O-Minimal Hybrid Systems*

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Abstract. An important approach to decidability questions for verification algorithms of hybrid systems has been the construction of a bisimulation. Bisimulations are finite state quotients whose reachability properties are equivalent to those of the original infinite state hybrid system. In this paper we introduce the notion of *o-minimal hybrid systems*, which are initialized hybrid systems whose relevant sets and flows are definable in an o-minimal theory. We prove that o-minimal hybrid systems always admit finite bisimulations. We then present specific examples of hybrid systems with complex continuous dynamics for which finite bisimulations exist.

Key words. Hybrid systems, Bisimulations, Model theory, O-minimality, Decidability.

1. Introduction

Hybrid systems consist of finite state machines interacting with differential equations. Various modeling formalisms, analysis, design, and control methodologies, as well as applications, can be found in [AHS], [AKNS1], [AKNS2], [GNRR], [HS], and [M1]. The theory of formal verification is one of the main approaches for analyzing properties of hybrid systems. The system to be analyzed is first modeled as a hybrid automaton, and the desired property is expressed using a formula from some temporal logic. Then model checking or deductive algorithms are used in order to guarantee that the system model indeed satisfies the desired property.

Verification algorithms are essentially reachability algorithms which check whether trajectories of the hybrid system can reach certain undesirable regions of the state space. Since hybrid systems have infinite state spaces, decidability of verification algorithms is very important. An important approach to decidability

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results for hybrid systems is the construction of special finite state quotients of the original infinite state system called *bisimulations*. Bisimulations are reachability-preserving quotient systems in the sense that checking a property on the quotient system is *equivalent* to checking the property on the original system. Even though the focus of this paper is on reachability properties, bisimulations preserve many other complex properties expressible in branching time logics. In this approach, showing that an infinite state hybrid system has a finite state bisimulation is the first step in proving that verification procedures are decidable. In [AD] a finite bisimulation was explicitly constructed for timed automata, and as a result reachability questions for such systems are decidable. Timed automata were the basis for showing that reachability for other classes of hybrid systems is decidable even though they do not admit a finite bisimulation themselves (multirate, initialized rectangular automata). These results as well as some undecidable questions are described in [ACH+], [AD], [H1], [HKPV], and the references therein. Computing finite bisimulations is clearly related to the problem of obtaining discrete abstractions of continuous systems which has been considered among others in [ASL], [CKN], and [RO] as well as in [CW2].

The common approach to obtaining bisimulations has been to utilize an algorithm which refines an initial partition of the state space until it becomes compatible with the system dynamics and the property to be preserved. Using this approach, there are three main issues that must be resolved:

1. When does the algorithm terminate after a finite number of iterations?
2. When does the resulting partition consist of a finite number of equivalence classes?
3. Are all the steps of the algorithm constructive?

Resolving all three issues results in a decidable problem. Attacking the first two issues has been solved either by explicitly providing an equivalence relation which is checked to be a bisimulation (timed automata), or by transforming the problem to one for which a bisimulation is known to exist (multirate, rectangular automata). The third issue is typically tackled using quantifier elimination techniques from mathematical logic.

In this paper we tackle the first two issues for a large class of new hybrid systems. The third issue has been recently addressed in [LPY]. In order to answer the first two questions, we need to identify classes of sets and flows of vector fields with finite, global intersection properties. This is provided by the concept of *o-minimal* (or *order-minimal*) theories in mathematical logic [PS], [vdD], [vdDM1], [vdDM2], [W]. Using this concept, we introduce the notion of *o-minimal hybrid systems* which are initialized hybrid systems whose relevant sets (guards, resets, etc.) and flows are definable in an o-minimal theory. We then prove that o-minimal hybrid systems always admit finite bisimulations. Examples show that relaxing the notion of o-minimality quickly leads into pathological situations. We list various o-minimal theories and the corresponding hybrid systems that are definable in them. This list captures hybrid systems with more complex continuous dynamics than those of timed automata but with more restrictive discrete dynamics.

In addition to generating more classes of hybrid systems with finite bisimula-

tions, the importance of this paper can be summarized by the following:

1. The results presented provide a unified framework for decidability analysis of hybrid systems.
2. Generation of more o-minimal theories immediately leads to new classes of o-minimal hybrid systems.
3. Constructive results within o-minimal theories immediately lead to decidability results.

By providing a purely model theoretic framework, we also extend the planar results of [LPS1] and [LPS2].

The outline of the paper is as follows: In Section 2 we review the notion of bisimulations of transitions systems. In Section 3 we define a general class of hybrid systems and describe the bisimulation algorithm as it applies to hybrid systems. Section 4 presents the notion of o-minimality from model theory which is used in Section 5 in order to define o-minimal hybrid systems and prove the main theorem. In Section 6 we list various classes of o-minimal hybrid systems, and finally Section 7 contains some conclusions.

2. Bisimulations of Transition Systems

We adopt here the terminology of [H1] slightly modified for our purposes. A transition system $T = (Q, \Sigma, \rightarrow, Q_O, Q_F)$ consists of a (not necessarily finite) set Q of states, an alphabet Σ of events, a transition relation $\rightarrow \subseteq Q \times \Sigma \times Q$, a set $Q_O \subseteq Q$ of initial states, and a set $Q_F \subseteq Q$ of final states. A transition $(q_1, \sigma, q_2) \in \rightarrow$ is denoted as $q_1 \xrightarrow{\sigma} q_2$. The transition system is finite if the cardinality of Q is finite and it is infinite otherwise. A region is a subset $P \subseteq Q$. Given $\sigma \in \Sigma$ we define the predecessor $Pre_\sigma(P)$ of a region P as

$$Pre_\sigma(P) = \{q \in Q \mid \exists p \in P \text{ and } q \xrightarrow{\sigma} p\}. \tag{2.1}$$

Given an equivalence relation $\sim \subseteq Q \times Q$ on the state space one can define a quotient transition system as follows. Let $Q/\sim$ denote the quotient space. For a region P we denote by $P/\sim$ the collection of all equivalence classes which intersect P. The transition relation $\rightarrow_\sim$ on the quotient space is defined as follows: for $Q_1, Q_2 \in Q/\sim$, $Q_1 \xrightarrow{\sigma}_\sim Q_2$ iff there exist $q_1 \in Q_1$ and $q_2 \in Q_2$ such that $q_1 \xrightarrow{\sigma} q_2$. The quotient transition system is then $T/\sim = (Q/\sim, \Sigma, \rightarrow_\sim, Q_O/\sim, Q_F/\sim)$.

Given an equivalence relation $\sim$ on Q, we call a set a $\sim$-block if it is a union of equivalence classes.

Definition 2.1. The equivalence relation $\sim$ is a *bisimulation* of T iff Q_O, Q_F are $\sim$-blocks and for all $\sigma \in \Sigma$ and all $\sim$-blocks P, the region $Pre_\sigma(P)$ is a $\sim$-block. In this case the systems T and $T/\sim$ are called *bisimilar*.

We also say that a partition is a bisimulation when its induced equivalence relation is a bisimulation. A bisimulation is called finite if it has a finite number of equivalence classes. Bisimulations are very important because bisimilar transition

systems preserve reachability properties in addition to other more complex properties expressible in branching time logics [H1]. Therefore, checking properties on the bisimilar transition system is equivalent to checking properties of the original transition system. This is very useful in reducing the complexity of various verification algorithms where Q is finite but very large. In addition, if T is infinite and $T/\sim$ is a finite bisimulation, then verification algorithms for infinite systems are guaranteed to terminate. This approach was successfully applied to timed automata [AD]. It should be noted that the notion of bisimulation is analogous to the notion of dynamic consistency [CW1], [CW2], [PLS]. If $\sim$ is a bisimulation, it can be easily shown that if $p \sim q$, then

B1: $p \in Q_F$ iff $q \in Q_F$, and $p \in Q_O$ iff $q \in Q_O$,

B2: if $p \xrightarrow{\sigma} p'$, then there exists q' such that $q \xrightarrow{\sigma} q'$ and $p' \sim q'$.

Based on the above characterization, given a transition system T, the following algorithm computes increasingly finer partitions of the state space Q. If the algorithm terminates, then the resulting quotient transition system is a finite bisimulation. The state space $Q/\sim$ is called a bisimilarity quotient.

> **Algorithm 1** (Bisimulation Algorithm for Transition Systems)
>
> **Set:** $Q/\sim = \{Q_O \cap Q_F, Q_O \backslash Q_F, Q_F \backslash Q_O, Q \backslash (Q_O \cup Q_F)\}$
> **while:** $\exists P, P' \in Q/\sim$ and $\sigma \in \Sigma$ such that $\varnothing \neq P \cap Pre_\sigma(P') \neq P$
>   **set:** $P_1 = P \cap Pre_\sigma(P')$, $P_2 = P \backslash Pre_\sigma(P')$
>   **refine:** $Q/\sim = (Q/\sim \backslash \{P\}) \cup \{P_1, P_2\}$
> **end while:**

Notice that each time the partition $Q/\sim$ is refined, the transitions are updated to account for the newly subdivided sets. When checking specific properties, such as reachability to the set Q_F, one might simplify the algorithm by starting with a coarser partition, for example, $\{Q_F, Q \backslash Q_F\}$. In general one should include in the initial partition all additional sets relevant to the verification problem of interest (such as safe or unsafe regions). The larger the initial class of sets the more difficult it is for the algorithm to terminate.

3. Bisimulations of Hybrid Systems

We focus on transition systems generated by the following class of hybrid systems.

Definition 3.1. A *hybrid system* is a tuple $H = (X, X_0, X_F, F, E, I, G, R)$ where:

- $X = X_D \times X_C$ is the state space with $X_D = \{q_1, \ldots, q_n\}$ and X_C a manifold.
- $X_0 \subseteq X$ is the set of initial states.
- $X_F \subseteq X$ is the set of final states.
- $F : X \to TX_C$ assigns to each discrete location $q \in X_D$ a vector field $F(q, \cdot)$.
- $E \subseteq X_D \times X_D$ is the set of edges (which induce discrete transitions as shown below).

- $I : X_D \to 2^{X_C}$ assigns to each location a set $I(q) \subseteq X_C$ called the invariant.
- $G : E \to X_D \times 2^{X_C}$ assigns to $e = (q_1, q_2) \in E$ a guard of the form $\{q_1\} \times U$, $U \subseteq I(q_1)$.
- $R : E \to X_D \times 2^{X_C}$ assigns to $e = (q_1, q_2) \in E$ a reset of the form $\{q_2\} \times V$, $V \subseteq I(q_2)$.

Trajectories of the hybrid system H originate at any $(q, x) \in X_0$ and consist of either continuous evolutions or discrete jumps. Continuous trajectories keep the discrete part of the state constant, and the continuous part evolves according to the vector field $F(q, \cdot)$ as long as (the continuous part of) the trajectory remains inside the invariant set $I(q)$. If the trajectory exits $I(q)$, then a discrete transition is *forced*. If, during the continuous evolution, a state $(q, x) \in G(e)$ is reached for some $e \in E$, then a discrete transition induced by e is *enabled*. The state of the hybrid system may then instantaneously jump from (q, x) to any $(q', x') \in R(e)$ and the continuous part of the trajectory then evolves according to the vector field $F(q', \cdot)$. Notice that even though the continuous evolution is deterministic, the discrete evolution may be nondeterministic. The discrete transitions allowed in our model are more restrictive than those in initialized rectangular automata [ACH+], [AD], [PV]. In rectangular automata, the continuous dynamics are decoupled and each component of the continuous part of the state may be either reset nondeterministically to an interval or remain the same. If, however, the dynamics of a particular component changes, then the reset map cannot be the identity map on that component. In this paper we restrict the reset maps in order to allow complex and fully coupled dynamics. Finally, we assume that our hybrid system model is *nonblocking*, that is, from every state either a continuous evolution or a discrete transition is possible.

Example 3.2. A typical hybrid system is shown in Fig. 1. The state space is $\{Q1, Q2\} \times \mathbb{R}^2$. The initial states are of the form $\{Q1\} \times \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 1 < y < 2\}$. The discrete dynamics consists of two transitions along the edges $e_1 = (Q1, Q2)$ and $e_2 = (Q2, Q1)$. Within location $Q1$, the continuous variables x and y evolve according to a differential equation as long as $(x, y) \in I(Q_1) = \{(x, y) \in \mathbb{R}^2 \mid x \leq 5\}$. Once $x > 5$, a discrete transition along e_1 is forced and x, y are nondeterministically reset to values in fixed sets. The system then
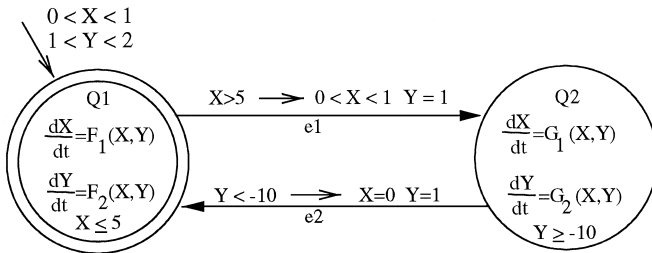

Fig. 1. A typical hybrid automation.

evolves according to the vector field associated with $Q2$. The evolution from that point on is similar. We would like to find out whether the system will reach the set of final states $\{Q_2\} \times \{(x, y) \in \mathbb{R}^2 \mid x < -5\}$.

Every hybrid system $H = (X, X_0, X_F, F, E, I, G, R)$ generates a transition system $T = (Q, \Sigma, \rightarrow, Q_O, Q_F)$ by setting $Q = X$, $Q_O = X_0$, $Q_F = X_F$, $\Sigma = E \cup \{\tau\}$, and $\rightarrow = (\bigcup_{e \in E} \xrightarrow{e}) \cup \xrightarrow{\tau}$ where:

Discrete Transitions: $(q, x) \xrightarrow{e} (q', x')$ for $e \in E$ iff $(q, x) \in G(e)$ and $(q', x') \in R(e)$.

Continuous Transitions: $(q_1, x_1) \xrightarrow{\tau} (q_2, x_2)$ iff $q_1 = q_2$ and there exists $\delta \geq 0$ and a curve $x : [0, \delta] \to M$ with $x(0) = x_1$, $x(\delta) = x_2$ and for all $t \in [0, \delta]$ it satisfies $x' = F(q_1, x(t))$ and $x(t) \in I(q_1)$.

The continuous τ transitions are time-abstract transitions, in the sense that the time it takes to reach one state from another is ignored. Having defined the continuous and discrete transitions $\xrightarrow{\tau}$ and $\xrightarrow{e}$ allows us to define formally $Pre_\tau(P)$ and $Pre_e(P)$ for $e \in E$ and any region $P \subseteq X$ using (2.1). Furthermore, the structure of the discrete transitions allowed in our hybrid system model result in

$$Pre_e(P) = \begin{cases} \varnothing & \text{if} \quad P \cap R(e) = \varnothing, \\ G(e) & \text{if} \quad P \cap R(e) \neq \varnothing, \end{cases} \tag{3.1}$$

for all $e \in E$ and regions P. Therefore, if the sets $R(e)$ and $G(e)$ are blocks of any partition of the state space, then no partition refinement is necessary in the bisimulation algorithm due to any discrete transitions. This fact, in a sense, decouples the continuous and discrete components of the hybrid system. In turn, this implies that the initial partition in the bisimulation algorithm should contain the invariants, guards, and reset sets, in addition to the initial and final sets. This allows us to carry out the algorithm independently for each location.

More precisely, define for any region $P \subseteq X$ and $q \in X_D$ the set $P_q = \{x \in X_C : (q, x) \in P\}$. For each location $q \in X_D$ consider the finite collection of sets

$$\mathscr{A}_q = \{I(q), (X_0)_q, (X_F)_q\} \cup \{G(e)_q, R(e)_q : e \in E\}, \tag{3.2}$$

which describes the initial and final states, guards, invariants, and resets associated with location q. Let $\mathscr{S}_q$ be the coarsest partition of X_C compatible with the collection $\mathscr{A}_q$ (by compatible we mean that each set in $\mathscr{A}_q$ is a union of sets in $\mathscr{S}_q$). The (finite) partition $\mathscr{S}_q$ can be easily computed by successively finding the intersections between each of the sets in $\mathscr{A}_q$ and their complements. We define $(q, \mathscr{S}_q)$ to be the set $\{\{q\} \times P \mid P \in \mathscr{S}_q\}$. These collections $(q, \mathscr{S}_q)$ will be the starting partitions of the bisimulation algorithm. In addition, since by definition $Pre_\tau(P)$ applies to regions $P \subseteq X$, but not to its continuous projection P_q, we define for $Y \subseteq X_C$ the operator $Pre_q(Y) = (Pre_\tau(\{q\} \times Y))_q$. The general bisimulation algorithm for transition systems then takes the following form for the class of hybrid systems that are considered in this paper.

Algorithm 2 (Bisimulation Algorithm for Hybrid Systems)

Set: $X/\sim \; = \bigcup_q (q, \mathscr{S}_q)$
for: $q \in X_D$
   while: $\exists P, P' \in \mathscr{S}_q$ such that $\varnothing \neq P \cap Pre_q(P') \neq P$
     Set: $P_1 = P \cap Pre_q(P'); \; P_2 = P \backslash Pre_q(P')$
     refine: $\mathscr{S}_q = (\mathscr{S}_q \backslash \{P\}) \cup \{P_1, P_2\}$
   end while:
   end for:

It is clear from the structure of the bisimulation algorithm that the iteration is carried out independently for each discrete location. In order for the above algorithm to terminate, the partition refinement process must terminate for each discrete location $q \in X_D$. It therefore suffices to look at one continuous slice of the hybrid system at a time and see whether we can construct a finite bisimulation that is consistent with all relevant sets of each location q as well as with the trajectories of the vector field $F(q, \cdot)$.

The following example shows that, even in apparently simple situations, Algorithm 2 does not terminate.

Example 3.3. Consider the hybrid system with only one discrete location q and let F be the linear vector field $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x}$ on $\mathbb{R}^2$. Assume the partition of $\mathbb{R}^2$ consists of the following three sets (see Fig. 2): $P_1 = \{(x,0) : 0 \leq x \leq 4\}$, $P_2 = \{(x,0) : -4 \leq x < 0\}$, $P_3 = \mathbb{R}^2 \backslash (P_1 \cup P_2)$. The trajectories of F are spirals moving away from the origin. The first iteration of the algorithm partitions P_2 into $P_4 = P_2 \cap Pre_q(P_1) = \{(x,0) : x_1 \leq x < 0\}$ and $P_2 \backslash Pre_q(P_1)$. Here $x_1 < 0$ is the x-coordinate of the first intersection point of the spiral through $(4,0)$ with P_2. The second iteration subdivides P_1 into $P_5 = P_1 \cap Pre_q(P_4) = \{(x,0) : 0 \leq x \leq x_2\}$

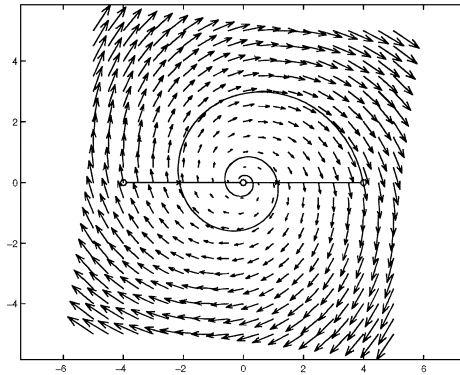**Fig. 2.** Algorithm 2 does not terminate.

and $P_1 \backslash Pre_q(P_4)$ where $x_2 > 0$ is the x-coordinate of the next point of intersection of the spiral with P_1. This process continues indefinitely since the spiral intersects P_1 in infinitely many points, and therefore the algorithm does not terminate.

From the above example it is clear that the critical problem one must investigate is how the trajectories of $F(q, \cdot)$ interact with the sets $\mathcal{S}_q$ for a single location q. This requires that the trajectories of the vector field $F(q, \cdot)$ have "nice" intersection properties with such sets. Since the goal is to obtain finite partitions, it will become important that we restrict the study to classes of sets with global "finiteness" properties, for example, sets with finitely many connected components. In the next section we identify such classes of sets and vector fields using the concept of o-minimality from model theory.

4. Model Theory

This section provides a brief introduction to mathematical logic and model theory. A clear expository survey of model theoretic results that are relevant to this paper is [M2]. For a more detailed treatment of model theory the reader is referred to [H2] and [vD].

4.1. Languages and Formulas

A *language* is a set of symbols separated into three groups: relations, functions, and constants. The sets $\mathcal{L}_0 = \{<, +, -, \{0, 1\}\}$, $\mathcal{L}_R = \{<, +, -, \cdot, C\}$, and $\mathcal{L}_{\exp} = \{<, +, -, \cdot, \exp, C\}$ are examples of languages where $<$ (less than) is the relation, $+$ (plus), $-$ (minus), $\cdot$ (product), and $\exp$ (exponentiation) are the functions, and 0 (zero), 1 (one), and the elements of set C are the constants.

Let $\mathcal{V} = \{x, y, z, x_0, x_1, \ldots\}$ be a countable set of *variables*. The set of *terms* of a language is inductively defined as follows. A term θ is a variable, a constant, or $F(\theta_1, \ldots, \theta_m)$, where F is an m-ary function and θ_i, $i = 1, \ldots, m$, are terms. For instance, $x - 2y + 3$ is a term of $\mathcal{L}_0$ since any integer can be obtained by adding or subtracting the constant 1. Any polynomial with coefficients in C is a term in $\mathcal{L}_R$.

The *atomic formulas* of a language are of the form $\theta_1 = \theta_2$, or $R(\theta_1, \ldots, \theta_n)$, where θ_i, $i = 1, \ldots, n$, are terms and R is an n-ary relation. For example, $xy > 0$ and $x^2 + c = d$ (with $c, d \in C$) are atomic formulas of $\mathcal{L}_R$. The set of (*first-order*) *formulas* is recursively defined as follows. Every atomic formula is a formula. If φ_1 and φ_2 are formulas, then $\varphi_1 \wedge \varphi_2$, $\neg \varphi_1$, $\forall x : \varphi_1$, or $\exists x : \varphi_1$ are formulas where x is a variable, $\wedge$ (conjunction) and $\neg$ (negation) are the boolean connectives, and $\forall$ (for all) and $\exists$ (there exists) are the quantifiers.

If $C = \mathbb{Z}$, then examples of $\mathcal{L}_R$-formulas are $\forall x \, \forall y : xy > 0$, $\exists x : x^2 - 2 = 0$, and $\exists w : xw^2 + yw + z = 0$. The occurrence of a variable in a formula is *free* if it is not inside the scope of a quantifier; otherwise, it is *bound*. For example, x, y, and z are free and w is bound in the last example. We often write $\varphi(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n$ are the free variables of the formula φ. A *sentence* of $\mathcal{L}_R$ is a formula with no free variables. The first two examples are sentences.

4.2. Models

A *model* of a language consists of a nonempty set S and an interpretation of the relations, functions, and constants. For example, using $S = \mathbb{R}$, $C = \mathbb{R}$, and the usual interpretation of the symbols $<, +, -, \cdot$, provides a model of $\mathscr{L}_R$ which we denote by $(\mathbb{R}, <, +, -, \cdot)$. A set $Y \subseteq S^n$ is *definable* in a language if there exists a formula $\varphi(x_1, \ldots, x_n)$ such that $Y = \{(a_1, \ldots, a_n) \in S^n \mid \varphi(a_1, \ldots, a_n)\}$. For example, over $\mathbb{R}$, the formula $x^2 - 2 = 0$ defines the set $\{\sqrt{2}, -\sqrt{2}\}$. A function f is definable if its graph is a definable set. The collection of definable sets is closed under Boolean operations and taking forward or inverse images under definable functions.

4.3. Theories

A *theory* is a subset of sentences. Any model of a language defines a theory: *the set of all sentences which hold in the model*. By abuse of notation the theory $(\mathbb{R}, <, +, -, \cdot)$ refers to the collection of formulas of $\mathscr{L}_R$ which are true in the model (and similarly for other languages). While many of the concepts here apply to more general models, in all that follows we consider only models over $\mathbb{R}$ (and assume that all real numbers are constants).

Definition 4.1. The theory of $\mathscr{L}$ is *o-minimal* (order minimal) if every definable subset of $\mathbb{R}$ is a finite union of points and intervals (possibly unbounded).

The class of o-minimal theories is quite rich. In [T] it was shown that the theory of the real numbers as a real closed field, $(\mathbb{R}, <, +, -, \cdot)$, admits elimination of quantifiers. This, together with an analysis of the structure of sets definable by quantifier free formulas shows that the theory is o-minimal. Tarski was also interested in the extension of the theory of the real numbers by the exponential function, $\mathbb{R}_{\exp} = (\mathbb{R}, <, +, -, \cdot, \exp)$ (i.e., there is an additional symbol in the language for the exponential function). While such theory does not admit elimination of quantifiers, it was shown in [W] that such theory is o-minimal. Another important extension is obtained as follows. Assume f is a real-analytic function in a neighborhood of the cube $[-1, 1]^n \subset \mathbb{R}^n$. Let $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if} \quad x \in [-1, 1]^n, \\ 0 & \text{otherwise.} \end{cases}$$

We call such functions *restricted analytic functions*. These functions are useful to describe the behavior of some periodic trajectories. For example, the functions sin and cos restricted to a period are sufficient to define closed orbits of some linear systems (see Section 6.3). The theory $\mathbb{R}_{\exp, \text{an}} = (\mathbb{R}, <, +, -, \cdot, \exp, \{\hat{f}\})$ is an extension of $\mathbb{R}_{\exp}$ where there is a symbol for each restricted analytic function. In [vdDM1] it was shown that $\mathbb{R}_{\exp, \text{an}}$ is also o-minimal. More recently it was shown in [S2] that so-called Pfaffian extensions of o-minimal theories are also o-minimal.

Table 1. O-minimal theories.

Name	Theory	Sample definable sets	Sample definable trajectories
$\mathbb{R}_{lin}$	$(\mathbb{R}, <, +, -)$	Polyhedral sets	Linear trajectories
$\mathbb{R}_{alg}$	$(\mathbb{R}, <, +, -, \cdot)$	Semialgebraic sets	Polynomial trajectories
$\mathbb{R}_{an}$	$(\mathbb{R}, <, +, -, \cdot, \{\hat{f}\})$	Subanalytic sets	Polynomial trajectories
$\mathbb{R}_{exp}$	$(\mathbb{R}, <, +, -, \cdot, \exp)$	Semialgebraic sets	Exponential trajectories
$\mathbb{R}_{exp, an}$	$(\mathbb{R}, <, +, -, \cdot, \exp, \{\hat{f}\})$	Subanalytic sets	Exponential trajectories

Table 1 summarizes o-minimal theories (even very recent ones) along with some *examples* of sets and vector field trajectories that are definable in these theories. We examine the connection between these o-minimal extensions and different classes of a hybrid system in Section 6.

Many geometric properties of the above theories can be found in [vdDM2] and the book [vdD]. We present below those properties of o-minimal theories that are used in the proof of the main theorem.

We assume a theory which is an extension of $(\mathbb{R}, <, +, -)$ is given. Definability refers to this theory.

Definition 4.2. We define a *cell* in $\mathbb{R}^n$ inductively as follows:

1. The cells in $\mathbb{R}$ are just the points $\{c\}$ with $c \in \mathbb{R}$ and the open intervals (a, b), $-\infty \le a < b \le +\infty$.
2. Let $C \subset \mathbb{R}^n$ be a cell and let $f, g : C \to \mathbb{R}$ be definable continuous functions such that $f < g$ on C. Then $(f, g) = \{(x, r) \in C \times \mathbb{R} : f(x) < r < g(x)\} \subseteq \mathbb{R}^{n+1}$ is a cell in $\mathbb{R}^{n+1}$. Also, for each definable function $f : C \to \mathbb{R}$, the graph of f and the sets $(-\infty, f) = \{(x, r) \in C \times \mathbb{R} : r < f(x)\}$, $(f, +\infty) = \{(x, r) \in C \times \mathbb{R} : f(x) < r\}$, and $C \times \mathbb{R}$ are cells in $\mathbb{R}^{n+1}$.

A geometric view of cells is as fibers over their projections, as shown in Fig. 3.
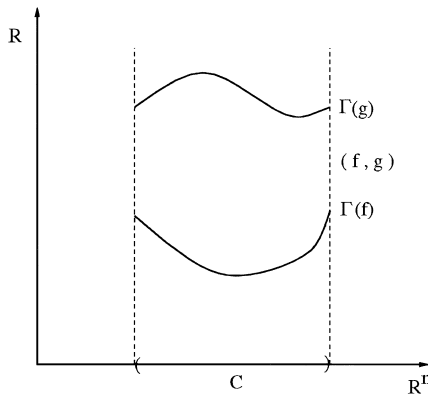
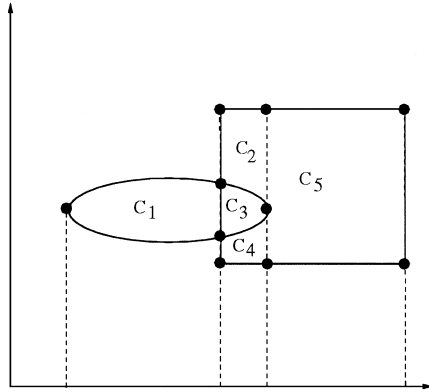**Fig. 3.** Inductive definition of cells.

Fig. 4.   Illustration of the Cell Decomposition Theorem.

Theorem 4.3.   *Assume we are given an o-minimal theory which is an extension of* $(\mathbb{R}, <, +, -)$. *Then*:

1. **(Cell Decomposition)** *Given any finite family* $\{A_1, \ldots, A_l\}$ *of definable subsets of* $\mathbb{R}^n$ *there exists a finite partition of* $\mathbb{R}^n$ *into cells so that each* A_i *is a union of such cells* [KPS], [vdD].
2. *Any definable set has a finite number of connected components, each of which is a definable set. Moreover, if* $A \subset \mathbb{R}^n \times \mathbb{R}$ *is definable, then there exists a positive integer N such that for each* $x \in \mathbb{R}^n$ *the number of connected components of* $A_x = \{t \in \mathbb{R} : (x, t) \in A\}$ *is less than N. (A consequence of cell decomposition.)*
3. *If A is definable and connected, then it is arcwise connected, that is, every two points in A can be connected by a continuous definable curve* [vdD].
4. **(Monotonicity Theorem)** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a definable function. Then there are points* $a_1 < \cdots < a_k$ *in* $\mathbb{R}$ *such that on each subinterval* (a_j, a_{j+1}) *with* $a_0 = -\infty$, $a_{k+1} = +\infty$, *the function f is either constant, or strictly monotone and continuous* [vdD].

The above Cell Decomposition Theorem is illustrated in Fig. 4, where a square and an ellipsoid are decomposed into 5 two-dimensional cells, 12 one-dimensional cells, and 10 zero-dimensional cells. The Cell Decomposition Theorem is used to provide the initial partition of Algorithm 2. It is also the first step in the proof of the main theorem.

Another application of o-minimality in a system theoretic context can be found in [S1], where o-minimality is used for input/output distinguishability of neural networks.

5. O-Minimal Hybrid Systems

In this section we prove the main theorem and give specific examples of new classes of hybrid systems which admit a finite bisimulation. We first review the notion of complete vector fields and their flows.

Definition 5.1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field on $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, let $\gamma_x(t)$ denote the integral curve of F which passes through x at $t = 0$, that is, $\dot{\gamma}_x(t) = F(\gamma_x(t))$ and $\gamma_x(0) = x$. We say that F is complete if, for every x, $\gamma_x(t)$ is defined for all t. For such an F, the flow of F is the function $\Phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by $\Phi(x, t) = \gamma_x(t)$.

In all subsequent analysis all vector fields are assumed smooth. We can now define the class of hybrid systems studied in this paper.

Definition 5.2. A hybrid system $H = (X, X_0, X_F, F, E, I, G, R)$ is said to be o-minimal if

- $X_C = \mathbb{R}^n$ (and thus $X = X_D \times \mathbb{R}^n$),
- for each $q \in X_D$ the vector field $F(q, \cdot)$ is complete, and
- for each $q \in X_D$ the family of sets $\mathscr{A}_q = \{I(q), (X_0)_q, (X_F)_q\} \cup \{G(e)_q, R(e)_q : e \in E\}$ and the flow of $F(q, \cdot)$ are definable in an o-minimal extension of $(\mathbb{R}, <, +, -)$.

The main theorem below uses the finiteness properties described in the previous section to construct finite bisimulations for o-minimal hybrid systems.

Theorem 5.3. *Every o-minimal hybrid system admits a finite bisimulation. In particular, the bisimulation algorithm, Algorithm 2, terminates for o-minimal hybrid systems.*

Proof. Before the detailed proof, we begin by providing a general outline. We assume a fixed o-minimal extension $\bar{\mathscr{R}}$ of $(\mathbb{R}, <, +, -\}$, in which all relevant objects are definable, is given. From now on, definable means definable in $\bar{\mathscr{R}}$. We start by applying the Cell Decomposition Theorem on each family $\mathscr{A}_q$. As mentioned in Section 3, due to (3.1), the special form of $Pre_e(P)$ shows that no partition refinement is necessary in the bisimulation algorithm due to any discrete transitions $e \in E$. This fact allows us to carry out the algorithm independently for each location and construct the bisimulation quotient on each set $\{q\} \times X_C$ separately. Therefore, we assume a finite partition $\mathscr{P}$ of $\mathbb{R}^n$ into definable sets and a vector field F whose flow is definable are given. Moreover, we drop the dependence on q and simply write Pre for Pre_q.

We then perform an initial finite refinement $\tilde{\mathscr{P}}$ of $\mathscr{P}$ which has the property that the intersection of any trajectory with each set has one connected component. Because of this property we can use a slight variation of the iterative step of the bisimulation algorithm to construct a finite partition $\mathscr{B}$ which is a further refinement, and satisfies the bisimulation condition, namely, that, for any $B \in \mathscr{B}$, the set $Pre(B)$ is a finite union of sets in $\mathscr{B}$. This guarantees that the bisimulation algorithm terminates and will conclude the proof.

To start the detailed proof, first notice that if $f : \mathbb{R} \to \mathbb{R}^n$ is continuous, periodic, and not constant, then f is not definable. Indeed, for such f there is $y \in \mathbb{R}^n$ such that the set $R = \{x \in \mathbb{R} : f(x) = y\}$ consists of an infinite number of isolated points. On the other hand, if f is definable, then so is R, but this contradicts o-minimality.

Let $\Phi(x,t) = \gamma_x(t)$ denote the flow of F. Since this is definable by hypothesis, we conclude from the above comment that, for each $x \in \mathbb{R}^n$, $\gamma_x(\cdot)$ is either constant or injective (injectivity follows by uniqueness of solutions of differential equations). We need the following lemma.

Lemma 5.4. *Let F be as above, and let γ be an integral curve of F. Define $\Gamma = \mathrm{Im}(\gamma) = \{\gamma(t) : t \in \mathbb{R}\}$. Let S be a definable set and let C be a connected component of $\Gamma \cap S$. If $t_0, t_1 \in \mathbb{R}$ are such that $\gamma(t_0), \gamma(t_1) \in C$, then $\gamma(t) \in C$ for all $t_0 \leq t \leq t_1$.*

Proof. Since C is definable and connected, it is also arcwise connected. Let $\beta : [0,1] \to C$ be continuous and such that $\beta(0) = \gamma(t_0)$ and $\beta(1) = \gamma(t_1)$. If γ is constant there is nothing to prove. We can then assume γ is injective and $F(\gamma(t)) \neq 0$ for all t. Therefore, the restriction of γ to any compact interval $[a,b]$ is a homeomorphism between $[a,b]$ and $\gamma([a,b])$. If $\beta([0,1]) \subseteq \gamma([a,b])$, then $\gamma^{-1} \circ \beta$ is continuous and so $\gamma^{-1} \circ \beta([0,1])$ is an interval containing t_0, t_1. Therefore, for all $t \in [t_0, t_1]$, $\gamma(t) \in \beta([0,1]) \subseteq C$ as desired.

Assume then that $\beta([0,1])$ is not contained in the image under γ of any finite interval. Hence there exist a sequence $\{t_n\}$ with $|t_n| \to \infty$ and $\gamma(t_n) \in \beta([0,1])$ for all n. By taking a subsequence if necessary we may assume that $\gamma(t_n) \to \tilde{x} \in \beta([0,1])$ and either $t_n \to \infty$ or $t_n \to -\infty$. Therefore, $\tilde{x} = \gamma(\tilde{t})$ for some $\tilde{t} \in \mathbb{R}$. We will show that this is a contradiction.

Since each component $\gamma_j(t)$ of $\gamma(t)$ is a real-valued definable function, it is eventually monotone (by the Monotonicity Theorem). Then, assuming for simplicity $t_n \to \infty$, we have $\lim_{t \to \infty} \gamma_j(t) = \tilde{x}_j$ and by continuity $\lim_{t \to \infty} F(\gamma(t)) = F(\tilde{x})$. Since $\dot{\gamma}_j(t) = F_j(\gamma(t))$, $\lim_{t \to \infty} \dot{\gamma}_j(t)$ exists, and must therefore equal zero. This contradicts the fact that $F(\gamma(\tilde{t})) \neq 0$. ∎

We now continue with the proof of the main theorem. Given a set S, we define $H = \{(x,t) \in \mathbb{R}^{n+1} : \Phi(x,t) \in S\}$. If S is definable, then H is definable. Moreover, by o-minimality there exists $N_S \in \mathbb{N}$ such that the number of connected components of $H_x = \{t : (x,t) \in H\}$ is less than N for all $x \in \mathbb{R}^n$. This implies that if S is definable and Γ_x denotes the trajectory of F passing through x, then the number of connected components of $\Gamma_x \cap S$ is bounded above by a constant independent of x. We choose $N \in \mathbb{N}$ larger than the corresponding N_S for all sets $S \in \mathscr{P}$.

We begin the construction of the partition $\mathscr{B}$ by subdividing each set S in $\mathscr{P}$ as follows. Let

$$S_0 = \{x \in S : \forall t \geq 0 \ \gamma_x(t) \in S\},$$

$$S_1 = \{x \in S \backslash S_0 : \forall t \geq 0 (\gamma_x(t) \notin S \backslash S_0 \Rightarrow \forall t' \geq t \ \gamma_x(t') \notin S \backslash S_0)\},$$

$$\vdots$$

$$S_i = \{x \in S \backslash (S_0 \cup \cdots \cup S_{i-1}) :$$
$$\forall t \geq 0 \ (\gamma_x(t) \notin S \backslash (S_0 \cup \cdots \cup S_{i-1}) \Rightarrow \forall t' \geq t \ \gamma_x(t') \notin S \backslash (S_0 \cup \cdots \cup S_{i-1}))\},$$

$$\vdots$$

The set S_i is clearly definable for every i. For $i \geq 1$ the set S_i consists of those x for which γ_x leaves the set $S \backslash (S_0 \cup \cdots \cup S_{i-1})$ and never returns to it.

Claim. $S_k = \emptyset$ for $k \geq N$.

To prove the claim it suffices to show that if $x \in S_i$ with $i \geq 1$, then $\Gamma_x \cap S$ has at least i connected components. To prove this we use a couple of lemmas.

Lemma 5.5.   *Let S and S_i, $i \geq 0$, be as above. Let I be an interval and let $\gamma(\cdot)$ be an integral curve of F such that $\gamma(I) \subseteq S$. If $\gamma(t_0) \in S_i$ for some $t_0 \in I$, then $\gamma(I) \subseteq S_i$.*

Proof.   We proceed by induction. The statement is clearly true for S_0. Assume it holds for $i \leq k$. Let $\gamma(I) \subseteq S$, $t_0 \in I$, and $\gamma(t_0) \in S_{k+1}$. Then $\gamma(t_0) \in S \backslash (S_0 \cup \cdots \cup S_k)$. For any $t \in I$, if $\gamma(t) \in S_0 \cup \cdots \cup S_k$, then there is $j \leq k$ such that $\gamma(t) \in S_j$. By the inductive hypothesis, $\gamma(I) \subseteq S_j$, but this contradicts $\gamma(t_0) \notin S_j$. Therefore we have $\gamma(I) \subseteq S \backslash (S_0 \cup \cdots \cup S_k)$. Let $t \in I$ and $t' > t$ be such that $\gamma(t') \notin S \backslash (S_0 \cup \cdots \cup S_k)$. Then $t' \notin I$ and so $t' > t_0$. Since $\gamma(t_0) \in S_{k+1}$ we conclude that for any $t'' > t'$ we get $\gamma(t'') \notin S \backslash (S_0 \cup \cdots \cup S_k)$. This shows that $\gamma(t) \in S_{k+1}$.   ∎

Lemma 5.6.   *If $x \in S_i$ for $i \geq 2$, then there exist $t_1 > s_1 > t_2 > \cdots > s_{i-2} > t_{i-1} > s_{i-1} > 0$ such that $\gamma_x(s_j) \notin S$ and $\gamma_x(t_j) \in S_j$ for $j = 1, \ldots, i-1$.*

Proof.   We proceed by induction. Let $x \in S_2$. Then $x \in S \backslash (S_0 \cup S_1) \subseteq S \backslash S_1$. Therefore there exist $t > s > 0$ such that $\gamma_x(s) \notin S \backslash S_0$ and $\gamma_x(t) \in S \backslash S_0$. We cannot have $\gamma_x(s) \in S_0$ because then we would also have $\gamma_x(t) \in S_0$. Therefore $\gamma_x(s) \notin S$. We set $s_1 = s$. If $\gamma_x(t) \in S_1$, then we set $t_1 = t$. Otherwise, there exist $t' > s' > t$ such that $\gamma_x(s') \notin S \backslash S_0$ and $\gamma_x(t') \in S \backslash S_0$. Since $x \in S_2$, $\gamma_x(s) \notin S \backslash (S_0 \cup S_1)$, and $t' > s$ we must have $\gamma_x(t') \notin S \backslash (S_0 \cup S_1)$. Therefore $\gamma_x(t') \in S_1$ and we set $t_1 = t'$. This completes the proof for the case $i = 2$.

Assume now the conclusion holds for i and let $x \in S_{i+1}$. In particular, $x \in S \backslash S_i$, and there are $t > s > 0$ such that $\gamma_x(s) \notin S \backslash (S_0 \cup \cdots \cup S_{i-1})$ and $\gamma_x(t) \in S \backslash (S_0 \cup \cdots \cup S_{i-1})$. If $\gamma_x(s) \in S_j$ for some $j \leq i-1$ and $\gamma_x(\bar{s}) \in S$ for all $s \leq \bar{s} \leq t$, then Lemma 5.5 would imply that $\gamma_x(t) \in S_j$ which is not true. Therefore there exists $\bar{s}$, $s \leq \bar{s} < t$, such that $\gamma_x(\bar{s}) \notin S$. We set $s_i = \bar{s}$.

If $\gamma_x(t) \in S_i$, then we set $t_i = t$. Otherwise, there exist $t' > s' > t$ such that $\gamma_x(s') \notin S \backslash (S_0 \cup \cdots \cup S_{i-1})$ and $\gamma_x(t') \in S \backslash (S_0 \cup \cdots \cup S_{i-1})$. Since $x \in S_{i+1}$, $\gamma_x(\bar{s}) \notin S \backslash (S_0 \cup \cdots \cup S_i)$, and $t' > \bar{s}$ we must have $\gamma_x(t') \notin S \backslash (S_0 \cup \cdots \cup S_i)$. Therefore $\gamma_x(t') \in S_i$ and we set $t_i = t'$.

By the inductive hypothesis there exist $\tilde{t}_1 > \tilde{s}_1 > \cdots > \tilde{t}_{i-1} > \tilde{s}_{i-1} > 0$ such that $\gamma_{\gamma_x(t_i)}(\tilde{s}_j) \notin S$, $\gamma_{\gamma_x(t_i)}(\tilde{t}_j) \in S_j$, for $j = 1, \ldots, i-1$. Setting $s_j = \tilde{s}_j + t_i$, $t_j = \tilde{t}_j + t_i$ for $j = 1, \ldots, i-1$ we get the desired conclusion.   ∎

The last lemma together with Lemma 5.4 proves that if $x \in S_i$, then $\Gamma_x \cap S$ has at least i connected components. This, in turn, proves the claim.

Notice that Lemmas 5.4 and 5.5 together imply that if $x \in S_i$, then $\Gamma_x \cap S_i$ has exactly one connected component.

By carrying out the subdivision into the sets S_i for all $S \in \mathscr{P}$ we obtain a new finite partition $\tilde{\mathscr{P}}$ of $\mathbb{R}^n$ with the property

(P) For each $S \in \tilde{\mathscr{P}}$, and each trajectory γ of F such that $\gamma(t_0), \gamma(t_1) \in S$ we have $\gamma(t) \in S$ for all t with $t_0 \leq t \leq t_1$. In particular, for each $x \in S$, the set $\Gamma_x \cap S$ has exactly one connected component.

We denote by $\rho = \rho(\tilde{\mathscr{P}})$ the number of sets in $\tilde{\mathscr{P}}$ and write $\tilde{\mathscr{P}} = \{S_i : i = 1, \ldots, \rho\}$.

We introduce two functions, I and C, acting on pairs of sets, defined by

$$I(A, B) = A \cap Pre(B),$$

$$C(A, B) = A \backslash Pre(B).$$

It is clear that if A and B are definable, then $I(A, B)$ and $C(A, B)$ are definable. Notice also that for each A, B the sets $I(A, B), C(A, B)$ form a partition of A.

For each i, $1 \leq i \leq \rho$, consider all the partitions of S_i defined by

$$I(S_i, Q(S_{j_1}, Q(S_{j_2}, \ldots, Q(S_{j_{k-1}}, S_{j_k}) \cdots))), \tag{5.1}$$

$$C(S_i, Q(S_{j_1}, Q(S_{j_2}, \ldots, Q(S_{j_{k-1}}, S_{j_k}) \cdots))), \tag{5.2}$$

where Q is either I or C and $1 \leq j_l \leq \rho$ for $l = 1, \ldots, k$ and $k \leq \rho$. This is a finite collection of finite partitions. We let $\mathscr{B}$ denote the coarsest partition of $\mathbb{R}^n$ compatible with all such partitions.

Claim. $\mathscr{B}$ is a bisimulation.

The intuitive basis for this proof is the fact that the partitions constructed so far are done "along the flow of F." That is, two sets in $\mathscr{B}$ which are subsets of the same set in $\tilde{\mathscr{P}}$ cannot be connected by a trajectory of F.

To prove the claim first notice that the sets in $\mathscr{B}$ are (finite) intersections of sets of the form (5.1) or (5.2). Notice also that by construction $\mathscr{B}$ is a refinement of $\mathscr{P}$.

To check the bisimulation property let $B \in \mathscr{B}$, $B \subseteq S_i \in \tilde{\mathscr{P}}$, be written as

$$B = \bigcap_{l=1}^{m} P_l,$$

where each P_l is of the form (5.1) or (5.2). We want to prove first that

$$Pre(B) = \bigcap_{l=1}^{m} Pre(P_l). \tag{5.3}$$

The inclusion $Pre(B) \subseteq \bigcap_{l=1}^{m} Pre(P_l)$ is straightforward. For the other one let $x \in \bigcap_{l=1}^{m} Pre(P_l)$. For each l there exists $t_l \geq 0$ such that $\gamma_x(t_l) \in P_l$. Each set P_l is of the form $I(S_i, A_l)$ or $C(S_i, A_l)$ for some A_l's. Hence, $\gamma_x(t_l) \in S_i$ for all l. We

now want to show that indeed $\gamma_x(t_l) \in B$ for all t_l. Consider the following property of a set A:

(Q) For any trajectory γ of F, if $\gamma(s_0) \in A \subseteq S \in \tilde{\mathscr{P}}$, then, for all s with $\gamma(s) \in S$, $\gamma(s) \in A$.

We show that if a set A has Property (Q), then so do $I(S', A)$ and $C(S', A)$ for any $S' \in \tilde{\mathscr{P}}$. Let $\gamma(s_0) \in I(S', A) \subseteq S'$. Then $\gamma(s_0) \in S'$ and there exists $t \geq s_0$ such that $\gamma(t) \in A$. If $\gamma(t) \in S'$, then we have $S = S'$ since both belong to $\tilde{\mathscr{P}}$. By Property (Q) $\gamma(s) \in A \subseteq Pre(A)$ for all s such that $\gamma(s) \in S'$. Therefore $\gamma(s) \in I(S', A)$ for all such s. On the other hand, if $\gamma(t) \notin S'$, then $A \cap S' \subseteq S \cap S' = \varnothing$. Let $\gamma(s) \in S'$. By Property (P) applied to S' we get that $s \leq t$. However, then $\gamma(s) \in Pre(A) \cap S'$ as desired. The proof for $C(S', A)$ is analogous.

Proceeding by induction it is easy to show that the sets P_l have Property (Q) and this completes the proof of (5.3).

Notice also, that $Pre(A \cup B) = Pre(A) \cup Pre(B)$ for all sets A, B.

To complete the proof that $\mathscr{B}$ is a bisimulation we only need to show that for each l, and each set $S \in \tilde{\mathscr{P}}$, the set $S \cap Pre(P_l)$ is a union of sets in $\mathscr{B}$. The set $S \cap Pre(P_l) = I(S, P_l)$ is of the form (5.1) with $k \leq \rho + 1$. If $k < \rho + 1$ we already know that $I(S, P_l)$ is a union of sets in $\mathscr{B}$. We only need to consider the case $k = \rho + 1$.

There are two possibilities for $I(S, P_l)$:

1. There are two or more occurrences of C in $I(S, P_l)$.
2. There are $\rho + 1$ occurrences of I in $I(S, P_l)$, and, therefore, at least one $S_i \in \tilde{\mathscr{P}}$ is repeated as an argument of I.

In case 1 the following two formulas, and boolean algebra, show how to rewrite $I(S, P_l)$ either with fewer terms or using only I:

$$C(S_3, C(S_2, S_1)) = C(S_3, S_2) \cup I(S_3, I(S_2, S_1)), \qquad (5.4)$$

$$C(S_3, I(S_2, S_1)) = C(S_3, S_2) \cup I(S_3, C(S_2, S_1)). \qquad (5.5)$$

Both formulas can be proved with arguments similar to the ones above, relying on Property (P). We give the proof of formula (5.4), the other one is analogous. That the left side is included in the right side does not require any special property of the sets S_i. Indeed, $x \in C(S_3, C(S_2, S_1)) = S_3 \backslash Pre(S_2 \backslash Pre(S_1))$ means

$$x \in S_3 \wedge (\forall t \geq 0(\gamma_x(t) \notin S_2 \vee (\gamma_x(t) \in S_2 \wedge \exists t' \geq t(\gamma_x(t') \in S_1)))).$$

Therefore, if, for all $t \geq 0$, $\gamma_x(t) \notin S_2$, then $x \in S_3 \backslash Pre(S_2)$. On the other hand, if there is $t \geq 0$ such that $\gamma_x(t) \in S_2$ and $t' \geq t$ such that $\gamma_x(t') \in S_1$, then $x \in Pre(S_2 \cap Pre(S_1)) = I(S_3, I(S_2, S_1))$ and the inclusion is proved.

For the other inclusion, first notice that $C(S_3, S_2) = S_3 \backslash Pre(S_2) \subseteq S_3 \backslash Pre(S_2 \backslash Pre(S_1)) = C(S_3, C(S_2, S_1))$. Let now $x \in I(S_3, I(S_2, S_1))$. So $x \in S_3$ and there exist $t' \geq t \geq 0$ such that $\gamma_x(t) \in S_2$ and $\gamma_x(t') \in S_1$. To show the desired inclusion we need to show that, for all $t'' \geq 0$, $\gamma_x(t'') \notin S_2 \backslash Pre(S_1)$. We do so by contradiction. Suppose there exists $t'' \geq 0$ such that $\gamma_x(t'') \in S_2 \backslash Pre(S_1)$. Since $\gamma_x(t') \in S_1$ and $\gamma_x(t'') \notin Pre(S_1)$ we must have $t'' > t'$. Moreover, using $\gamma_x(t) \in S_2$, $\gamma_x(t'') \in S_2$, $t \leq t' \leq t''$, and property (P) applied to S_2 we get $\gamma_x(t') \in S_2$. Since

the S_i's form a partition and $\gamma_x(t') \in S_1 \cap S_2$ we also get $S_1 = S_2$. However, then $S_2 \backslash Pre(S_1) = \varnothing$ which contradicts $\gamma_x(t'') \in S_2 \backslash Pre(S_1)$. This concludes the proof of the inclusion and of formula (5.4).

Finally, we consider case 2. If the two occurrences of the same S_i are consecutive, then the expression may be rewritten with fewer terms $(I(S_i, I(S_i, A)) = I(S_i, A)$ for any set A). If the occurrences of S_i alternate with a different S_j, then we use Property (P) to conclude that $I(S, P_l) = \varnothing$ (since $I(S_i, I(S_j, S_i)) = \varnothing$ for $i \neq j$). This concludes the proof that $\mathscr{B}$ is a bisimulation. ∎

In the next section we list various classes of o-minimal hybrid systems.

6. Classes of O-Minimal Hybrid Systems

In this section we apply Theorem 5.3 to several special classes of o-minimal hybrid systems. For each o-minimal theory of Table 1, we provide examples of definable, o-minimal hybrid systems.

6.1. $\mathbb{R}_{\mathrm{lin}} = (\mathbb{R}, <, +, -)$

The definable sets in this theory capture polyhedral sets whereas the definable flows capture linear flows. It is a well-known fact that this theory is not only o-minimal but also decidable. Therefore, the definable o-minimal hybrid systems do not only admit finite bisimulations but there is also an effective procedure to compute them. This immediately leads to decidability of reachability for o-minimal hybrid systems defined in $\mathbb{R}_{\mathrm{lin}}$. In particular, it captures timed automata [AD] in the special case where all reset maps are constant.

6.2. $\mathbb{R}_{\mathrm{alg}} = (\mathbb{R}, <, +, -, \cdot)$

It was shown in [T] that $\mathbb{R}_{\mathrm{alg}}$ is decidable. In fact, the decision procedure consisted of two parts: first an algorithm for eliminating quantifiers, and second an algorithm for deciding quantifier free formulas. Because of these results, the definable sets in $\mathbb{R}_{\mathrm{alg}}$ (with parameters) are the *semialgebraic sets*, which are defined as Boolean combinations of sets of the form $\{x : p(x) < 0\}$ and $\{x : p(x) = 0\}$ where $p(x)$ is a polynomial. The definable flows in this theory are semialgebraic. Therefore, the o-minimal hybrid systems corresponding to this theory are hybrid systems H where all sets and flows are semialgebraic. Moreover, if all polynomials involved in the description of the hybrid system have rational coefficients, we obtain a new class of decidable hybrid systems.

The o-minimality of this theory can also be used to show the existence of finite bisimulations in special cases when the flow is not definable. This was illustrated in [LPS1] for the case of planar hybrid systems whose vector fields admit definable Hamiltonians. This captures the decidability result of [CV].

6.3. $\mathbb{R}_{\mathrm{an}} = (\mathbb{R}, <, +, -, \cdot, \{\hat{f}\})$

In order to describe the definable sets in this theory, we need the notions of *semianalytic* and *subanalytic* sets. We provide below an informal definition of these

notions. For precise definitions and properties the reader is referred to [BM]. We say that a bounded subset S of $\mathbb{R}^n$ is semianalytic in $\mathbb{R}^n$ if for every $x \in \mathbb{R}^n$ there exists a neighborhood U of x such that $U \cap S$ is a boolean combination of sets of the form $\{x : f(x) < 0\}$ and $\{x : f(x) = 0\}$ where f is an analytic function on U. Roughly speaking, a local description of a semianalytic set is analogous to that of a semialgebraic set with analytic functions replacing polynomials. A bounded subset S of $\mathbb{R}^n$ is subanalytic in $\mathbb{R}^n$ if it is the image of a relatively compact semi-analytic set T under an analytic map (defined on $\overline{T}$). The bounded subanalytic sets in $\mathbb{R}^n$ are definable in this theory.

Even though polynomial flows are definable in this theory, since the functions $\hat{f}$ are zero outside a compact set, these functions cannot be used to define complete flows. However, the *Pre* operator corresponding to some periodic flows may still be definable. Consider, for example, a hybrid system H whose vector fields are diagonalizable linear vector fields with purely imaginary eigenvalues and all relevant sets are definable in $\mathbb{R}_{an}$. Since the restriction of sin on $[-\pi, \pi]$ is definable, the *Pre* operator corresponding to F is definable. This leads to the following theorem which generalizes the planar result in [LPS1].

Theorem 6.1. *Let H be a hybrid system for which all relevant sets are (bounded) subanalytic and all vector fields are diagonalizable linear vector fields with purely imaginary eigenvalues. Then H admits a finite bisimulation.*

$$6.4. \quad \mathbb{R}_{exp} = (\mathbb{R}, <, +, -, \cdot, \exp)$$

The main difference between $\mathbb{R}_{exp}$ and the previous theories, besides enriching the class of definable sets, is the fact that the symbol exp represents a globally defined function. This allows new classes of definable flows. In particular, the flows of linear vector fields with real eigenvalues are definable. The following theorem is then a special case of Theorem 5.3.

Theorem 6.2. *Let H be a hybrid system for which all relevant sets are semi-algebraic and all vector fields are linear with real eigenvalues. Then H admits a finite bisimulation.*

It is not known if the theory of $\mathbb{R}_{exp}$ is decidable, although in [MW] it was shown that it would be a consequence of Schanuel's conjecture in number theory.

$$6.5. \quad \mathbb{R}_{exp, an} = (\mathbb{R}, <, +, -, \cdot, \exp, \{\hat{f}\})$$

This theory extends both $\mathbb{R}_{an}$ and $\mathbb{R}_{exp}$. We can therefore combine Theorems 6.1 and 6.2 to obtain the following result.

Theorem 6.3. *Let H be a hybrid system for which all relevant sets are (bounded) subanalytic and all vector fields are of one of the following two forms:*

- *Linear vector fields with real eigenvalues.*
- *Diagonalizable linear vector fields with purely imaginary eigenvalues.*

Then H admits a finite bisimulation.

The above theorem extends the planar results in [LPS1] to $\mathbb{R}^n$. Note that relaxations of Theorem 6.3 would allow spiraling, linear vector fields which are not definable in $\mathbb{R}_{\exp,an}$. As was shown by Example 3.3, such systems, in general, do not admit finite bisimulations.

6.6. Other Extensions

It is shown in [S2] that extensions of o-minimal theories by Pfaffian functions are also o-minimal. While this theory provides new globally defined functions, there are no easily described classes of vector fields whose flows are definable in it. The search for such classes is a topic for current research.

7. Conclusions

In this paper we presented a unified framework for tackling decidability questions of hybrid systems. We introduced the notion of o-minimal hybrid systems as initialized hybrid systems whose relevant sets and flows are definable in an o-minimal theory. We showed that all o-minimal hybrid systems admit finite bisimulations. Various examples from recently discovered o-minimal theories were presented. In addition, they extend the class of hybrid systems which admit finite bisimulations by enriching the class of relevant sets and incorporating more complex dynamics at each discrete location.

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