group initially deviates from their Nash strategies, the second group then plays Nash in response, the first group then plays Nash in response to the second group, and so on. We can approximate the deviation in group a's controls in response to a deviation in group b's controls as

$$\begin{bmatrix}\partial\vec{P}_a\\ \partial\vec{K}_a\end{bmatrix} = -\begin{bmatrix}F_{a,P_\bullet} & F_{a,K_\bullet}\\ G_{a,P_\bullet} & G_{a,K_\bullet}\end{bmatrix}^{-1}\begin{bmatrix}F_{a,P_\blacktriangle} & F_{a,K_\blacktriangle}\\ G_{a,P_\blacktriangle} & G_{a,K_\blacktriangle}\end{bmatrix}\begin{bmatrix}\partial\vec{P}_b\\ \partial\vec{K}_b\end{bmatrix}. \tag{44}$$

Since $F_{a,P_\blacktriangle}$ and $G_{a,P_\blacktriangle}$ are both $\mathbf{0}$, we only need to consider the perturbations in K_a and K_b . Thus similar to Equation (23), we can calculate

$$\partial\vec{K}_a = -\left(H_{P_\bullet K_\bullet}\right)^{-1}\left(H_{P_\bullet K_\blacktriangle}\right)\partial\vec{K}_b \tag{45}$$

where

$$H_{P_a K_a} := G_{a,K_a} - G_{a,P_a} F_{a,P_a}^{-1} F_{a,K_a} \tag{46}$$
$$H_{P_a K_b} := G_{a,K_b} - G_{a,P_a} F_{a,P_a}^{-1} F_{a,K_b}. \tag{47}$$

If ∂K_b was initially a response to a previous perturbation in ∂K_a, we can calculate the total effect of one back and forth iteration on group a's controls, ∂K_a^+, as

$$\partial\vec{K}_a^+ = \left(H_{P_\bullet K_\bullet}^{-1} H_{P_\bullet K_\blacktriangle}\right)\left(H_{P_\blacktriangle K_\blacktriangle}^{-1} H_{P_\blacktriangle K_\bullet}\right)\partial\vec{K}_a. \tag{48}$$

We want to find a condition such that this update map from ∂K_a to ∂K_a^+ is a contraction. To this end, we state the following lemma.

Lemma 3. *For a block matrix*

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (49)$$

$$|A| \geq |A_{11}| \, . \quad (50)$$

The proof is trivial and is omitted. The result states that the norm of a matrix cannot increase by removing any number of rows or columns.

Theorem 3. *If H_{PK} satisfies the conditions of Theorem 2 and*

$$|H_{PK}| \leq C, \quad (51)$$

then $H_{P_\bullet K_\bullet}$ is invertible and

$$\left| H^{-1}_{P_\bullet K_\bullet} H_{P_\bullet K_\flat} \right| \leq \frac{C}{t} \quad (52)$$

for any two index subsets $\mathcal{I}_a, \mathcal{I}_b \subseteq \mathcal{I}$.

Proof: A simple calculation shows that the blocks of $H_{P_\bullet K_\bullet}$ have the same form as Equation (35) except for $i \in \mathcal{I}_a$ and $j \in \mathcal{I}_a$. Thus applying Lemma 2 to $H_{P_\bullet K_\bullet}$, we have that it is invertible and the norm of its inverse is bounded by $\frac{1}{t}$ if

$$\sigma_{\min}\left(G_{i,K_i}\right) \geq nt + \sum_{j \in \mathcal{I}_\bullet, j \neq i} \left| G_{i,K_j} - G_{i,P_i}\left(F_{i,P_i}\right)^{-1} F_{i,K_j} \right| \quad (53)$$

for all $i \in \mathcal{I}_a$. Clearly, if H_{PK} satisfies (36), then $H_{P_\bullet K_\bullet}$ satisfies (53). Since $H_{P_\bullet K_\flat}$ is a subblock of H_{PK} by Lemma 3, $|H_{P_\bullet K_\flat}| \leq |H_{PK}| \leq C$ and Equation (52) holds. ■

We now state a condition that guarantees that two groups of players iteratively playing Nash in response to each other after an initial perturbation in one group's controls will converge back to the Nash equilibrium.

Theorem 4. *Let $\mathcal{I}_a \subseteq \mathcal{I}$ and $\mathcal{I}_b \subseteq \mathcal{I}$ be index sets such that $\mathcal{I}_a \cup \mathcal{I}_b = \mathcal{I}$ and $\mathcal{I}_a \cap \mathcal{I}_b = \emptyset$. If H_{PK} satisfies the conditions in Theorem 3 for $t > 1$ and $C < 1$ (and thus $\left|H^{-1}_{PK}\right| |H_{PK}| < 1$), then for small enough initial perturbations in K_a or K_b, players $\mathcal{I}_a$ and $\mathcal{I}_b$ iteratively playing Nash in response to each other will eventually converge back to the original Nash equilibrium.*

Proof: We show that the update map in Equation (48) is a contraction. By Theorem 3, both

$$\left| H^{-1}_{P_a K_a} H_{P_a K_b} \right| < 1 \qquad \text{and} \qquad \left| H^{-1}_{P_\flat K_\flat} H_{P_\flat K_\bullet} \right| < 1 \quad (54)$$

and thus $\left|\partial \vec{K}_a^+\right| < \alpha \left|\partial \vec{K}_a\right|$ for some $\alpha < 1$. The same holds true for $\partial \vec{K}_b$. ■

This result is similar to the one presented in [7] and could be considered an extension to LQ games where there are more than two players.

Numerical techniques for calculating Nash equilibria of LQ games are limited (i.e. the only methods proven to converge are limited to the two player case where the R_i's are diagonal.) We conjecture that if the R_i's are forced to be diagonal, the set of pricing matrices that induces a desired equilibrium K is unique which does not give us freedom to select between prices. In addition, from running the simulations for Section IV, we also conjecture that non-zero off-diagonal terms in the R_i's are critical for designing prices that achieve stability of desired Nash strategies. For these reasons, we leave numerical simulations of the results in Section V for future work.

VI. Conclusion

In this paper, we presented additions to the convex pricing design program presented in [5] that can be used to design prices that reduce the effect of parameter perturbations on the Nash equilibrium of the game as well as ensure that the Nash equilibrium is stable with respect to small perturbations in the players' strategies. In the future, we plan to explore the trade-offs between the various objectives such as the revenue neutral objective and the robustness objective, better characterize the feasible set of prices that achieve different levels of robustness, explore better numerical techniques for calculating Nash equilibria of LQ games, and possibly explore designing prices that induce global stability of Nash equilibria in LQ games as opposed to just local stability.

References

[1] P. Antmann, "Reducing technical and non-technical losses in the power sector," World Bank Group Energy Sector, Tech. Rep., 2009.

[2] F. Cleveland, "Cyber Security Issues for Advanced Metering Infrasttructure (AMI)," in *Power and Energy Society General Meeting-Conversion and Delivery of Electrical Energy in the 21st Century*. IEEE, 2008, pp. 1–5.

[3] A. Ramrez-Arias, F. Rodrguez, J. Guzman, and M. Berenguel, "Multiobjective hierarchical control architecture for greenhouse crop growth," *Automatica*, vol. 48, no. 3, pp. 490 – 498, 2012.

[4] T. B. Smith, "Electricity Theft: A Comparative Analysis," *Energy Policy*, vol. 32, no. 18, pp. 2067 – 2076, 2004.

[5] L. Ratliff, S. Coogan, D. Calderone, and S. S. Sastry, "Pricing in linear-quadratic dynamic games," in *Fiftieth Annual Allerton Conference on Communication, Control, and Computing*, 2012.

[6] K. Zhou and J. C. Doyle, *Essentials of robust control*. Prentice Hall Upper Saddle River, NJ, 1998, vol. 104.

[7] A. Bressan and Z. Wang, "On the stability of the best reply map for noncooperative differential games," *Analysis and Applications*, vol. 10, no. 02, pp. 113–132, 2012.

[8] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed., ser. Classics in Applied Mathematics. SIAM, 1999.

[9] S. Li and T. Başar, "Distributed algorithms for the computation of noncooperative equilibria," *Automatica*, vol. 23, no. 4, pp. 523–533, 1987.

[10] M. Konstantinov, V. Angelova, P. Petkov, D. Gu, and V. Tsachouridis, "Perturbation bounds for coupled matrix riccati equations," *Linear algebra and its applications*, vol. 359, no. 1, pp. 197–218, 2003.

[11] J. M. Lee, *Introduction to smooth manifolds*. Springer, 2002, vol. 218.

[12] K. B. Petersen and M. S. Pedersen, "The matrix cookbook," nov 2012, version 20121115. [Online]. Available: http://www2.imm.dtu.dk/pubdb/p.php?3274

[13] J. M. Ortega, *Numerical analysis: A second course*. Society for Industrial Mathematics, 1987, vol. 3.

[14] J. Löfberg, "YALMIP : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. [Online]. Available: http://users.isy.liu.se/johanl/yalmip