OPTIMAL COORDINATED MANEUVERS FOR THREE DIMENSIONAL AIRCRAFT CONFLICT RESOLUTION*

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In this paper, we study the problem of designing optimal coordinated maneuvers for multiple aircraft conflict resolution. We propose an energy function to select among all the conflict-free maneuvers the optimal one. The introduced cost function incorporates a priority mechanism that favors those maneuvers where aircraft with lower priority assume more responsibility in resolving the predicted conflicts. The energy-minimizing resolution maneuvers may involve changes of heading and speed, as well as of altitude. However, vertical maneuvers are penalized with respect to horizontal ones for the sake of passenger comfort. A geometric construction and a numerical algorithm for computing the optimal resolution maneuvers are given in the two aircraft case. As for the multi-aircraft case, an approximation scheme is proposed to compute a suboptimal two-legged solution. Extensive examples are presented to illustrate the effectiveness of the proposed algorithms.

Introduction

The main concern of Air Traffic Management (ATM) systems is guaranteeing *safety*. This is achieved by avoiding the occurrence of *conflicts*, i.e., of those situations where two aircraft come closer to each other than a minimum allowed horizontal separation R *and* a minimum allowed vertical separation H at the same time. Currently, R is set equal to 5 nautical miles (nmi) in en-route airspace, and 3 nmi inside the Terminal Radar Approach Control facilities (TRACONs), whereas H is 2000 feet (ft) above the altitude of $29,000$ ft (FL290), and 1000 ft below FL290. Conflict avoidance is typically decomposed into two phases:

- *conflict detection*, where potential conflicts that may arise in the future are detected based on the available information on the aircraft current positions, headings, and flight plans;
- *conflict resolution*, where the flight plans of the aircraft involved in the detected conflicts are replanned so as to prevent that any conflict actually occurs.

In this paper we shall focus on conflict resolution.

The existing approaches to aircraft conflict resolution can be classified according to various criteria. The interested reader is referred to[1] for an up-to-date survey on the different conflict resolution approaches proposed in the literature. In the following we shall review briefly some of the most relevant ones.

Based on the level of coordination or on the level of mutual trust among participating aircraft, conflict resolution methods can be classified as *noncooperative* and *cooperative*[2].
In the noncooperative case, the aircraft involved in the encounter do not exchange information on their intentions and do not trust one another at all, hence the worst case approach is adopted. In the solution proposed in[3,4], the two-aircraft conflict resolution problem is formulated as each aircraft playing a zero-sum noncooperative game against disturbances that model the uncertainty in the other aircraft intentions, with the value function being the aircraft distance. The differential game methodology is also used in[5] for determining the safe region for aircraft approaching closely spaced parallel runways.
In the cooperative conflict resolution case, the current positions and intentions of the aircraft are assumed to be perfectly known to a supervising central controller. Each aircraft completely trusts the central controller (and hence all the other aircraft), and follows its advice. The cooperative conflict resolution problem is typically formulated as an optimization problem, where the flight plans of all the aircraft are designed so as to avoid conflicts while minimizing a certain cost function. Contributions in the literature

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belonging to this class include[6–11], to mention only a few.
In between the extremes of noncooperative and cooperative conflict resolution there is the *probabilistic* conflict resolution approach. In this approach, each aircraft position is assumed to be distributed according to some probabilistic law, which models the presence of disturbances affecting the aircraft motion as well as the partial confidence of each aircraft in the available information on the intentions of the other aircraft. Contributions belonging to this category include [12,13] for the two aircraft case, and [14,15] for the multiple aircraft case.

Based on whether vertical maneuvers are employed or not, conflict resolution methods can be classified as *two-dimensional* (2D) or *three-dimensional* (3D), with the former being a particular case of the latter. Typically, conflicts are resolved by resorting to three different actions: turn, climb/descend, and accelerate/decelerate, which affect the aircraft heading, altitude, and speed, respectively. Resolution strategies adopting one of these actions or a combination of them are analyzed and compared in terms of cost and efficacy in[16]. It is found that climb/descend is the most efficient action for resolving short-term conflicts, since the horizontal separation requirement is much more stringent than the vertical one. Vertical maneuvers are actually used to resolve imminent conflicts in the Traffic Alert and Collision Avoidance System (TCAS[17,18]) currently operating on board of all commercial aircraft carrying more than thirty passengers. On the other hand, excessive changes of altitude are likely to cause discomfort to passengers and are not much compatible with the current vertically layered structure of the airspace. These facts together with the relative simplicity of dealing with the two dimensional case have caused most of the approaches proposed in the literature to focus on 2D conflict resolution, assuming level flight and horizontal resolution maneuvers.

In this paper, we address the problem of *optimal cooperative 3D conflict resolution involving multiple aircraft*. Conflict situations involving more than two aircraft may actually occur in areas with high traffic density. For example, it may happen that by solving a two aircraft conflict without taking into consideration the surrounding aircraft, a new conflict with a third aircraft is generated (domino effect). Nevertheless, only a few of the existing treatments on conflict resolution deal with the multiple aircraft case, since resolving them is intrinsically more difficult than dealing with the two aircraft case. In[14,15,19] the potential and vortex field method is used to determine multi-aircraft coordinated maneuvers, which, however, are not guaranteed to be safe. An alternative approach consists in formulating the multiple aircraft conflict resolution problem as a constrained optimization problem. The contributions belonging to this category[6–8,10,20,21] differ for the model and cost function used, and also for the method adopted to solve the resultant optimization problem, e.g., genetic algorithms[6], semidefinite programming combined with a branch-and-bound search[7], and sequential quadratic programming (SQP) using a linear approximation of the feasible region[10]. The last method is the closest in spirit to the approximation scheme we shall propose in this paper, though our problem formulation is different. In[20], time-optimal cooperative conflict resolution for multiple aircraft flying at the same altitude with constant speed and bounded curvature is studied using optimal control techniques, and a numerical algorithm is proposed for computing a suboptimal solution.

This paper is organized as follows. First we formulate precisely the optimal conflict resolution problem we shall deal with. In particular, we describe the energy cost function used for selecting among all the conflict-free coordinated maneuvers the optimal one. This cost function favors resolution maneuvers of not only shorter travel distance, but also less speed variation, thus taking into account important practical factors such as fuel consumption and passenger comfort. Also, the energy function depends on some parameters that allow one to assign different priorities to the aircraft and penalize excessive vertical maneuvers.
A necessary condition for a conflict-free coordinated maneuver to be optimal is then derived through a variational analysis. Compared with conventional variational problems, special attention has to be paid to the presence of the conflict-free constraint. We show that in the two aircraft case the derived necessary condition is sufficient to give a geometric characterization of the optimal resolution maneuvers. A numerical procedure is proposed to compute them, and simulation results are presented. In the multiple aircraft case, the original constrained optimization problem is approximated by a finite dimensional convex optimization problem with linear constraints. This is achieved by considering two-legged coordinated maneuvers specified by a set of waypoints, and making a linear approximation of the region to which the waypoints should belong for the corresponding two-legged coordinated maneuver to be conflict-free. We describe how a not-too-conservative inner approximation scheme can be carried out, and discuss the effect of various parameters on the optimal resolution maneuvers by simulation examples.
We then point out the limitations of the proposed approach and suggest some methods to alleviate them. In particular, we introduce additional (convex) constraints on the waypoints position so as to avoid sharp turns near the waypoints, which would eventually cause the two-legged maneuvers to be not flyable in practice. Conclusions are given at the end of the paper.

As a general remark, note that, in order to make the comprehension of the technical derivations easier, we give a geometric interpretation of the obtained results, wherever possible. Also, we put emphasis on the implementative aspects of the proposed algorithms.

Problem formulation

Consider a single aircraft, say aircraft i, flying from position $a_i \in \mathbb{R}^3$ at time t_0 to position $b_i \in \mathbb{R}^3$ at time t_f. Set $T \triangleq [t_0, t_f]$, and denote by $\mathbf{P}_i$ the set of all *maneuvers* for aircraft i, where a maneuver is defined to be a continuous and piecewise C^1 map, say α_i, from T to $\mathbb{R}^3$ satisfying $\alpha_i(t_0) = a_i$ and $\alpha_i(t_f) = b_i$[1].

The *energy* of a maneuver $\alpha_i \in \mathbf{P}_i$ is then defined as

$$J(\alpha_i) = \frac{1}{2} \int_{t_0}^{t_f} ||\dot{\alpha}_i(t)||^2 \, dt. \tag{1}$$

Denote by $L(\alpha_i)$ the length of the curve α_i, i.e., $L(\alpha_i) = \int_{t_0}^{t_f} ||\dot{\alpha}_i(t)|| \, dt$. Then, by the Cauchy-Schwartz inequality,[22] $J(\alpha_i)$ satisfies

$$J(\alpha_i) \geq \frac{1}{2} \frac{L(\alpha_i)^2}{(t_f - t_0)}.$$

The equality holds if and only if the speed $\|\dot{\alpha}_i(t)\|$ is constant, and in this case the energy $J(\alpha_i)$ is proportional to the square of the length of α_i. This implies that the maneuver with the least energy for a single aircraft is the constant-speed motion along the line segment from its starting to its destination position. If the aircraft is forced to move along some fixed curve other than the line segment, then the parameterization of the curve with the least energy is the one with constant speed, and the minimal energy is proportional to the square of the curve length. As a result, the least energy maneuver between two points in the presence of *static* obstacles is the shortest curve joining the two points without crossing the obstacles, parameterized proportionally to its arc length. This observation will be used in a later section when dealing with the two aircraft case.

In the following developments, we assume that a group of aircraft flying in a certain region of the airspace has been isolated so that only conflicts among aircraft within this group need to be considered during the time interval of interest. This assumption, although impractical, is commonly adopted in the literature.

Suppose that the group under consideration is composed of n aircraft numbered from 1 to n. Set $\mathbf{P}(\mathbf{a}, \mathbf{b}) \triangleq \prod_{i=1}^{n} \mathbf{P}_i$, where $\mathbf{a} \triangleq (a_1, \ldots, a_n)$ and $\mathbf{b} \triangleq (b_1, \ldots, b_n)$. Then, each element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbf{P}(\mathbf{a}, \mathbf{b})$ represents a *joint maneuver* (n-maneuver or simply maneuver when there is no ambiguity) for the n-aircraft system with starting position $\mathbf{a}$ and destination position $\mathbf{b}$. A joint maneuver $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{P}(\mathbf{a}, \mathbf{b})$ is said to be *conflict-free* if, for all the duration of the encounter, none of the aircraft enters the cylindrical protection zone of radius R and height $2H$ surrounding any other aircraft. If for an arbitrary $c \in \mathbb{R}^3$ we denote by $c_{xy} \in \mathbb{R}^2$ and $c_z \in \mathbb{R}$ its components on the horizontal xy plane and the vertical z-axis respectively, then the conflict-free condition is equivalent to the condition that there is no pair of indices (i, j), $1 \leq i < j \leq n$, such that $\|\alpha_{i,xy}(t) - \alpha_{j,xy}(t)\| < R$ and $|\alpha_{i,z}(t) - \alpha_{j,z}(t)| < H$ for some $t \in T$.

We denote by $\mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ the set of all *conflict-free (joint) maneuvers* for the n-aircraft system with starting position $\mathbf{a} = (a_1, \ldots, a_n)$ and destination position $\mathbf{b} = (b_1, \ldots, b_n)$. Throughout the paper we assume that each pair of points in the n-tuple $(a_1, \ldots, a_n)$ satisfies either the horizontal or the vertical separation condition so that there is no conflict for the n-aircraft system at time t_0. Similarly for $(b_1, \ldots, b_n)$. As a result, the set $\mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ is nonempty. Conflict-free maneuvers in $\mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ will occasionally be called *resolution maneuvers.*

The performance of each n-maneuver $\alpha \in \mathbf{P}(\mathbf{a}, \mathbf{b})$ can be characterized in terms of the cost function:

$$J_\mu(\alpha) \triangleq \sum_{i=1}^{n} \mu_i J(\alpha_i), \tag{2}$$

where $J(\alpha_i)$ is the energy of α_i defined in equation (1), and $\mu_1, \ldots, \mu_n$ are positive real numbers adding up to 1 that represent the priorities of the aircraft. Given a joint maneuver $\alpha \in \mathbf{P}(\mathbf{a}, \mathbf{b})$, we call $J_\mu(\alpha)$ its *μ-energy.*

In absence of the separation constraint, the joint maneuver minimizing the μ-energy is clearly the one where each aircraft flies at constant speed along the straight line joining its starting to its destination position. If we consider the separation requirement, then the conflict-free maneuver with minimal μ-energy will still tend to be straight and smooth, which has important practical implications in terms of, for example, passenger comfort and fuel consumption. Observe that, by choosing different coefficients μ_i, $i = 1, \ldots, n$, one can assign different priorities to the n aircraft. In particular, one should associate smaller μ_i's to those aircraft with higher maneuverability so that they will assume a larger responsibility in resolving the conflict.

Our goal is then to solve the constrained optimization problem:

$$\text{Minimize } J_\mu(\alpha) \text{ subject to } \alpha \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b}). \tag{3}$$

Each solution α^* to problem (3) is called an *optimal (resolution) maneuver* for the multi-aircraft system.

Note that, in this formulation, it can be expected that the optimal resolution maneuvers will mainly utilize the vertical dimension for almost all encounters

[1]Piecewise C^1 means that there is finite subdivision of T such that the map α_i is continuously differentiable till the first order on each open subinterval. Then $\dot{\alpha}_i(t)$, denotes the first derivative of α_i at those t where it is well defined, i.e., at all except a finite number of $t \in T$.

since the minimum allowed vertical distance H is much smaller than the minimum allowed horizontal distance R. However, vertical maneuvers are usually the least comfortable ones for passengers. This is the reason why we now redefine the energy of a maneuver α_i in equation (1) as follows:

$$J(\alpha_i) = \frac{1}{2}\int_{t_0}^{t_f} [\|\dot{\alpha}_{i,xy}(t)\|^2 + \eta^2|\dot{\alpha}_{i,z}(t)|^2]\,dt, \qquad (4)$$

where $\eta \geq 1$ is a coefficient introduced to penalize vertical maneuvers. This modification does not add further difficulties to the solution of problem (3), since the minimization of the new cost function can be easily reduced to the minimization of the one without penalty by scaling the z-axis by a factor of η. The μ-energy with penalty η of a joint maneuver is in fact equal to the μ-energy without penalty of the scaled version of the same joint maneuver, and optimal solutions to the scaled problem can be scaled back to give the optimal solutions to the original problem.

After scaling, the protection zone becomes a cylinder of radius R and height $2\eta H$, hence for large values of η, horizontal resolution maneuvers are more likely to be invoked. In particular, in the level flight case, when $\eta \to \infty$ the problem degenerates into the 2D resolution problem studied in[21] .

Without loss of generality, we shall then consider $\eta = 1$ in the following developments.

The μ-alignment condition

In this section we derive a necessary condition for a conflict-free maneuver to be optimal, which will then be used to obtain the optimal resolution maneuvers in the two aircraft case.

Consider the destination position $\mathbf{b} = (b_1, \ldots, b_n)$. For each $w \in \mathbb{R}^3$ we denote by $\mathbf{b} + w$ the n-tuple $(b_1 + w, \ldots, b_n + w)$, which can be thought of as a new destination position of the n-aircraft system.

Definition 1 *The tilt operator* $\mathcal{T}_w : \mathbf{P}(R, H; \mathbf{a}, \mathbf{b}) \to \mathbf{P}(R, H; \mathbf{a}, \mathbf{b} + w)$ *is a map such that for any* $\alpha \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$, $\beta = \mathcal{T}_w(\alpha) \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b} + w)$ *is defined by* $\beta_i(t) = \alpha_i(t) + \frac{t-t_0}{t_f-t_0}w$, $\forall t \in T$, $i = 1, \ldots, n$.

It is easily verified that $\mathcal{T}_w \circ \mathcal{T}_{-w} = \mathcal{T}_{-w} \circ \mathcal{T}_w = id$, where $\circ$ denotes map composition and id the identity map. Hence $\mathcal{T}_w$ is a bijection. Moreover,

Proposition 1 *Suppose that* $\alpha^* \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ *is an optimal solution to problem (3). Then* $\beta^* = \mathcal{T}_w(\alpha^*)$ *minimizes* $J_\mu(\beta)$ *subject to* $\beta \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b} + w)$.

Proof: For any $\beta \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b} + w)$, let $\alpha = \mathcal{T}_{-w}(\beta)$. Then $\alpha \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$, and $J_\mu(\beta)$ can be expressed as

$$\begin{aligned} J_\mu(\beta) &= \frac{1}{2}\int_{t_0}^{t_f} \sum_{i=1}^n \mu_i \Big\|\dot{\alpha}_i(t) + \frac{w}{t_f - t_0}\Big\|^2\,dt \\ &= J_\mu(\alpha) + \frac{w^T(\sum_{i=1}^n \mu_i(b_i - a_i) + w/2)}{t_f - t_0}. \end{aligned} \qquad (5)$$

Notice that the second term in the last expression is a constant independent of β. Denote it by C. From equation (5) and the optimality of α^*, it follows that $J_\mu(\beta) \geq J_\mu(\alpha^*) + C$, $\forall \beta \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b} + w)$, where equality holds if $\alpha = \alpha^*$, i.e. if $\beta = \beta^*$. This concludes the proof. ∎

The starting and destination positions $\mathbf{a}$ and $\mathbf{b}$ of an n-aircraft system are said to be *μ-aligned* if they have the same μ-centroid, i.e., if $\sum_{i=1}^n \mu_i a_i = \sum_{i=1}^n \mu_i b_i$. For arbitrary $\mathbf{a}$ and $\mathbf{b}$, set $\mathbf{b}' = \mathbf{b} + w$ where $w = \sum_{i=1}^n \mu_i(a_i - b_i)$, then $\mathbf{a}$ and $\mathbf{b}'$ are μ-aligned.

We next introduce the drift operation on joint maneuvers, which generates a conflict-free maneuver when applied to a conflict-free maneuver. This operator is used in the proof of Proposition 2.

Definition 2 *Let* $\gamma : T \to \mathbb{R}^3$ *be a continuous and piecewise* C^1 *map such that* $\gamma(t_0) = \gamma(t_f) = 0$. *Then the drift operator* $\mathcal{D}_\gamma : \mathbf{P}(R, H; \mathbf{a}, \mathbf{b}) \to \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ *is a map such that for any* $\alpha \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$, $\beta = \mathcal{D}_\gamma(\alpha) \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ *is defined by* $\beta_i(t) = \alpha_i(t) + \gamma(t)$, $\forall t \in T$, $i = 1, \ldots, n$.

Using the fact that the μ-energy of an optimal resolution maneuver cannot decrease under the perturbation of any drift operation, one can derive the optimality condition in Proposition 2 below. The proof starts by considering the μ-aligned case, and then proceeds to the case of arbitrary $\mathbf{a}$ and $\mathbf{b}$ by using the conclusion of Proposition 1.

Proposition 2 *Assume that* $\alpha^* \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ *is an optimal solution to problem (3). Then for all* $t \in T$,

$$\sum_{i=1}^n \mu_i \alpha_i^*(t) = \sum_{i=1}^n \mu_i a_i + \frac{t - t_0}{t_f - t_0}\Big(\sum_{i=1}^n \mu_i b_i - \sum_{i=1}^n \mu_i a_i\Big),$$

which in the case of μ-aligned $\mathbf{a}$ *and* $\mathbf{b}$ *reduces to*

$$\sum_{i=1}^n \mu_i \alpha_i^*(t) = \sum_{i=1}^n \mu_i a_i = \sum_{i=1}^n \mu_i b_i, \quad \forall t \in T.$$

Proof: We start by considering the case when $\mathbf{a}$ and $\mathbf{b}$ are μ-aligned. Consider a continuous and piecewise C^1 map $\gamma : T \to \mathbb{R}^3$ satisfying $\gamma(t_0) = \gamma(t_f) = 0$. For each $\lambda \in \mathbb{R}$ define $\beta_\lambda \triangleq \mathcal{D}_{\lambda\gamma}(\alpha^*)$. Note that $\beta_\lambda \in$

$\mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ for all $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} & J_\mu(\beta_\lambda) - J_\mu(\alpha^*) \\ =& \frac{1}{2}\int_{t_0}^{t_f} \sum_{i=1}^{n} \mu_i \|\dot{\alpha}_i^*(t) + \lambda\dot{\gamma}(t)\|^2 \, dt - J_\mu(\alpha^*) \\ =& \frac{\lambda^2}{2}\int_{t_0}^{t_f} \|\dot{\gamma}(t)\|^2 \, dt + \lambda \int_{t_0}^{t_f} \dot{\gamma}(t)^T \sum_{i=1}^{n} \mu_i \dot{\alpha}_i^*(t) \, dt \, . \end{aligned}$$

The difference $J_\mu(\beta_\lambda) - J_\mu(\alpha^*)$ is a quadratic function of λ which, by the optimality of α^*, must be nonnegative for all $\lambda \in \mathbb{R}$. Hence $\int_{t_0}^{t_f} \dot{\gamma}(t)^T \sum_{i=1}^{n} \mu_i \dot{\alpha}_i^*(t) \, dt = 0$ must hold for any choice of γ such that $\gamma(t_0) = \gamma(t_f) = 0$. Since $\mathbf{a}$ and $\mathbf{b}$ are μ-aligned, we can choose $\gamma(t) = \sum_{i=1}^{n} \mu_i \alpha_i^*(t) - \sum_{i=1}^{n} \mu_i a_i$. Given that α^* is piecewise C^1, this leads to $\sum_{i=1}^{n} \mu_i \dot{\alpha}_i^*(t) = 0$ for almost all $t \in T$, and hence, by integration, to the desired conclusion for the μ-aligned case.

For the general case when $\mathbf{a}$ and $\mathbf{b}$ are not necessarily μ-aligned, let $w = \sum_{i=1}^{n} \mu_i (a_i - b_i)$. Then $\mathcal{T}_w$ maps α^* to $\beta^* = \mathcal{T}_w(\alpha^*)$ which, by Proposition 1, minimizes $J_\mu(\beta)$ over all $\beta \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b}')$ with $\mathbf{b}' = \mathbf{b} + w$. Since $\mathbf{a}$ and $\mathbf{b}'$ are μ-aligned, we know that

$$\sum_{i=1}^{n} \mu_i \beta_i^*(t) = \sum_{i=1}^{n} \mu_i a_i, \quad \forall t \in T.$$

The desired conclusion is then obtained by using the relation $\alpha^* = \mathcal{T}_{-w}(\beta^*)$. ■

Note that the result in Proposition 2 can be restated by saying that the μ-centroid of α^* moves at constant speed along the straight line joining the μ-centroid of $\mathbf{a}$ to the μ-centroid of $\mathbf{b}$. This property will allow us to derive the solution to problem (3) in the two aircraft case.

In the sequel, we shall focus on the μ-aligned case, since in fact by Proposition 1 solving problem (3) for $\mathbf{a}$ and $\mathbf{b}$ is equivalent to solving problem (3) for the μ-aligned $\mathbf{a}$ and $\mathbf{b}' = \mathbf{b} + \sum_{i=1}^{n} \mu_i (a_i - b_i)$.

Two aircraft case

Assume that $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are μ-aligned and denote by c their common μ-centroid, i.e., $c = \mu_1 a_1 + \mu_2 a_2 = \mu_1 b_1 + \mu_2 b_2$. By Proposition 2, an optimal 2-maneuver $\alpha^* = (\alpha_1^*, \alpha_2^*) \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ satisfies

$$\alpha_1^*(t) - c = -\frac{\mu_2}{\mu_1}(\alpha_2^*(t) - c), \quad \forall t \in T, \qquad (6)$$

from which it easily follows that the energies of α_1^* and α_2^* are related by $\mu_1^2 J(\alpha_1^*) = \mu_2^2 J(\alpha_2^*)$. Hence problem (3) becomes finding among all conflict-free maneuvers satisfying equation (6) the one that minimizes the energy of the maneuver for a single aircraft, say, aircraft 1. The separation constraint can be simplified as well, since by equation (6) it is equivalent to the condition that the curve $\alpha_1^*(\cdot)$ never enters the cylinder W_μ of radius $R_\mu = \mu_2 R$ and height $2H_\mu = 2\mu_2 H$ centered symmetrically around the μ-centroid c.

As a result of these simplifications, problem (3) is equivalent to:

$$\begin{aligned} & \text{Minimize } J(\alpha_1) \text{ subject to} \\ & \alpha_1 \in \mathbf{P}_1, \alpha_1(t) \in \mathbb{R}^3 \setminus W_\mu, \forall t \in T, \end{aligned} \qquad (7)$$

which consists in finding the minimum energy maneuvers of aircraft 1 in the presence of the static obstacle W_μ. From the discussion following definition (1) of the energy of a maneuver, we then know that a solution to problem (7) is a constant-speed motion along a shortest curve joining a_1 to b_1 while avoiding the obstacle W_μ. Under the feasibility assumption, both a_1 and b_1 belong to $\mathbb{R}^3 \setminus W_\mu$, and such a curve can be computed efficiently by an algorithm whose description is postponed to a later section. Once α_1^* is computed, then α_2^* can be obtained from α_1^* through equation (6), thus concluding the treatment of the μ-aligned case.

For not necessarily μ-aligned $\mathbf{a}$ and $\mathbf{b}$, by Proposition 1 an optimal solution $\alpha^* \in \mathbf{P}(R, H; \mathbf{a}, \mathbf{b})$ to problem (3) is given by:

$$\begin{cases} \alpha_1^*(t) = \gamma_1^*(\mathbf{a}, \mathbf{b} + w)(t) - \frac{t - t_0}{t_f - t_0} w \\ \alpha_2^*(t) = \gamma_2^*(\mathbf{a}, \mathbf{b} + w)(t) - \frac{t - t_0}{t_f - t_0} w \end{cases}, \quad \forall t \in T, \qquad (8)$$

where $(\gamma_1^*(\mathbf{a}, \mathbf{b} + w), \gamma_2^*(\mathbf{a}, \mathbf{b} + w))$ denotes an optimal conflict-free maneuver in $\mathbf{P}(R, H; \mathbf{a}, \mathbf{b} + w)$ with $w = \mu_1 a_1 - \mu_1 b_1 + \mu_2 a_2 - \mu_2 b_2$ (note that $\mathbf{a}$ and $\mathbf{b} + w$ are μ-aligned).

The optimal solutions depend on the choice of the priority coefficients μ_1 and μ_2. Consider the case when the priority of aircraft 1 is much larger than that of aircraft 2 so that $\mu_2 \simeq 0$. In the μ-aligned case this implies that $a_1 \simeq b_1$, and the radius and height of the cylinder W_μ are approximately 0. Therefore γ_1^* is nearly a zero motion. For not necessarily μ-aligned $\mathbf{a}$ and $\mathbf{b}$, from the first equation in (8) it follows that an optimal maneuver for aircraft 1 is almost a constant-speed motion along the line segment from a_1 to b_1. Hence, as expected, aircraft 1 behaves as if there were no other aircraft flying in the same region, whereas aircraft 2 is the one assuming the responsibility of avoiding conflicts.

Some examples of optimal 2-maneuvers

In this section, we present some examples of two-aircraft encounters, and discuss the influence of various factors on the corresponding optimal resolution maneuvers. In all the examples, the coordinates of the aircraft positions are measured in nmi, $R = 5$ nmi and $H = 0.3292$ nmi.

We start by considering a two-aircraft encounter where $a_1 = (0, 20, 1)$, $b_1 = (40, 20, 1)$, and $a_2 = (20, 0, 1)$, $b_2 = (20, 40, 1)$, so that the two straight lines connecting the starting and destination positions

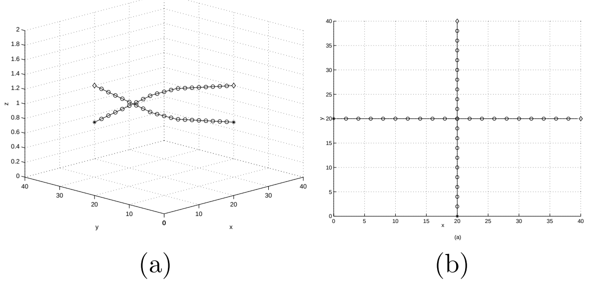


Fig. 1 An optimal resolution maneuver for an orthogonal two-aircraft encounter ($\eta = 5$ and $\mu_1 = \mu_2 = 0.5$): (a) 3D representation; (b) top view.

of each aircraft are on the same horizontal plane and cross each other at a right angle. These two lines represent the *ideal trajectories* of the two aircraft.
Figure 1 shows an optimal maneuver in the case when the two aircraft have the same priority ($\mu_1 = \mu_2 = 0.5$) and $\eta = 5$. Starting and destination positions of the two aircraft are marked with stars and diamonds respectively, whereas the circles represent the aircraft positions at equally spaced time instants. Hence the denser the circles, the slowlier the motions. The top view in (b) shows that the conflict is resolved by vertical deviations from the ideal trajectories.

Figure 2 represents optimal resolution maneuvers for the same two-aircraft orthogonal encounter under three different sets of aircraft priorities and the same η ($\eta = 5$). Although the optimal maneuvers in all three cases have the same top view (shown in the right-hand side of Figure 1), the vertical deviation of aircraft 1 from its ideal trajectory decreases as its priority increases. In other words, aircraft 2 with smaller priority will assume more responsibility in resolving the conflict. In the extreme case when $\mu_1 = 1$ and $\mu_2 = 0$, the optimal resolution maneuver will be such that aircraft 1 flies along its ideal trajectory, while aircraft 2 assumes all the responsibility of avoiding conflicts with aircraft 1. These conclusions on the effect of the priority coefficients on the optimal resolution maneuvers hold in general for multi-aircraft encounters.

As for the effect of the vertical penalty factor, note that in Figure 1, where $\eta = 5$ and $\mu_1 = \mu_2 = 0.5$, the conflict is resolved using only vertical deviations from the ideal trajectories. In contrast, if η is set equal to 15 ($\mu_1 = \mu_2 = 0.5$), the conflict is resolved using only horizontal deviations (Figure 3). The explanation is that, in order to obtain the optimal resolution maneuvers, we have to scale the z-axis by a factor of η. When η is large so that the height of the cylindrical obstacle becomes much larger than its radius, a shortest curve between two points across the cylinder is more likely to be a curve around the side of the cylinder rather than around its top or bottom. Therefore the larger the vertical penalty factor η, the more likely it is that an optimal resolution maneuver will consist of horizontal deviations from the ideal trajectories. In general,

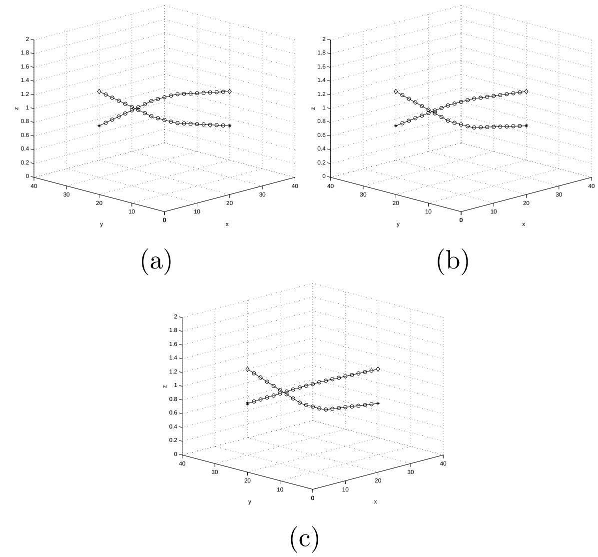


Fig. 2 Optimal resolution maneuvers for the orthogonal two-aircraft encounter with $\eta = 5$ and (a) $\mu_1 = 0.5$, $\mu_2 = 0.5$; (b) $\mu_1 = 0.7$, $\mu_2 = 0.3$; (c) $\mu_1 = 0.9$, $\mu_2 = 0.1$.

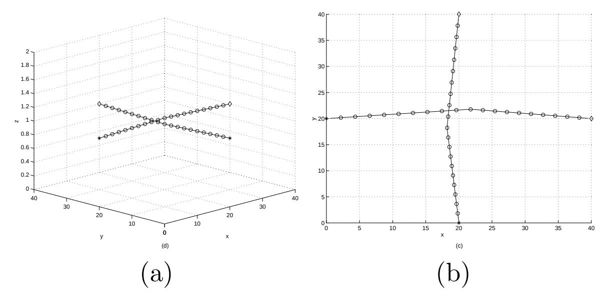


Fig. 3 An optimal resolution maneuver for the orthogonal two-aircraft encounter with $\eta = 15$ ($\mu_1 = \mu_2 = 0.5$): (a) 3D representation; (b) top view.

for encounters involving two or more aircraft with the aircraft initial and destination positions all at about the same altitude, there are two extreme cases: When η is very large, the problem degenerates into a planar conflict resolution problem, where only horizontal deviations are allowed in resolving the conflict; When η is close to 0, then only vertical deviations are used in the optimal resolution maneuvers and their top view consists of straight line segments.

Shortest curve between two points in $\mathbb{R}^3$ avoiding a cylindrical obstacle

In this section we describe briefly how to compute a shortest curve in $\mathbb{R}^3$ connecting two points while avoiding a cylindrical obstacle. This is to complete the solution to problem (3) in the two aircraft case.

Consider a cylinder of radius r and heigh $2h$ centered at the origin:

$$D = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 < r^2 \text{ and } |z| < h\}.$$

Given two points a and b in $\mathbb{R}^3 \setminus D$, our objective is to

$$\text{Find a shortest curve in } \mathbb{R}^3 \setminus D \text{ connecting } a \text{ and } b. \tag{9}$$

We require the curve to be continuous and piecewise C^1 so that its arc length is well defined.

It is obvious that, when a and b are *visible* to each other in the sense that the line segment joining a and b does not intersect the obstacle D, the shortest curve between a and b is the straight line segment joining them, hence the solution to problem (9) is trivial. Suppose now that a and b are not visible to each other.

Curves that are locally distance minimizing are called *geodesics*. A shortest curve between two points is necessarily a geodesic. In this sense, problem (9) is a special instance of the general problem of finding distance-minimizing geodesics in manifolds with (non-smooth) boundary, which is studied to some extent in[23, 24]. In particular,

Proposition 3 *A shortest curve in $\mathbb{R}^3 \setminus D$ connecting a and b can be decomposed into three segments: a straight line segment from a to a point $p \in \partial D$, a geodesic segment of ∂D from p to a point $q \in \partial D$, and a straight line segment from q to b. Moreover, the two line segments are contained entirely in the interior of $\mathbb{R}^3 \setminus D$ except for their end points p and q.*

If the shortest curve between a and b is viewed as a path traveled from a to b, then Proposition 3 says that the curve will enter and exit ∂D exactly once, at positions p and q respectively. We then call p and q *entry point* and *exit point*, respectively. As a result of Proposition 3, solving problem (9) is equivalent to determining the entry point p, the exit point q, and the distance-minimizing geodesic segment on ∂D between p and q. In certain cases, some or all of the three segments in Proposition 3 can degenerate into points.

Notice that D is a subset of the cylinder Q defined by $Q = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 < r^2\}$. We can then distinguish three cases:

Case 1. Both a and b are outside of Q, and at least one of them has z-coordinate in $[-h, h]$;

Case 2. Both a and b are outside of Q, and none of them has z-coordinate in $[-h, h]$;

Case 3. At least one of a and b belongs to Q,

In each one of these cases a solution to problem (9) can assume only a finite number of qualitatively different configurations (see Figure 4). Within each admissible configuration, one can reduce problem (9) to a simple optimization problem over a compact region of $\mathbb{R}^1$ or at most $\mathbb{R}^2$, and solve it numerically using softwares such as MATLAB. The (global) optimum is obtained by choosing among the so-obtained curves the one with the smallest length. For further details, see[25].

Optimal two-legged maneuvers

The approach adopted in the previous section to compute optimal resolution maneuvers in the two aircraft case cannot be easily generalized to the multiple aircraft case since there are too many configurations to be considered. Therefore, in this section we simplify the problem by considering two-legged maneuvers specified by a set of waypoints.

Reformulation of the problem

Consider an n-aircraft system with starting position $\mathbf{a} = (a_1, \dots, a_n)$ and destination position $\mathbf{b} = (b_1, \dots, b_n)$. Fix an epoch $t_c \in T$ such that $t_0 < t_c < t_f$. For each aircraft i, $i = 1, \dots, n$, choose a waypoint $c_i \in \mathbb{R}^3$. A *two-legged maneuver* with waypoint c_i for aircraft i is a maneuver consisting of two stages: first from a_i at time t_0 to c_i at time t_c, and then from c_i at time t_c to b_i at time t_f, moving at constant velocity in both stages. Denote by $\mathbf{P}_{i,2}$ the set of all two-legged maneuvers for aircraft i, and with $\mathbf{P}_2(\mathbf{a}, \mathbf{b}) = \prod_{i=1}^n \mathbf{P}_{i,2}$ the set of all *two-legged joint maneuvers* for the n-aircraft system. Denote by $\mathbf{P}_2(R, H; \mathbf{a}, \mathbf{b})$ the subset of $\mathbf{P}_2(\mathbf{a}, \mathbf{b})$ consisting of all those elements of $\mathbf{P}_2(\mathbf{a}, \mathbf{b})$ that are conflict-free. We assume that the epoch t_c is fixed, so that each maneuver in $\mathbf{P}_2(\mathbf{a}, \mathbf{b})$ (and hence in $\mathbf{P}_2(R, H; \mathbf{a}, \mathbf{b})$) is uniquely specified by its waypoints $(c_1, \dots, c_n)$.

Now we try to solve the following problem:

$$\text{Minimize } J_\mu(\alpha) \text{ subject to } \alpha \in \mathbf{P}_2(R, H; \mathbf{a}, \mathbf{b}). \tag{10}$$

One of the reasons why it makes sense studying problem (10) instead of the general problem (3) is related to the ATM practice: it is far simpler for the central controller to transmit the aircraft trajectory information in the form of waypoints and time to reach them rather than continuous trajectories. From a methodological point of view, since each maneuver in $\mathbf{P}_2(R, H; \mathbf{a}, \mathbf{b})$ is parameterized by a vector of waypoints $(c_1, \dots, c_n)$, then problem (10) is a finite dimensional optimization problem, which is much easier to deal with than the variational problem (3).

In the two-legged case, both the cost function and the constraints in problem (10) can be simplified, and a (suboptimal) solution can be computed. We start by considering the cost function, and postpone the discussion on the constraints to later on.

Let α be a two-legged joint maneuver in $\mathbf{P}_2(\mathbf{a}, \mathbf{b})$ with waypoints $(c_1, \dots, c_n)$. Then α is specified by

$$\alpha_i(t) = \begin{cases} a_i + (c_i - a_i)\frac{t-t_0}{t_c-t_0}, & t_0 \le t \le t_c \\ b_i + (c_i - b_i)\frac{t-t_f}{t_c-t_f}, & t_c < t \le t_f \end{cases}, \tag{11}$$

for $i = 1, \dots, n$. It is easy to verify that the μ-energy of α with $\eta = 1$ is given by

$$J_\mu(\alpha) = \frac{t_f - t_0}{(t_f - t_c)(t_c - t_0)} \sum_{i=1}^n \mu_i \|c_i - c_i^u\|^2 + C, \tag{12}$$

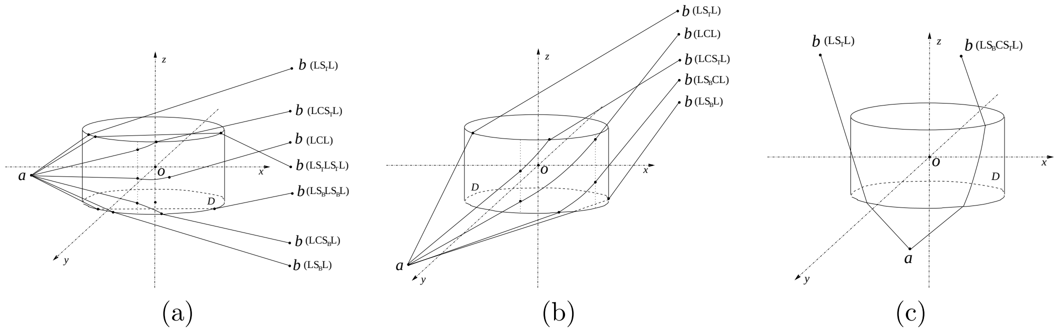


Fig. 4 Possible configurations for the solutions in (a) case 1; (b) case 2; (c) case 3.

where C is a constant and

$$c_i^u = \frac{(t_f - t_c)a_i + (t_c - t_0)b_i}{t_f - t_0}, \quad i = 1, \ldots, n, \tag{13}$$

are the optimal waypoints when minimizing $J_\mu(\alpha)$ without the conflict-free constraint. As a result of (12), problem (10) is equivalent to

$$\text{Minimize} \sum_{i=1}^{n} \mu_i \|c_i - c_i^u\|^2 \tag{14}$$

for all $\alpha \in \mathbf{P}_2(R, H; \mathbf{a}, \mathbf{b})$ with waypoints $(c_1, \ldots, c_n)$. Note that the cost function to be optimized is quadratic in the optimization variables $(c_1, \ldots, c_n)$.

Constraints on the waypoints

The condition that the two-legged joint maneuver $\alpha \in \mathbf{P}_2(\mathbf{a}, \mathbf{b})$ with waypoints $(c_1, \ldots, c_n)$ is conflict-free can be expressed in terms of constraints on $(c_1, \ldots, c_n)$. These constraints are in general nonconvex. We now study how they can be simplified and approximated by appropriate linear constraints.

Since $\alpha \in \mathbf{P}_2(R, H; \mathbf{a}, \mathbf{b})$ is equivalent to the condition that there is no conflict between any aircraft pair, we focus on aircraft 1 and 2, and temporarily ignore the presence of other aircraft. The following result can be proven (the interested reader is referred to[25] for details).

Proposition 4 *The condition that there is no conflict between aircraft 1 and aircraft 2 in $\alpha \in \mathbf{P}_2(\mathbf{a}, \mathbf{b})$ is equivalent to the condition that the waypoints c_1 and c_2 satisfy: $c_1 - c_2$ is visible to both $a_1 - a_2$ and $b_1 - b_2$ in $\mathbb{R}^3$ in the presence of the open cylindrical obstacle W of radius R and height $2H$ centered at the origin.*

Set $\Delta a = a_1 - a_2$, $\Delta b = b_1 - b_2$, and $\Delta c = c_1 - c_2$. By Proposition 4, the feasible region of Δc consists of those points in $\mathbb{R}^3$ visible to both Δa and Δb in the presence of the obstacle W. Such a region has a complex shape and, in particular, is not convex. Hence problem (14) is in essence a nonconvex optimization problem, which is not only difficult to solve, but may also admit multiple solutions. It is then natural to look for some convex approximation of the feasible region.

In a safety-critical context such as in ATM systems, it is necessary that the approximated region is strictly contained in the original feasible region (*inner approximation*) so as to ensure absolute safety. On the other hand, the approximation should be as tight as possible so that the computed solutions are close to be optimal. The approximation scheme introduced below satisfies these requirements. Moreover, since it only uses the fact that W is convex, it can be easily generalized to the case when the protection zone has an arbitrary convex shape, not necessarily cylindrical.

In the following we assume that both Δa and Δb belong to the interior of $\mathbb{R}^3 \setminus W$, which is satisfied in all situations in practice. We then distinguish two different cases depending on whether Δa and Δb are visible to each other in the presence of the obstacle W.

Δa and Δb are visible to each other. Suppose that the line segment joining Δa and Δb does not intersect W. In this case there is no conflict between aircraft 1 and aircraft 2 if they both fly at constant speed along their ideal trajectories, which are the two-legged joint maneuver with waypoints c_1^u and c_2^u defined in (13). Notice that $\Delta c^u = c_1^u - c_2^u$ is on the line segment between Δa and Δb, hence outside of W. From this it follows that the approximated feasible region of Δc should include Δc^u and as much region in $\mathbb{R}^3$ as possible, provided it is visible to both Δa and Δb. One such choice is described next.

Let L_{ab} be the line segment between Δa and Δb (end points included), and let $\overline{W}$ be the closure of W, which is a closed cylinder. Since both L_{ab} and $\overline{W}$ are compact and convex subsets of $\mathbb{R}^3$, there exists a point u in L_{ab} and a point v in W such that $\|u - v\| = \inf\{\|x - y\| : x \in L_{ab}, y \in \overline{W}\}$. If $u \neq v$, then through point v there is a unique plane P orthogonal to the straight line between u and v. P divides $\mathbb{R}^3$ into two closed half spaces which intersect each other at P. The definition of u and v together with the convexity of L_{ab} and $\overline{W}$ implies that L_{ab} is contained in one half space, while $\overline{W}$ is contained in the other half space. We denote by P^+ the closed half space containing L_{ab}. If $u = v$, then u (hence v) is located on ∂W. In this case we can choose any tangent plane to ∂W at u that separates L_{ab} and $\overline{W}$, and define P^+ to be the side of it containing L_{ab}. Note that here we use the term "tangent planes" of ∂W in its generalized sense, i.e., those planes which intersect ∂W and have W on one side of them exclusively. In the special case when $u = v$ and u is on the sharp edges of ∂W, there might be a family of such tangent planes, and we can choose any one of them in defining P^+, provided it separates L_{ab} and $\overline{W}$.

The closed half space P^+ thus obtained satisfies the condition that it contains Δc^u and that all its points are visible to both Δa and Δb. Therefore we can use P^+ as the approximated feasible region of Δc. This in essence imposes a single linear constraint on c_1 and c_2 in the form $n^T(c_1 - c_2 - v) \geq 0$ for some vector $n \in \mathbb{R}^3$.

The points u and v can be computed by using standard optimization algorithms.

<u>Δa and Δb are not visible to each other.</u> Let p and q be the entry point and the exit point of a shortest curve in $\mathbb{R}^3 \setminus W$ from Δa to Δb as defined after Proposition 3.

Since Δa is in the interior of $\mathbb{R}^3 \setminus W$ by assumption, p is located on the contour of W with respect to a viewer situated at Δa. Among all the planes that are tangent to ∂W at p, let P_a be the one which passes through Δa. The choice of P_a is unique unless p is on the sharp edges of ∂W *and* Δa has the same z-coordinate as p. In the latter case, we can choose an arbitrary tangent plane. Let P_a^+ be the closed half space determined by the side of P_a that does not contain W. Points in P_a^+ are visible to Δa. In a similar way we can define P_b^+ based on the tangent plane to ∂W at q that passes through Δb. Points in P_b^+ are visible to Δb. Therefore points in $P^+ \triangleq P_a^+ \cap P_b^+$ are visible to both Δa and Δb, and can be used as the approximated feasible set for Δc. This translates into two linear constraints on c_1 and c_2.

In summary, given Δa and Δb, one or two linear inequalities can be used to approximate the constraint that Δc is visible to both Δa and Δb in the presence of obstacle W. Such a linear approximation should be carried out for all aircraft pairs, thus leading to the following approximated version of problem (14):

$$\min_{c_i - c_j \in P_{ij}^+, 1 \leq i < j \leq n} \sum_{i=1}^{n} \mu_i \|c_i - c_i^u\|^2, \qquad (15)$$

where P_{ij}^+ is the linear approximation of the feasible set for $c_i - c_j$ computed based on $a_i - a_j$ and $b_i - b_j$ as described above. Problem (15) is a linearly constrained quadratic programming problem, which can be efficiently solved by many software packages.

Some examples of multi-aircraft encounters

Consider a three-aircraft encounter where $a_1 = (0, 50, 4)$, $b_1 = (100, 50, 4)$, $a_2 = (50, 0, 4)$, $b_2 = (50, 100, 4)$, $a_3 = (100, 100, 5)$, and $b_3 = (0, 0, 3)$, i.e., aircraft 1 and aircraft 2 are flying at the same altitude with cross-path angle of 90°, whereas aircraft 3 dives across that altitude and has a path angle of 135° with both aircraft 1 and aircraft 2. All the three aircraft have identical priority and $t_c = (t_0 + t_f)/2$. We choose $R = 10$ nmi to make the resolution maneuvers more evident in the plots.

Figure 5 shows the solutions to problem (15) corresponding to two different values of η. Specifically, plotted in (a) is the snapshot at a time instant near t_c of the two-legged joint maneuver solving problem (15) for $\eta = 5$. Its top view is shown in (b). The cylinders with radius $R/2$ and height H in (a) represent *half* the size of the protection zones surrounding each aircraft, so that two aircraft are in a conflict situation if and only if the corresponding cylinders intersect each other. Similarly, (c) and (d) represent a snapshot of the solution to problem (15) with $\eta = 50$. As in the two-aircraft case, larger value of η will force the aircraft to adopt horizontal maneuvers to resolve the conflict.

Figure 6 shows the simulation results for a four-aircraft encounter with $a_1 = (0, 100, 4)$, $b_1 = (100, 0, 4)$, $a_2 = (20, 80, 4)$, $b_2 = (80, 20, 4)$, $a_3 = (95, 95, 4)$, $b_3 = (0, 0, 4)$, $a_4 = (70, 65, 4)$, and $b_4 = (20, 25, 4)$. The four aircraft are divided into two groups, each consisting of two aircraft one overtaking the other, with the path angle between the two groups being 90°. We choose $R = 10$ nmi, $H = 0.3292$ nmi, and $t_c = (t_0 + t_f)/2$. All aircraft have equal priority. A snap shot of the solution to problem (15) at a time instant near t_c is represented in (a) and (b) for $\eta = 5$, and in (c) and (d) for $\eta = 50$. (c) and (d) can be thought of as the restricted solution to problem (10) when the motion of each aircraft is required to be contained in the plane at altitude 4.

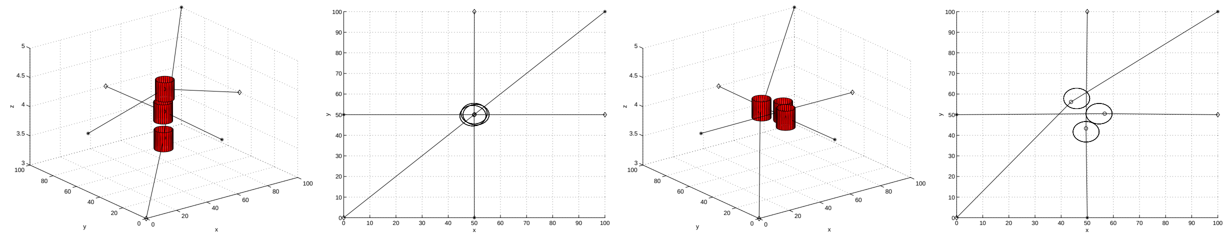
Further constraints on the waypoints

So far we have assumed that the two-legged maneuver obtained by solving the optimization problem (15) is flyable. In practice, this is generally not the case because of the abrupt turn and the change of speed when an aircraft passes through its waypoint. In the following we shall propose practical constraints on the waypoints to alleviate such drawbacks, at least to a certain extent. In order for the optimization problem to be computationally tractable, it is important that the introduced constraints are convex.

We start by considering the speed constraint. Suppose that the speed of each aircraft during both stages of its maneuver cannot exceed a certain threshold v_{max}. Recall that t_c is the time epoch corresponding to the middle waypoints. Then the speed constraint for aircraft i can be expressed as:

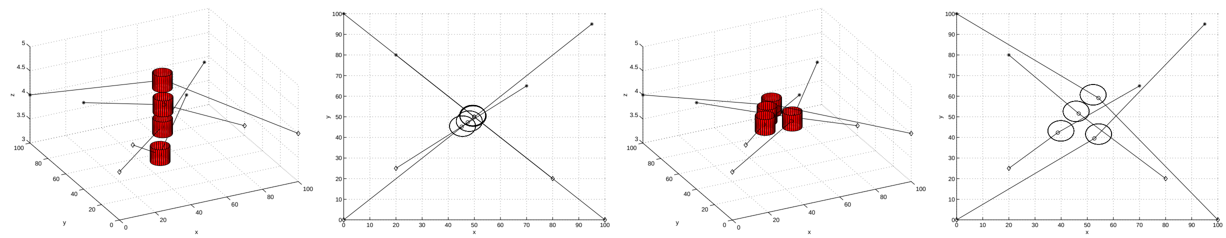
$$\begin{aligned} \|a_i - c_i\| &\leq v_{max}(t_c - t_0), \\ \|b_i - c_i\| &\leq v_{max}(t_f - t_c). \end{aligned} \qquad (16)$$

Note that constraint (16) implies that c_i must be-



(a) (b) (c) (d)

Fig. 5 Two-legged resolution maneuvers for a three-aircraft encounter ($\mu_1 = \mu_2 = \mu_3 = 1/3$): (a) 3D representation and (b) top view when $\eta = 5$; (c) 3D representation and (d) top view when $\eta = 50$.



(a) (b) (c) (d)

Fig. 6 Two-legged resolution maneuvers for a four-aircraft encounter ($\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/4$): (a) 3D representation and (b) top view when $\eta = 5$; (c) 3D representation and (d) top view when $\eta = 50$.

long to the intersection of two spheres, one centered at a_i and the other centered at b_i. Hence the speed constraint is convex. Different v_{max} can be used for aircraft with different capabilities.

A further practical constraint is that on the turning angle. Here, we consider a simplified version of this constraint. More precisely, we project each aircraft maneuver onto the xy, yz, and xz planes, and require that each projection satisfies the condition that the angle between the two segments composing it does not exceed a certain threshold θ_{max}.

Note that both the speed and turning angle constraints can be expressed using a *second order cone constraint* of the form

$$\|\hat{A}s + \hat{b}\| \leq \hat{c}s + \hat{d}, \tag{17}$$

for some matrix $\hat{A}$, vectors $\hat{b}$, $\hat{c}$, and constant $\hat{d}$ of suitable dimensions, where s denotes the optimization variable. Therefore, the optimization problem (15) together with the speed and the simplified turning angle constraints becomes a Second Order Cone Programming (SOCP) problem, which can be solved by using softwares such as SOCP[26]. Note that as before, the vertical discount factor η can be incorporated into these two constraints.

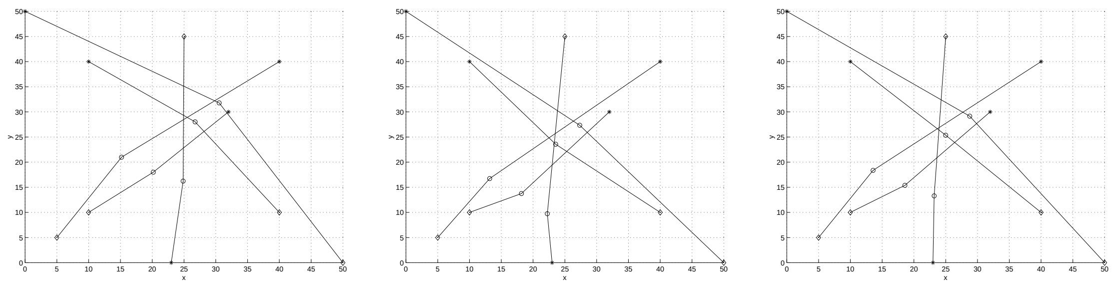
Figure 7 shows the effect of the speed and turning angle constraints on a five-aircraft encounter. Here we choose $t_0 = 0$ min, $t_f = 10$ min, $t_c = 5$ min, $\eta = 50$, $R = 5$ nmi, and we assign the same priority to all the aircraft. The solution to problem (15) without any additional constraint is shown in (a), the solution with the speed constraint of $v_{max} = 7.102$ nmi/min is shown in (b), whereas the solution with the turning angle constraint $\theta_{max} = \pi/10$ on the xy plane projection is reported in (c). As expected, the aircraft which experiments the largest speed and turning angle in case (a) (the one starting from the top left corner and ending in the bottom right corner) tends to have a straighter motion under the additional constraints on either the speed or the turning angle.

Further adjustments can be introduced to improve the flyability of the generated maneuvers. For example, one can consider multi-legged maneuvers and adopt an iterative procedure to get an approximated optimal solution for the multi-legged version of the conflict resolution problem. Furthermore, to avoid sharp turns at time t_0, one can choose the starting epoch to be $t_0 + \Delta$ for some positive Δ, and use the time interval $[t_0, t_0 + \Delta]$ as buffer for possible heading adjustments.

Conclusions

In this paper we study the problem of designing optimal conflict-free maneuvers for multi-aircraft encounters. An algorithm is proposed for solving the resultant constrained optimization problem in the two aircraft case. When more than two aircraft are involved, we consider two-legged maneuvers defined by a set of waypoints. The original optimization problem is then reduced to a finite dimensional convex optimization problem with linearly approximated conflict-free constraints on the waypoints. Path flyability is taken into account by introducing maximum speed and turning angle constraints.

Still, much work needs to be done for the implementation of an optimal resolution algorithm that proves to be effective in most practical situations.



(a) (b) (c)

Fig. 7 Two-legged resolution maneuvers for a five-aircraft encounter ($\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1/4$, $\eta = 50$): (a) no additional constraint; (b) speed constraint with $v_{max} = 7.102$ nmi/min; (c) turning angle constraint with $\theta_{max} = \pi/10$.

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