Dynamic Tolling in Arc-based Traffic Assignment Models

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***Abstract*— Tolling in traffic networks offers a popular measure to minimize overall congestion. Existing toll designs primarily focus on congestion in *route-based* traffic assignment models (TAMs), in which travelers make a single route selection from source to destination. However, these models do not reflect real-world traveler decisions because they preclude deviations from a chosen route, and because the enumeration of all routes is computationally expensive. To address these limitations, our work focuses on *arc-based* TAMs, in which travelers sequentially select individual arcs (or edges) on the network to reach their destination. We first demonstrate that *marginal pricing*, a tolling scheme commonly used in route-based TAMs, also achieves socially optimal congestion levels in our arc-based formulation. Then, we use *perturbed best response* dynamics to model the evolution of travelers' arc selection preferences over time, and a marginal pricing scheme to capture the social planner's adaptive toll updates in response. We prove that our adaptive learning and marginal pricing dynamics converge to a neighborhood of the socially optimal loads and tolls. We then present empirical results that verify our theoretical claims.**

I. INTRODUCTION

Mitigating congestion on transportation networks is a key concern in urban planning, since the selfish behavior of individual drivers often significantly increases driving time and pollution levels. Congestion pricing (tolling) is an increasingly popular tool for regulating traffic flows ([1, 2]). The design of tolls that can effectually induce socially optimal traffic loads requires a realistic traffic assignment model (TAM) that captures travelers' routing preferences.

The classical literature on congestion pricing [3–5] often considers *route-based TAMs*, in which travelers make a single route selection at the origin node of the network, and do not deviate from their selected route until they reach the destination node. However, route-based modeling often requires enumerating all routes in a network, which may be computationally impractical, and do not capture correlations between the total costs of routes that share arcs. To address these issues, this work uses an *arc-based TAM* [6–11] to capture travelers' routing decisions. In this framework, travelers navigate through a traffic network by sequentially selecting among outgoing edges at each intermediate node. Designing tolls for arc-based TAMs is relatively under-studied, with the only exception of [11] where the authors show that, similar to route based TAMs, marginal tolling also achieves social optimality in arc-based TAMs.

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The basic philosophy of toll design is to steer the equilibrium behavior of agents towards social optimality by adding external incentives to their utility functions. However, a key assumption in this setting is that agents always adopt the equilibrium behavior, regardless of the incentives applied. This is not realistic, as real-world agents typically update their strategies from their initial strategies based on repeated interactions, only eventually converging to an equilibrium outcome [12]. While there exist learning rules for route-based TAMs which provably converge to the equilibrium strategies [13, 14], the development of analogous learning mechanisms for arc-based TAMs is relatively recent, e.g., in [10], which introduces a perturbed best response based dynamics. Consequently, it is necessary to study tolling in the presence of such dynamic adaptation rules by travelers.

Many prior works design tolls in dynamic environments by using reinforcement learning to iteratively update the toll on each arc. Chen et al. formulated the toll design problem as a Markov Decision Process (MDP) with high-dimensional state and action spaces, and apply a novel policy gradient algorithm to dynamically design tolls [15]. Mirzaei et al. used policy gradient methods to design incremental tolls on each link based on the difference between the observed and free-flow travel times [16]. Qiu et al. cast dynamic tolling into the framework of cooperative multi-agent reinforcement learning, and then applies graph convolutional networks to tractably solve the problem [17]. Likewise, Wang et al. use a cooperative actor-critic algorithm to tractably update a dynamic tolling scheme [18]. However, these methods operate on high-dimensional spaces, and are thus often computationally expensive. Moreover, they typically lack theoretical guarantees of convergence. The work most closely related to ours is [13] which studies dynamic tolling on parallel-link networks.

In this work, we study tolling in the arc-based TAM detailed in [10]. We show that there exists a unique toll that induces socially optimal congestion levels. Furthermore, we propose an adaptive tolling dynamics that steers the travelers' routing preferences towards socially optimal congestion levels on the network. Specifically, we implement *marginal cost tolling*, via a discrete-time dynamic tolling scheme that adjusts tolls on arcs, with the following key features:

1) Tolls are adjusted at each time step towards the direction of the current marginal cost of travel latency.
2) Tolls are updated at a much slower rate compared to the rate at which travelers update arc selections at each non-destination node (timescale separation).
3) The toll update of each arc only depends on "local information" (in particular, the flow on each arc), and

does not require the traffic authority to access the demands of travelers elsewhere on the network.

This form of adaptive tolling was first introduced in [13] to study dynamic tolling scheme for parallel-link networks. This work extends the scope of that tolling scheme to bidirectional traffic networks, in the context of arc-based TAMs.

We show that the tolling dynamics converges to a neighborhood of a fixed toll vector, the corresponding equilibrium flows of which we prove to be socially optimal. We also show that the travelers' arc selections converge to a neighborhood of this socially optimal equilibrium flow. Our proof is based on the constant step-size two-timescale stochastic approximation theory [19], which allows us to decouple the toll and arc selection dynamics, and establish their convergence via two separate Lyapunov-based proofs. Although marginal tolling provably leads to socially efficient traffic allocation in a route-based TAM framework [5], to the best of our knowledge, this work presents the first marginal tolling scheme that induces socially optimal traffic flows in an arc-based setting.

The rest of the paper is outlined as follows: In Section II we present the transportation network model we consider in this work and summarize the required preliminaries from [10] on arc-based TAM. Furthermore, we also introduce the equilibrium concept we consider in this work, along with the notion of social optimality. In Section III, we present properties of the optimal tolls which induce social optimality in this setup. In Section IV, we introduce the tolling dynamics and present the convergence results. In Section V, we present a numerical study which corroborate the theoretical findings of this paper. Finally we conclude this paper in VI and present some directions of future research.

Notation: For each positive integer $n \in \mathbb{N}$, we denote $[n] := \{1, \cdots, n\}$. For each $i \in [n]$ in an Euclidean space $\mathbb{R}^n$, we denote by e_i the i-th standard unit vector. Finally, let $\mathbf{1}\{\cdot\}$ denote the indicator function, which returns 1 if the input is true and 0 otherwise.

II. SETUP

Consider a traffic network described by a directed graph $G_O = (I_O, A_O)$, where I_O and A_O denote nodes and arcs, respectively. An example is shown in Figure 1 (top left); note that G_O can contain bidirectional arcs. Let the *origin nodes* and *destination nodes* be two disjoint subsets of I_O. To simplify our exposition, we assume that I_O contains only one origin $o \in I$ and one destination $d \in I$, although the results presented below straightforwardly extend to the multiple origin-destination-pair scenario. Travelers navigate through the network, from origin o to destination d, by sequentially selecting arcs at every intermediate node. This process produces congestion on each arc, which in turn determines travel times. The *cost* on each arc is then obtained by summing the travel time and toll. Specifically, each arc $a \in A_O$ is associated with a toll $p_a \in \mathbb{R}^{|A_O|}$, and a positive, strictly increasing *latency function* $s_a : [0, \infty) \to [0, \infty)$, which gives travel time as a function of traffic flow. The

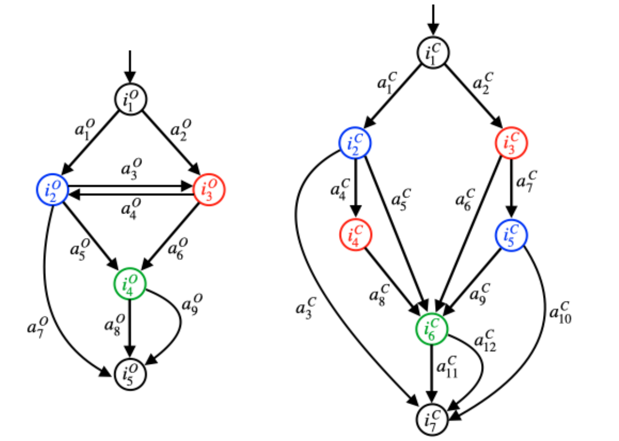

Fig. 1: Example of a single-origin single-destination original network G_O (top left, with superscript O), and its corresponding condensed DAG, or CoDAG, representation G (top right, with superscript C). Arc correspondences between the two networks are given by Table I, while node correspondences are indicated by color.

cost on arc $a \in A_O$ is then given by:

$$c_a(w_a, p_a) = s_{\tilde{a}}(w_a) + p_a.$$

Finally, let the demand of (infinitesimal) travelers entering from origin node o be denoted by g_o.

Note that sequential arc selection on networks with bidirectional arcs can result in a cyclic route. For example, a traveler navigating the left traffic network in Figure 1 using sequential arc selection may cycle between nodes i_2^O and i_3^O. To resolve this issue, we consider arc selection on the *condensed DAG (CoDAG) representation* of the original network G_O, a directed acyclic graph (DAG) representation, as proposed in [10]. The Condensed DAG representation preserves all acyclic routes from origin o to destination d in G_O, but precludes cyclic routes by design. Details regarding the construction and properties of CoDAG representations are provided in [10], Section II.

TABLE I: Arc correspondences between the graphs in Figure 1: The original network (top left) and the CoDAG (top right).

Original	a_1^O	a_2^O	a_3^O	a_4^O	a_5^O	a_6^O	a_7^O	a_8^O	a_9^O
CoDAG	a_1^T	a_2^T	a_4^T	a_7^T	a_5^T a_9^T	a_6^T a_8^T	a_3^T a_{10}^T	a_{11}^T	a_{12}^T

We define $[\cdot] : A \to A_O$ to be a map from each CoDAG arc $a \in A$ to the corresponding arc in the original graph, $[a] \in A_O$ (as shown in Table I). For each arc $a \in A$, let i_a and j_a denote the start and terminal nodes, and for each node $i \in I$, let $A_i^-, A_i^+ \subset A$ denote the set of incoming and outgoing arcs.

A. Cost Model

Below, we assume that every traveler has access to G_O, and to the same CoDAG representation $G = (I, A)$ of G_O; in particular, G is used to perform sequential arc selection to generate acyclic routes. The travelers' aggregative arc selections generate network congestion. Specifically, for each $a \in A$, let the *flow* or *congestion level* on arc a be denoted by w_a, and let the total flow on the corresponding arc in the original network be denoted, with a slight abuse of notation, by $w_{[a]} := \sum_{a' \in [a]} w_{a'}$[1]. Travelers perceive the cost on each arc $a \in A$ as:

$$\begin{aligned}\tilde{c}_{[a]}(w_{[a]}, p_{[a]}) &:= c_{[a]}(w_{[a]}, p_{[a]}) + \nu_a \\ &= s_{[a]}(w_{[a]}) + p_{[a]} + \nu_a,\end{aligned}$$

where ν_a is a zero-mean random variable. At each non-destination node $i \in I \backslash \{d\}$, travelers select among outgoing nodes $a \in A_i^+$ by comparing their perceived cost-to-go $\tilde{z}_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \to \mathbb{R}$, given recursively by:

$$\begin{aligned}\tilde{z}_a(w, p) &:= \tilde{s}_{[a]}(w_{[a]}) + p_{[a]} + \min_{a' \in A_{j_a}^+} \tilde{z}_{a'}(w, p), \quad j_a \neq d, \\ \tilde{z}_a(w, p) &:= \tilde{s}_{[a]}(w_{[a]}) + p_{[a]}, \qquad j_a = d.\end{aligned} \tag{1}$$

Consequently, the fraction of travelers who arrives at $i \in I \backslash \{d\}$ and choose arc $a \in A_i^+$ is given by:

$$P_{ij_a} := \mathbb{P}(\tilde{z}_a \leq \tilde{z}_{a'}, \forall a' \in A_i^+). \tag{2}$$

An explicit formula for the probabilities $\{P_{ij_a} : a \in A_i^+\}$, in terms of the statistics of $\tilde{z}_a$, is provided by the discrete-choice theory [20]. In particular, define $z_a(w) := \mathbb{E}[\tilde{z}_a(w)]$ and $\epsilon_a := \tilde{z}_a(w) - z_a(w)$, and define the latency-to-go at each node by:

$$\varphi_i(\{z_{a'}(w, p) : a' \in A_i^+\}) = \mathbb{E}\left[\min_{a' \in A_i^+} \tilde{z}_{a'}(w, p)\right]. \tag{3}$$

Then, from discrete-choice theory [20]:

$$P_{ij_a} = \frac{\partial \varphi_i}{\partial z_a}(z), \quad i \in I \backslash \{d\}, a \in A_i^+, \tag{4}$$

where, with a slight abuse of notation, we write $\varphi_i(z)$ for $\varphi_i(\{z_{a'} : a' \in A_i^+\})$.

To obtain a closed-form expression of φ, we employ the *logit Markovian model* [6, 7], under which the noise terms ϵ_a are described by the Gumbel distribution with scale parameter β. As a result, the expected minimum cost-to-go $z_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \to \mathbb{R}$, associated with traveling on each arc $a \in A$, assumes the following form:

$$\begin{aligned}&z_a(w, p) \\ &= s_{[a]}\left(\sum_{\bar{a} \in [a]} w_{\bar{a}}\right) + p_{[a]} - \frac{1}{\beta} \ln\left(\sum_{a' \in A_{j_a}^+} e^{-\beta z_{a'}(w,p)}\right).\end{aligned} \tag{5}$$

Note that (5) is well-posed, as z_a can be recursively computed from the destination back to the origin ([10], Section III).

[1]Unlike existing TAMs, in our model, the latency of arcs in G can be coupled, since multiple copies of the same arc in G_O may exist in G.

B. CoDAG Equilibrium

Here, we define the *condensed DAG (CoDAG) equilibrium* (Definition 1), based on the CoDAG representation of the original traffic network. Specifically, we show that the CoDAG equilibrium exists, is unique, and solves a strictly convex optimization problem (Theorem 1).

*Definition 1 (**Condensed DAG Equilibrium**):* Fix a toll vector $p \in \mathbb{R}^{|A_O|}$, and fix $\beta > 0$. We call an arc-flow vector $\bar{w}^\beta(p) \in \mathbb{R}^{|A|}$ a *Condensed DAG (CoDAG) equilibrium at* p if, for each $i \in I \backslash \{d\}$, $a \in A_i^+$:

$$\bar{w}_a^\beta(p) \tag{6}$$

$$= \left(g_i + \sum_{a' \in A_i^+} \bar{w}_{a'}^\beta(p)\right) \frac{\exp(-\beta z_a(\bar{w}^\beta(p), p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(\bar{w}^\beta(p), p))}, \tag{7}$$

where $g_i = g_0 \cdot \mathbf{1}(i = o)$, and $w \in \mathcal{W}$, where:

$$\begin{aligned}\mathcal{W} := \Bigg\{ w \in \mathbb{R}^{|A|} : \sum_{a \in A_i^+} w_a = \sum_{a \in A_i^-} w_a, \forall i \neq o, d, \\ \sum_{a \in A_o^+} w_a = g_o,\ w_a \geq 0, \forall a \in A \Bigg\}\end{aligned} \tag{8}$$

characterizes the conservation of flow in the CoDAG G. Note that $\mathcal{W}$ is convex and compact.

At a CoDAG equilibrium $\bar{w}^\beta(p)$, the fraction of travelers at any intermediate node $i \in I \backslash \{d\}$ who selects an arc $a \in A_i^+$ is given by $\bar{\xi}_a^\beta(p)$, as defined below:

$$\bar{\xi}_a^\beta(p) := \frac{\bar{w}_a^\beta(p)}{\sum_{a' \in A_i^+} \bar{w}_{a'}^\beta(p)}.$$

The CoDAG equilibrium bears some resemblance to the Markovian Traffic Equilibrium (MTE) introduced in Baillon and Cominetti [7]. However, the CoDAG formulation by design precludes the possibility of assigning cyclic routes, and is capable of capturing couplings between arcs in the CoDAG G that correspond to the same arc in the original network G_O (see [10], Remark 6).

Below, we show that, given any CoDAG representation G of an original network G_O and any fixed toll vector $p \in \mathbb{R}^{|A_O|}$, the CoDAG equilibrium exists and is unique. Specifically, the CoDAG equilibrium is the unique minimizer of a strictly convex optimization problem over a compact set. This characterization provides powerful insight into the mathematical properties of the CoDAG equilibrium flow, and its dependence on the toll vector. These properties will be used in our work to establish the existence of an optimal toll (Theorem 2) and the convergence of our discrete-time toll dynamics to the optimal toll (Theorem 3).

For each $[a] \in A_O$, define $F : \mathcal{W} \times \mathbb{R}^{|A_O|} \to \mathbb{R}$ by:

$$\begin{aligned}&F(w, p) \\ &= \sum_{[a] \in A_O} \int_0^{w_{[a]}} \left[s_{[a]}(u) + p_{[a]}\right] du\end{aligned}$$

$$+\frac{1}{\beta}\sum_{i\neq d}\left[\sum_{a\in A_i^+} w_a \ln w_a - \left(\sum_{a\in A_i^+} w_a\right)\ln\left(\sum_{a\in A_i^+} w_a\right)\right]. \tag{9}$$

Theorem 1: For each fixed toll vector $p \in \mathbb{R}^{|A_O|}$, the corresponding CoDAG equilibrium $\bar{w}^\beta(p) \in \mathcal{W}$ exists, is unique, and is the unique minimizer of $F(\cdot, p)$ over $\mathcal{W}$.

Proof: (**Proof Sketch**) The proof parallels that of [10], Theorem 1 and Lemma 1. For details, please see [10], Section III and Appendix B. ∎

C. Social Optimality

We now describe the socially optimal flow which would lead to the most efficient use of the transportation network. More specifically, we define below the notion of *perturbed social optimality* considered in our work.

*Definition 2 (**Perturbed Socially Optimal Flow**):* We define a *perturbed socially optimal flow* with regularization parameter $\beta > 0$ to be a minimizer of the following convex optimization problem:

$$\min_{w\in\mathcal{W}} \quad \sum_{[a]\in A_O} w_{[a]}\cdot s_{[a]}(w_{[a]})$$
$$+\frac{1}{\beta}\sum_{i\neq d}\left[\sum_{a\in A_i^+} w_a \ln w_a - \left(\sum_{a\in A_i^+} w_a\right)\ln\left(\sum_{a\in A_i^+} w_a\right)\right],$$

with $\mathcal{W}$ given by (8), and $w_{[a]} := \sum_{a'\in[a]} w_{a'}$, as defined above.

In words, perturbed social optimality is characterized as the total latency experienced by travelers on each arc of the CoDAG G, augmented by an entropy term with regularization parameter β which captures stochasticity in the travelers' arc selections.

III. OPTIMAL TOLL: EXISTENCE AND UNIQUENESS

Below, we characterize the *optimal toll* $\bar{p} \in \mathbb{R}^{|A_O|}$ for which the corresponding CoDAG equilibrium $\bar{w}^\beta(\bar{p})$ is perturbed socially optimal (see Definition 2). Throughout the rest of the paper, we call $\bar{p}$ the *optimal toll*.

Theorem 2: There exists a unique toll vector $\bar{p} \in \mathbb{R}^{|A_0|}$ that satisfies the following fixed-point equation:

$$\bar{p}_{[a]} = \bar{w}^\beta_{[a]}(\bar{p})\cdot\frac{ds_{[a]}}{dw}\bar{w}^\beta_{[a]}(\bar{p}), \quad \forall\, a \in A. \tag{10}$$

Moreover, $\bar{w}^\beta(\bar{p})$, the CoDAG equilibrium flow distribution corresponding to $\bar{p}$, is the perturbed socially optimal flow with regularization β.

To prove Theorem 2, we first show that $\bar{w}^\beta(p)$ is continuous and monotonic in the toll p (Lemmas 1 and 2). Then, we use these properties to establish the existence and uniqueness of a toll vector $\bar{p} \in \mathbb{R}^{|A_O|}$ satisfying the fixed-point equation (10) (Lemma 3). Finally, we prove that the CoDAG equilibrium flow allocation $\bar{w}^\beta(\bar{p})$ corresponding to $\bar{p}$ is perturbed socially optimal (Lemma 4).

Below, we begin by establishing that the CoDAG equilibrium $\bar{w}^\beta(p)$ is a continuously differentiable and monotonic function of the toll $p \in \mathbb{R}^{|A_O|}$.

Lemma 1: $\bar{w}^\beta(p)$ is continuously differentiable in p.

Proof: (**Proof Sketch**) For each fixed toll vector $p \in \mathbb{R}^{|A_O|}$, the corresponding CoDAG equilibrium $\bar{w}^\beta(p)$ uniquely solves the KKT conditions of the optimization problem of minimizing $F(\cdot, p)$ over $\mathcal{W}$ (Theorem 1). We write these KKT conditions as an *implicit function* $J : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \to \mathbb{R}^{|A|}$ of the flow and tolls (w, p):

$$J(w, p) = \mathbf{0},$$

where $\mathbf{0}$ denotes the $|A|$-dimensional zero vector. We can then derive an explicit expression for $\frac{d\bar{w}^\beta}{dp}(p)$ at each $p \in \mathbb{R}^{|A_O|}$ by proving that:

$$\frac{\partial J}{\partial w}\big(\bar{w}^\beta(p), p\big) \in \mathbb{R}^{|A|\times|A|}$$

is non-singular for each fixed p, and invoking the Implicit Function Theorem. For details, please see Appendix A.1 [21]. ∎

Lemma 2: For any $p, p' \in \mathbb{R}^{|A_0|}$:

$$\sum_{a\in A}\left(\bar{w}^\beta_a(p') - \bar{w}^\beta_a(p)\right)(p'_{[a]} - p_{[a]}) \leq 0.$$

Proof: (**Proof Sketch**) By Theorem 1, the CoDAG equilibrium $\bar{w}^\beta(p)$ is the unique minimizer of the strictly convex function $F(\cdot, p) : \mathcal{W} \to \mathbb{R}$ defined by (9). Thus, $\bar{w}^\beta(p)$ can be characterized by the first-order optimality conditions of this optimization problem. This in turn allows us to establish monotonicity. For details, please see Appendix A.2 [21]. ∎

We then use the above lemmas to prove that the fixed-point equation (10) yields a unique solution.

Lemma 3: There exists a unique $\bar{p} \in \mathbb{R}^{|A_O|}$ satisfying (10):

$$\bar{p}_{[a]} = \bar{w}^\beta_{[a]}(\bar{p})\cdot\frac{ds_{[a]}}{dw}\big(\bar{w}^\beta_{[a]}(\bar{p})\big), \qquad \forall\,[a] \in A_O.$$

Proof: (**Proof Sketch**) Existence follows from the Brouwer fixed point theorem, since $\bar{w}^\beta(p)$ is continuous in p (Lemma 1). Uniqueness follows via a contradiction argument; we show that the existence of two distinct fixed points of (10) would violate the monotonicity established by Lemma 2. For details, please see Appendix A.3 [21]. ∎

Finally, we prove that the CoDAG equilibrium flow corresponding to $\bar{p} \in \mathbb{R}^{|A_O|}$ is perturbed socially optimal.

Lemma 4: $\bar{w}^\beta(\bar{p})$ is perturbed socially optimal.

Proof: (**Proof Sketch**) This follows by comparing the KKT conditions satisfied by $\bar{w}^\beta(\bar{p})$ (Theorem 1) with the KKT conditions of the optimization problem that defines the perturbed socially optimal flow in Definition 2. For details, please see Appendix A.4 [21]. ∎

Together, Lemmas 1, 2, 3, and 4 prove Theorem 2.

IV. DYNAMICS AND CONVERGENCE

A. Discrete-time Dynamics

Here, we present discrete-time stochastic dynamics that describes the evolution of the traffic flow and tolls on the network. Formally, g_o units of traveler flow enter the network at the origin node o at each time step $n \geq 0$. At each non-destination node $i \in I\backslash\{d\}$, a $\xi_a[n]$ fraction of travelers chooses an outgoing arc $a \in A_i^+$. We shall refer to $\xi_a[n]$ as the *aggregate arc selection probability*. Consequently, the flow induced on any arc $a \in A$ satisfies:

$$W_a[n] = \left(g_{i_a} + \sum_{a' \in A_{i_a}^+} W_{a'}[n]\right) \cdot \xi_a[n]. \tag{11}$$

At the conclusion of every time step n, travelers reach the destination node d and observe a noisy estimate of the cost-to-go values and tolls on all arcs in the network (including arcs not traversed during that time step). Let $K_i > 0$ denote node-dependent constants, and let $\{\eta_i[n+1] \in \mathbb{R} : i \in I, n \geq 0\}$ be independent bounded random variables[2] in $[\underline{\mu}, \overline{\mu}]$, with $0 < \underline{\mu} < \mu < \overline{\mu} < 1/\max\{K_i : i \in I\backslash\{d\}\}$ and $\mathbb{E}[\eta_{i_a}[n+1]] = \mu$ at each node $i \in I$ and discrete time index $n \geq 0$. At each time $n+1$ and non-destination node $i \in I\backslash\{d\}$, a $\eta_i[n+1] \cdot K_i$ fraction of travelers at node $i \in I$ observes the latencies on each arc, and decides to switch to the outgoing arc that minimizes the (stochastic) observed cost-to-go. Meanwhile, $1 - \eta_i[n+1] \cdot K_i$ fraction of travelers selects the same arc they used at time step n. Thus, the arc selection probabilities evolve according to the following *perturbed best-response dynamics*:

$$\begin{aligned} &\xi_a[n+1] \\ &= \xi_a[n] + \eta_{i_a}[n+1] \cdot K_{i_a} \\ &\quad \cdot \left(- \xi_a[n] + \frac{\exp(-\beta\big[z_a(W[n], P[n])\big])}{\sum_{a' \in A_{i_a}^+} \exp(-\beta\big[z_{a'}(W[n], P[n])\big])}\right). \end{aligned} \tag{12}$$

We assume that $\xi_a[0] > 0$ for each $a \in A$, i.e., each arc has some strictly positive initial traffic flow. This captures the stochasticity in travelers' perception of network congestion that causes each arc to be assigned a nonzero probability of being selected.

At each time step $n+1 \geq 0$, the tolls $P_{[a]}[n] \in \mathbb{R}^{|A_O|}$ on each arc $[a] \in A_O$ are updated by interpolating between the tolls implemented at time step n, and the marginal latency of that arc given the flow at time step n. That is:

$$\begin{aligned} &P_{[a]}[n+1] \\ &= P_{[a]}[n] + \gamma \left(-P_{[a]}[n] + W_{[a]}[n] \cdot \frac{ds_{[a]}}{dw}(W_{[a]}[n])\right), \end{aligned} \tag{13}$$

with $\gamma \in (0,1)$[3], where with a slight abuse of notation, we denote $W_{[a]} := \sum_{a' \in [a]} W_{a'}$. Note that the update (13) is distributed, i.e., for each arc in the original network, the updated toll depends only on the flow of that arc, and not on the flow of any other arc. Moreover, we assume that $\gamma \ll \mu$, i.e., the toll updates (13) occur at a slower timescale compared to the arc selection probability updates (12).

To simplify our study of the convergence of the dynamics (12) and (13), we assume that the arc latency functions are affine in the congestion on the link.

[2] The random variables $\{\eta_a[n] : a \in A, n \geq 0\}$ are assumed to be independent of travelers' perception uncertainties.

[3] Our result also holds if γ is a random variable with bounded support.

Assumption 1: Each arc latency function $s_{[a]}$ is affine, i.e.,:

$$s_{[a]}(w_{[a]}) = \theta_{\tilde{a},1} w_{[a]} + \theta_{[a],0}, \tag{14}$$

for some $\theta_{[a],1}, \theta_{[a],0} > 0$.

Under Assumption 1, the toll dynamics (13) can be alternatively written as follows

$$P_{[a]}[n+1] = P_{[a]}[n] + \gamma \left(-P_{[a]}[n] + W_{[a]}[n] \cdot \theta_{[a],1}\right). \tag{15}$$

B. Convergence Results

In this subsection, we show that the arc selection probability and toll updates (12)-(15) converge in the neighborhood of the socially optimal flow $\bar{w}^\beta(\bar{p})$ and the corresponding toll $\bar{p}$ respectively.

Theorem 3: The joint evolution of arc selection probability and toll updates (12)-(15) satisfies

$$\begin{aligned} &\limsup_{n \to \infty} \mathbb{E}\big[\|\xi[n] - \bar{\xi}^\beta(\bar{p})\|_2^2 + \|P[n] - \bar{p}\|_2^2\big] \\ &= O\left(\mu + \frac{\gamma}{\mu}\right). \end{aligned}$$

Consequently, for each $\delta > 0$:

$$\begin{aligned} &\limsup_{n \to \infty} \mathbb{P}\big(\|\xi[n] - \bar{\xi}^\beta(\bar{p})\|_2^2 + \|P[n] - \bar{p}\|_2^2 \geq \delta\big) \\ &= O\left(\frac{\mu}{\delta} + \frac{\gamma}{\delta\mu}\right). \end{aligned}$$

To prove Theorem 3, we employ the theory of two-timescale stochastic approximation [22]. Consequently, the asymptotic behavior of (12)-(15) can be characterized by studying the convergence properties of the corresponding continuous-time dynamical system. Since the tolls are updated at a slower rate compared to the traffic flows ($\gamma \ll \mu$), we consider the evolution of continuous-time flows $w(t)$ under a fixed toll $p \in \mathbb{R}^{|A_O|}$, and continuous-time tolls $p(t)$ with flow converged at the corresponding CoDAG equilibrium $\bar{w}^\beta(p(t))$ at each time. Specifically, for any fixed toll $p \in \mathbb{R}^{|A_O|}$, on each arc $a \in A$, the arc selection probabilities evolve as follows:

$$w_a(t) = \xi_a(t) \cdot \left(g_{i_a} + \sum_{a' \in A_{i_a}^-} w_{a'}(t)\right), \tag{16}$$

$$\dot{\xi}_a(t) = K_{i_a} \cdot \left(-\xi_a(t) + \frac{\exp(-\beta \cdot z_a(w(t), p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta \cdot z_{a'}(w(t), p))}\right). \tag{17}$$

Meanwhile, on each arc $[a] \in A_O$ in the original network, we consider the following continuous-time toll dynamics:

$$\dot{p}_{[a]}(t) = -p_{[a]}(t) + \bar{w}_{[a]}^\beta(p(t)) \cdot \theta_{[a],1}. \tag{18}$$

We prove that, for each fixed toll $p \in \mathbb{R}^{|A_O|}$, the corresponding continuous-time ξ-dynamics (17) globally asymptotically converges to the corresponding CoDAG equilibrium $\bar{w}^\beta(p) \in \mathbb{R}^{|A|}$. Moreover, the continuous-time toll dynamics (18) globally converges to the optimal toll $\bar{p} \in \mathbb{R}^{|A_O|}$.

*Lemma 5 (**Informal**):* Suppose $w(0) \in \mathcal{W}$, i.e., the initial flow satisfies flow continuity. Under the continuous-time flow dynamics (17) and (16), if $K_i \ll K_{i'}$ whenever $\ell_i < \ell_{i'}$, the continuous-time traffic allocation $w(t)$ globally asymptotically converges to the corresponding CoDAG equilibrium $\bar{w}^\beta(p)$.

Proof: (**Proof Sketch**) The following proof sketch parallels that of [10], Lemma 2, and is included for completeness. Recall that Theorem 1 establishes $\bar{w}^\beta(p)$ as the unique minimizer of the map $F(\cdot, p) : \mathcal{W} \to \mathbb{R}$, defined by (9). We show that $F(\cdot, p)$ is a Lyapunov function for the continuous-time flow dynamics induced by (17). To this end, we first unroll the dynamics (17) using (16), as follows:

$$\begin{aligned}
&\dot{w}_a(t) \\
&= -K_{i_a} \cdot \left(1 - \frac{1}{K_{i_a}} \cdot \frac{\sum_{a' \in A_{i_a}^-} \dot{w}_{a'}(t)}{\sum_{\hat{a} \in A_{i_a}^+} w_{\hat{a}}(t)}\right) w_a(t) \\
&\quad + K_{i_a} \cdot \sum_{a' \in A_{i_a}^-} w_{a'}(t) \cdot \frac{\exp(-\beta z_a(w(t), p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(w(t), p))}.
\end{aligned}$$

Next, we establish that if $w(0) \in \mathcal{W}$, then for each $t \geq 0$:

$$\dot{F}(t) = \dot{w}(t)^\top \nabla_w F(w(t)) \leq 0.$$

The proof then follows from LaSalle's Theorem (see [23, Proposition 5.22]). For a precise statement of Lemma 5, please see Appendix B.1 [21]; for the proof of the analogous theorem in [10], please see [10] Appendix C.1. ■

Lemma 6: The continuous-time toll dynamics (18) globally exponentially converges to the CoDAG equilibrium $\bar{w}^\beta(\bar{p})$ corresponding to the optimal toll $\bar{p}$.

Proof: Define $D \in \mathbb{R}^{|A_O| \times |A_O|}$ to be the diagonal and symmetric positive definite matrix whose $[a]$-th diagonal element is given by:

$$\frac{ds_{[a]}}{dw}\left(\bar{w}_{[a]}^\beta(\bar{p})\right) = \theta_{[a],1} > 0,$$

for each $[a] \in A_O$. Note that D is independent of the toll p. Now, consider the Lyapunov function $V : \mathbb{R}^{|A_O|} \to \mathbb{R}$, defined by:

$$V(p) := \frac{1}{2}(p - \bar{p})^\top D^{-1}(p - \bar{p}).$$

The trajectory of the continuous-time toll dynamics (18), starting at $p(0)$, satisfies:

$$\begin{aligned}
&\dot{V}(p(t)) \\
&= (p(t) - \bar{p})^\top D^{-1} \dot{p}(t) \\
&= \sum_{[a] \in A_O} \frac{(p_{[a]}(t) - \bar{p}_{[a]})}{\theta_{[a],1}} \cdot \left(-p_{[a]}(t) + \theta_{[a],1} \bar{w}_{[a]}^\beta(p(t))\right) \\
&= \sum_{[a] \in A_O} \frac{(p_{[a]}(t) - \bar{p}_{[a]})}{\theta_{[a],1}} \\
&\qquad \cdot \left(-p_{[a]}(t) + \bar{p}_{[a]} - \bar{p}_{[a]} + \theta_{[a],1} \bar{w}_{[a]}^\beta(p(t))\right) \\
&= -2V(p(t)) \\
&\qquad + \sum_{[a] \in A_O} (p_{[a]}(t) - \bar{p}_{[a]}) \left(\bar{w}_{[a]}^\beta(p(t)) - \bar{w}_{[a]}^\beta(\bar{p})\right) \\
&\leq -2V(p(t)),
\end{aligned}$$

where the final inequality follows due to the monotonicity of the map $\bar{w}^\beta(\cdot)$ (Lemma 2). ■

To conclude the proof of Theorem 3, it remains to check that the discrete-time dynamics (12)-(15), and the continuous-time dynamics (17)-(18), satisfy the technical conditions in Lemmas 7 and 8. In particular, Lemma 7 establishes that flows and tolls are uniformly bounded across the arc and time indices, while Lemma 8 asserts that the continuous-time flow and toll dynamics maps are Lipschitz continuous.

Lemma 7: The continuous-time flow and toll dynamics induced by (12)-(15) satisfy:

1) For each $a \in A$: $\{M_a[n+1] : n \geq 0\}$ is a martingale difference sequence with respect to the filtration $\mathcal{F}_n := \sigma\big(\cup_{a \in A} (W_a[1], \xi[1], p[1], \cdots, W_a[n], \xi[n], p[n])\big)$.
2) There exist $C_w, C_m, C_p > 0$ such that, for each $a \in A$ and each $n \geq 0$, we have $W_a[n] \in [C_w, g_o]$, $P_a[n] \in [0, C_p]$, and $|M_a[n]| \leq C_m$.

Likewise, the continuous-time flow and toll dynamics induced by (17) and (18) satisfy:

3) For each $a \in A$, $t \geq 0$, we have $w_a(t) \in [C_w, g_o]$ and $p_a(t) \in [0, C_p]$.

Proof: Please see Appendix B.2 [21]. ■

Lemma 8: The continuous-time flow dynamics (16) and toll dynamics (18) satisfy:

1) The map $\bar{\xi}^\beta : \mathbb{R}^{|A_O|} \to \mathbb{R}^{|A|}$ is Lipschitz continuous.
2) For each $a \in A$, the restriction of the cost-to-go map $z_a : \mathcal{W} \times \mathbb{R}^{|A_O|} \to \mathbb{R}$ to the set of realizable flows and tolls, i.e., $\mathcal{W}' \times [0, C_p]^{|A_O|}$, is Lipschitz continuous.
3) The map from the probability transitions $\xi \in \prod_{i \in I \setminus \{d\}} \Delta(A_i^+)$ and the traffic flows $w \in \mathcal{W}$ is Lipschitz continuous.
4) For each $a \in A$, the restriction of the continuous dynamics transition map $\rho_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \to \mathbb{R}^{|A|}$, defined recursively as follows for each $a \in A$:
$$\rho_a(\xi, p) := -\xi_a + \frac{\exp(-\beta z_a(w, p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(w, p))}$$
to the set of realizable flows and tolls, i.e., $\mathcal{W}' \times [0, C_p]^{|A_O|}$, is Lipschitz continuous.
5) For each $a \in A$, the map $r_{[a]} : \mathbb{R}^{|A_O|} \times \mathbb{R}^{|A_O|}$, defined as follows for each $a \in A$:
$$r_{[a]}(p) := -p_{[a]} + \bar{w}_{[a]}^\beta(p) \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(p)),$$
is Lipschitz continuous.

Proof: Please see Appendix B.3 [21]. ■

V. EXPERIMENT RESULTS

This section presents experiments that validate the theoretical convergence results of Section IV. We present simulation results illustrating that, under (12)-(15), the traffic flows and tolls converge to a neighborhood of the socially optimal values, as claimed by Theorem 3.

Consider the network presented in Figure 1, following affine latency functions (14) with parameters given in Table II. To validate Theorem 3, we evaluate and plot the traffic

TABLE II: Parameters for simulation.

Notation	Default value
$\theta_{\tilde{a},0}$	0, 1, 0, 1, 1, 0, 1, 1, 1 (ordered by edge index)
$\theta_{\tilde{a},1}$	2, 1, 1, 1, 1, 1, 2, 2, 2 (ordered by edge index)
g_1	1
β	10
γ	0.02
$\eta_{i_a}[n]$	Uniform$(0, 0.1)$, $\forall a \in A, i \in I \backslash \{d\}$

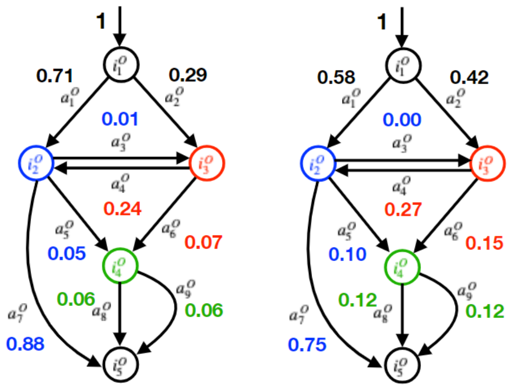

Fig. 2: Steady state traffic flow on each arc for the original network before (left) and after (right) tolls. Flows on arcs emerging from the same node are represented in the same color.

flow values $W_a[n]$ and toll values $P_a[n]$ on each arc $a \in A$ with respect to discrete time index $n \geq 0$. Figure 2 presents traffic flow values at the condensed DAG equilibrium (i.e., w^β) for the original network before and after tolls. Meanwhile, Figure 3 and 4 illustrate that w and p converge to the condensed DAG equilibrium in approximately 300 iterations. As in [10], flow convergence to the optimal allocation occurs even when the constants $\{K_i : i \in I\}$ are simply all set to 1. While the original traffic distribution is more concentrated on a few routes, tolls can distribute the traffic more evenly. This shows that tolls can improve overall social welfare by reducing congestion in over-utilized routes.

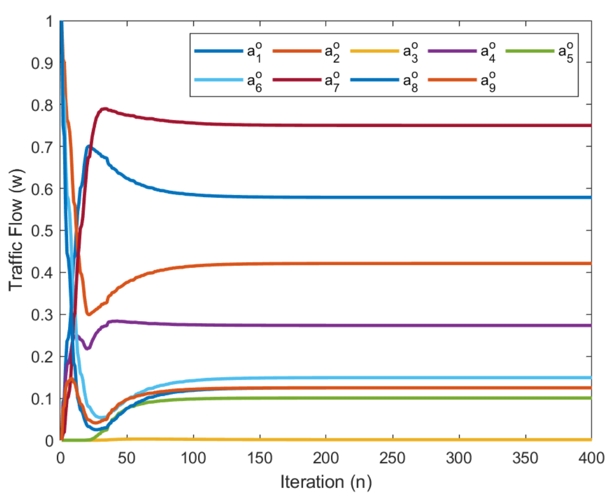

Fig. 3: Traffic flow W vs. time index n for the condensed DAG in Figure 1.

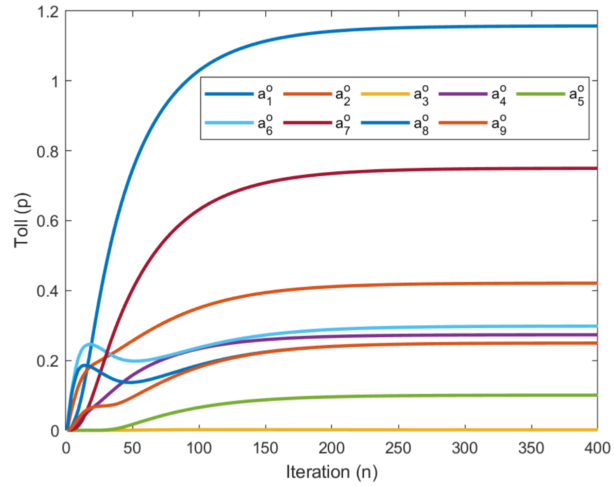

Fig. 4: Toll P vs. time index n for the condensed DAG in Figure 1.

VI. Conclusion and Future Work

This work introduces a discrete-time adaptive tolling scheme to minimize the total travel latency in a general traffic network with bidirectional edges. Our model assumes that, at each time, players near-instantaneously react via perturbed best response to the announced tolls. Accordingly, we formulate a two-timescale stochastic dynamical system that describes the joint evolution of traffic flow and tolls. We prove that the fixed point of these dynamics is unique and corresponds to the optimal traffic flow allocation from the perspective of minimizing the total travel time. Moreover, we prove that the stochastic dynamics converges to a neighborhood of the unique fixed point with high probability. Finally, we present simulation results that corroborate our theoretical findings.

Interesting avenues of future research include: (1) Extending our theoretical analysis to the setting where the latency function of each arc is not necessarily affine, (2) Developing tolling dynamics for the setting in which the central authority must learn the network latency functions and entropy regularization parameter $\beta > 0$ while simultaneously implementing an adaptive tolling scheme that converges to the optimal toll, and (3) Designing robust tolls for traffic networks in which some fraction of the population behaves unexpectedly or adversarially.

References

[1] Michael Manville. "How and Why Would Congestion Pricing Work?" In: (2021).

[2] Ana Ley. "How Might Congestion Pricing Actually Work in New York?" In: (2023).

[3] Hai Yang and Hai-Jun Huang. "Principle of Marginal-Cost Pricing: How Does it Work in a General Road Network?" In: *Transportation Research Part A: Policy and Practice* 32.1 (1998), pp. 45–54.

[4] Michael Florian and Donald Hearn. *Network Equilibrium and Pricing*. Vol. 23. Jan. 2003, pp. 373–411. DOI: 10.1007/0-306-48058-1_11.

[5] Tim Roughgarden. “Algorithmic Game Theory”. In: *Communications of the ACM* 53.7 (2010), pp. 78–86.

[6] Takashi Akamatsu. “Decomposition of Path Choice Entropy in General Transport Networks”. In: *Transportation Science* 31.4 (Nov. 1997), pp. 349–362. DOI: 10.1287/trsc.31.4.349.

[7] Jean-Bernard Baillon and Roberto Cominetti. “Markovian Traffic Equilibrium”. In: *Mathematical Programming* (Feb. 2008). DOI: 10.1007/s10107-006-0076-2.

[8] Yuki Oyama and Eiji Hato. “A Discounted Recursive Logit Model for Dynamic Gridlock Network Analysis”. In: *Transportation Research Part C: Emerging Technologies* 85 (2017), pp. 509–527.

[9] Yuki Oyama and Eiji Hato. “Prism-based Path Set Restriction for Solving Markovian Traffic Assignment Problem”. In: *Transportation Research Part B: Methodological* 122 (2019), pp. 528–546.

[10] Chih-Yuan Chiu, Chinmay Maheshwari, Pan-Yang Su, and Shankar Sastry. “Arc-based Traffic Assignment: Equilibrium Characterization and Learning”. In: *arXiv* (2023). eprint: 2304.04705.

[11] Noriko Kanekoa, Daisuke Fukudab, and Qian Gec. “Optimal Congestion Tolling Problem under the Markovian Traffic Equilibrium”. In: *Sustainability* (2021).

[12] Drew Fudenberg and David K Levine. *The Theory of Learning in Games*. Vol. 2. MIT press, 1998.

[13] Chinmay Maheshwari, Kshitij Kulkarni, Manxi Wu, and S. Shankar Sastry. “Dynamic Tolling for Inducing Socially Optimal Traffic Loads”. In: *2022 American Control Conference (ACC)*. 2022, pp. 4601–4607. DOI: 10.23919/ACC53348.2022.9867193.

[14] William H. Sandholm. *Population Games And Evolutionary Dynamics*. Economic Learning and Social Evolution, 2010.

[15] Haipeng Chen, Bo An, Guni Sharon, Josiah Hanna, Peter Stone, Chunyan Miao, and Yeng Soh. “DyETC: Dynamic electronic toll collection for traffic congestion alleviation”. In: *Proceedings of the AAAI Conference on Artificial Intelligence*. Vol. 32. 1. 2018.

[16] Hamid Mirzaei, Guni Sharon, Stephen Boyles, Tony Givargis, and Peter Stone. “Enhanced Delta-tolling: Traffic Optimization via Policy Gradient Reinforcement Learning”. In: *2018 21st International Conference on Intelligent Transportation Systems (ITSC)*. 2018, pp. 47–52. DOI: 10.1109/ITSC.2018.8569737.

[17] Wei Qiu, Haipeng Chen, and Bo An. “Dynamic Electronic Toll Collection via Multi-Agent Deep Reinforcement Learning with Edge-Based Graph Convolutional Networks”. In: *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19*. International Joint Conferences on Artificial Intelligence Organization, July 2019, pp. 4568–4574. DOI: 10.24963/ijcai.2019/635. URL: https://doi.org/10.24963/ijcai.2019/635.

[18] Yiheng Wang, Hexi Jin, and Guanjie Zheng. “CTRL: Cooperative Traffic Tolling via Reinforcement Learning”. In: *Proceedings of the 31st ACM International Conference on Information & Knowledge Management*. CIKM ’22. Atlanta, GA, USA: Association for Computing Machinery, 2022, pp. 3545–3554. ISBN: 9781450392365. DOI: 10.1145/3511808.3557112. URL: https://doi.org/10.1145/3511808.3557112.

[19] Vivek S Borkar. “Cooperative Dynamics and Wardrop Equilibria”. In: *Systems & control letters* 58.2 (2009), pp. 91–93.

[20] M. E. Ben-Akiva. *Discrete Choice Analysis: Theory and Application to Travel Demand*. Cambridge: MIT Press, 1985.

[21] Chih-Yuan Chiu, Chinmay Maheshwari, Pan-Yang Su, and Shankar Sastry. “Dynamic Tolling in Arc-based Traffic Assignment Models”. In: *arXiv* (2023). eprint: 2307.05466.

[22] Vivek Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.

[23] Shankar Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer, 1999.

APPENDIX

Please use the following link to access the ArXiv version with the appendix [21] (https://arxiv.org/pdf/2307.05466.pdf). The authors will make certain that this link stays active.