2020 59th IEEE Conference on Decision and Control (CDC)
Jeju Island, Republic of Korea, December 14-18, 2020

High Confidence Sets for Trajectories of Stochastic Time-Varying Nonlinear Systems

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Abstract— **We analyze stochastic differential equations and their discretizations to derive novel high probability tracking bounds for exponentially stable time varying systems which are corrupted by process noise. The bounds have an explicit dependence on the rate of convergence for the unperturbed system and the dimension of the state space. The magnitude of the stochastic deviations have a simple intuitive form, and our perturbation bounds also allow us to derive tighter high probability bounds on the tracking of reference trajectories than the state of the art. The resulting bounds can be used in analyzing many tracking control schemes.**

I. INTRODUCTION

Modern autonomous systems require safety assurances which scale gracefully with the dimension of the system. However, much of the prior work on computing safe tracking bounds for uncertain dynamical systems relies on adversarial or worst-case analysis (see e.g. [1], [2] and the references therein). Yet worst-case analysis often becomes too conservative and computationally expensive for high dimensional systems. This suggests the use of average-case stochastic safety guarantees to mitigate the curse of dimensionality. The behavior of dynamical systems under stochastic perturbations has been studied but mostly in settings where the structure gives rise to well known distributions (like e.g. linear systems perturbed with Gaussian noise [3]) or under strong stability assumptions which may be hard to verify [4]. Tractable finite-time error bounds for more complex classes of systems are, to the best of our knowledge, still missing from the literature.

In this work we derive new bounds on the iterates of stochastic, time-varying, nonlinear dynamical systems that satisfy a Lyapunov stability condition on their corresponding deterministic dynamics. The resulting bounds have explicit dependencies on problem-dependent parameters which make them amenable for use in practical scenarios.

Error bounds for reference tracking have been extensively studied in the literature, though often in the asymptotic regime [5]–[7] and with strong structural assumptions on the underlying dynamics (e.g. linear [7], [8]). Most similar to our work is a recent line of work on tracking error bounds of constant step-size stochastic approximation [9], in terms of bounds on the mean-squared error. Under slightly stronger assumptions on the noise and using a different set of analysis tools, we are able to bound the higher order moments of the error which yields correspondingly tighter tracking error bounds. Other related work includes work on constant step-sizes for linear dynamics arising from reinforcement learning schemes [8], [10], or the convergence of stochastic gradient schemes [11] where the assumed structure guarantees a contraction on the state space.

Finally, our work draws on a set of techniques most recently used to analyze the finite-time properties of Langevin stochastic differential equations used in Markov-Chain Monte-Carlo (MCMC) sampling algorithms (see e.g. [12]–[15]). However, the structure of the Langevin dynamics– namely the drift term of the diffusion being the gradient of a strongly concave potential function– is often stronger than is present in nonlinear control problems. As such, in this work we expand upon techniques used in prior work to derive finite-time guarantees for nonlinear dynamical systems with Lyapunov functions under Gaussian perturbations. Our results hold under weaker assumptions than Lyapunov-stability of stochastic differential equations [4], [16], which has been well researched in its own right. The conditions for stochastic stability remain hard to verify in general, and our assumptions – namely the stability of the deterministic system– by no means guarantee stability of the stochastic dynamics [4], [16].

The paper is organized as follows. In Section II we present the discrete dynamics and the continuous-time limit we seek to analyze along with our assumptions on the processes. In Section III we then develop two high-probability error bounds on the continuous time process by analyzing how a Lyapunov function evolves along trajectories of a stochastic differential equation. The first bound gives high probability bounds on level-sets of the Lyapunov function, while the second is on the individual iterates of the process. Our proof technique results in tighter bounds than prior work with explicit dependencies on problem-relevant parameters. In Section IV we show that the bounds on the continuous process can be used to derive error bounds for the discrete process of interest. In particular we show that the iterates of the discrete process satisfy a sub-Gaussian moment condition. We conclude in Section V with a brief discussion of potential directions for future work.

II. PRELIMINARIES

In many control tasks it is often desired to derive high confidence sets or tubes which are invariant under the dynamics. In this paper, we focus on deriving such sets for a dynamical system of the form:

$$x_{k+1} = x_k + h\left(f(x_k, kh) + w_{k+1}\right), \tag{1}$$

This work was supported by Defense Advanced Research Projects Agency Award FA8750-18-C-0101

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where $f : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ describes the dynamics of the system, $h > 0$ is the stepsize, and $w_{k+1} \sim \mathcal{N}(0, \hat{\Sigma}_{k+1})$ can be seen as un-modelled dynamics or unavoidable process noise. We assume throughout this paper that f is locally Lipschitz continuous in x and piecewise continuous in t. Key to our analysis, is in viewing (1) as the forwards Euler discretization with stepsize h of a stochastic differential equation (SDE) of the form:

$$dX_t = f(X_t, t)dt + Q_t dB_t, \tag{2}$$

where $Q_t = \Sigma_t^{\frac{1}{2}}$ and B_t is standard Brownian motion.

To derive high confidence bounds around trajectories of the discrete dynamics we analyze properties of the limiting continuous time dynamics and then bound the approximation error. Key to our analysis are properties of the deterministic continuous-time system: $\dot{x} = f(x, t)$. In particular, we assume that there exists a (global) Lyapunov function, $V : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ which satisfies the following standard assumptions (see e.g [17]):

Assumption 1 (Assumption on the Lyapunov Function). The Lyapunov function $V \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}_+)$ satisfies, for all $x \in \mathbb{R}^d$:

$$\alpha_1 ||x||^2 \leq V(x,t) \leq \alpha_2 ||x||^2 \quad \forall \quad t \geq 0 \tag{3}$$

$$\nabla_x V(x,t) \quad \text{is } L\text{-Lipshitz continuous in} \quad x \quad \forall \quad t \geq 0 \tag{4}$$

$$\nabla_x V(x,t)^T f(x,t) + \frac{dV(x,t)}{dt} \leq -\alpha_3 ||x||^2 \quad \forall\ t \geq 0. \tag{5}$$

We remark that without loss of generality we assume the minimum of the Lyapunov function to be at the origin. We further note that our assumption on time-varying dynamics and a time-varying Lyapunov function allows for the minimum of the Lyapunov function to be a time-varying process $x^*(t)$, but our analysis does not change. We further note that our assumptions imply that the minimum of the Lyapunov function is exponentially stable such that:

$$V(X_T, T) \leq V(X_0, 0) e^{-\frac{\alpha_3}{\alpha_2} T}$$

given our assumptions. Though this is a strong property, it is still weaker than following the gradient of a potential function as the Lyapunov function does not necessarily imply a contraction on the space of trajectories over arbitrary time horizons.

Remark 1. *We note that our assumption is that V is a Lyapunov function for the* deterministic *system. As such though it guarantees exponential stability of the deterministic system it does not necessarily certify exponential stochastic stability or pth moment exponential stochastic stability as is common in the analysis of stability of SDEs [4]. Such concepts imply that X tends to the minimum of the Lyapunov function in probability or almost surely which is a much stronger property than is implied by Assumption 1. Indeed our assumption only guarantees the existence of a limiting steady state distribution [16]. In this paper we derive properties of the finite time distribution.*

To conclude, our preliminaries section we make an assumption on the noise process Σ_t:

Assumption 2 (Assumption on the noise process). The process $\Sigma : \mathbb{R}_+ \to \mathbb{R}^{d \times d}$ satisfies:

$$\mathrm{spec}(\Sigma_t) \leq \sigma \quad \forall \quad t \geq 0,$$

where $\sigma > 0$ and spec denotes the spectrum of a matrix.

Given these assumptions, in the next section we derive high probability bounds on trajectories of the SDE which we subsequently use to derive the confidence sets for the discrete dynamics. We remark that throughout this paper all norms are the Euclidean norm unless stated otherwise.

III. CONSTRUCTING CONFIDENCE SETS

To construct confidence sets around trajectories of the SDE we proceed by analyzing how the Lyapunov function evolves along its trajectories. We construct a bound on the higher order moments of the Lyapunov function and use this to construct our high probability bounds. Our proof makes use of techniques and ideas recently used to analyze the convergence of Langevin MCMC sampling algorithms [18]–[20] but the analysis is complicated due to the fact that the drift is not the gradient of a strongly concave function. Despite this, we show that the standard assumptions on the Lyapunov function can substitute for this property and be used to derive high-probability bounds around the (potentially time-varying) minimum of the Lyapunov function.

Theorem 1. *Given Assumptions 1-2, initial condition x_0, and for a time $T \geq 0$, along trajectories of* (2) *for any $\delta \in (0, 1)$:*

$$\mathbb{P}\left(V(X_T, T) > \epsilon_1(\delta)\right) \leq \delta$$

where:

$$\epsilon_1(\delta) = 2e^{-\frac{\alpha_3}{\alpha_2}T+1} V(x_0, 0) + \frac{\alpha_2 \sigma L e}{\alpha_3}\left(d + \frac{8L \log \frac{1}{\delta}}{\alpha_1}\right).$$

Before presenting the proof of this Theorem, we first note that it guarantees that the Lyapunov function behaves like a sub-exponential random variable around its minimum along trajectories of the SDE,. The first term in $\epsilon_1(\delta)$ reflects an exponentially decaying dependence on the initial position, with the same decay rate, α_3/α_2 as for the deterministic dynamical system.

The second term has an inverse relationship with this decay rate, highlighting how tighter Lyapunov functions lead to tighter high-probability guarantees for the system. If the Lyapunov function were a control Lyapunov function, this decay rate could be seen to be proportional to the gain of the control. As such we observe how a higher gain in the control has a direct effect on the resulting confidence sets.

Proof. To derive our confidence sets we study how a scaled version of the Lyapunov function, $e^{ct}V(x_t, t)$, evolves along trajectories of the SDE (2), where $c > 0$ is a design choice we make later in the proof.

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Via Itô's formula:

$$e^{ct}V(x_t,t) = V(x_0,0) + T1 + T2 + T3 + T4$$

where:

$$T1 = \int_0^t ce^{cs}V(X_s,s)ds$$
$$T2 = \int_0^t e^{cs}\left(\nabla_x V(X_s,s)^T f(X_s,s) + \frac{dV(X_s,s)}{dt}\right)ds$$
$$T3 = \int_0^t \frac{e^{cs}}{2}\left(\text{tr}(\Sigma_s \nabla_x^2 V(X_s,s)\right)ds$$
$$T4 = \int_0^t e^{cs}\nabla_x V(X_s,s)^T Q_s dB_s.$$

Via our assumptions on the Lyapunov function, we can combine $T1$ and $T2$ to find that:

$$T1 + T2 \le \int_0^t e^{cs}(c\alpha_2 - \alpha_3)\|X_s\|^2 ds,$$

where we used (3) to upper bound $T1$ and (5) to upper bound $T2$. Further choosing $c = \frac{\alpha_3}{\alpha_2}$, lets lets us upper bound $T1 + T2$ by 0.

To upper bound $T3$, we use the fact that V has L-Lipshcitz gradients in x and our upper bound on the singular values of Σ_t to find that:

$$T3 \le \int_0^t \frac{\sigma d L e^{cs}}{2}ds \le \frac{\sigma d L e^{ct}}{2c},$$

where to derive this result we used Cauchy-Schwartz to upper bound the trace of the two matrices, the fact that $\sigma(\Sigma(x)) < \sigma$ and $\sigma(\nabla_x^2 V(X_s,s)) < L$, and trivially upper bounded $e^{ct}-1$ by e^{ct}. Finally, for $T4$, we note that this term is a Martingale which we define as M_t. Letting $c = \frac{\alpha_3}{\alpha_2}$ these upper bounds combine to give:

$$e^{ct}V(x_t,t) \le \underbrace{V(x_0,0) + \frac{\sigma d L e^{ct}}{2c}}_{:=U_t} + M_t. \tag{6}$$

To derive our high-probability guarantees we control the high moments of the supremum of $e^{ct}V(x_t,t)$. To begin, we invoke (6) to find that:

$$\mathbb{E}\left[\left(\sup_{t\le T} e^{ct}V(x_t,t)\right)^p\right]^{\frac{1}{p}} \le \mathbb{E}\left[\left(\sup_{t\le T} U_t + M_t\right)^p\right]^{\frac{1}{p}}$$
$$\le U_T + \mathbb{E}\left[\left(\sup_{t\le T}|M_t|\right)^p\right]^{\frac{1}{p}},$$

where in the last line we used the fact that U_t is a monotonically increasing function of t and the Minkowski inequality.

Thus, it remains to bound the second term on the right hand side above. To do so, we make use of the Burkholder-Gundy-Davis inequality [21] which bounds the supremum of a martingale over time by analyzing its quadratic variations:

$$\mathbb{E}\left[\left(\sup_{t\le T}|M_t|\right)^p\right]^{\frac{1}{p}} \le (8p)^{\frac{p}{2}}\underbrace{\mathbb{E}\left[\langle M_t, M_t\rangle^{\frac{p}{2}}\right]}_{I}. \tag{7}$$

Expanding on I gives:

$$I = \mathbb{E}\left[\left(\int_0^T e^{2ct}\nabla_x V(X_s,s)^T\Sigma(X_s)\nabla_x V(X_s,s)ds\right)^{\frac{p}{2}}\right]$$
$$\overset{i}{\le} \mathbb{E}\left[\left(\int_0^T \sigma L^2 e^{2ct}\|X_s\|^2 ds\right)^{\frac{p}{2}}\right]$$
$$\overset{ii}{\le} \mathbb{E}\left[\left(\frac{\sigma L^2 e^{cT}}{c}\sup_{t\le T} e^{ct}\|X_t\|^2\right)^{\frac{p}{2}}\right].$$

where i is a result of the upper bound on $\Sigma(X_s)$ when $c_2 = 0$ and the fact that $\nabla_x V(x,s)$ is L-Lipschitz and $\nabla_x V(x,s) = 0$ when $x = 0$ by assumption which implies that $\|\nabla_x V(x,s)\| \le L\|X\|$. The second inequality follows by upper bounding the integral by the $\sup_{t\le T} e^{ct}\|X_t\|^2$ and solving the remaining integral $\int_0^T e^{cs}ds$.

Combining all the upper bounds gives:

$$\mathbb{E}\left[\left(\sup_{t\le T} e^{ct}V(x_t,t)\right)^p\right]^{\frac{1}{p}}$$
$$\le U_T + \mathbb{E}\left[\left(\left(\frac{8p\sigma L^2 e^{cT}}{c}\right)^{\frac{1}{2}}\left(\sup_{t\le T} e^{ct}\|X_t\|^2\right)^{\frac{1}{2}}\right)^p\right]^{\frac{1}{p}}.$$

We further develop this upper bound by using Young's inequality on the quantity inside of the expectation:

$$\left(\frac{8p\sigma L^2 e^{cT}}{c}\right)^{\frac{1}{2}}\left(\sup_{t\le T} e^{ct}\|X_t\|^2\right)^{\frac{1}{2}}$$
$$\le \frac{1}{2\alpha_1}\left(\frac{8p\sigma L^2 e^{cT}}{c}\right) + \frac{\alpha_1}{2}\left(\sup_{t\le T} e^{ct}\|X_t\|^2\right).$$

Plugging this in and using Minkowski's inequality again gives:

$$\mathbb{E}\left[\left(\sup_{t\le T} e^{ct}V(x_t,t)\right)^p\right]^{\frac{1}{p}}$$
$$\le U_T + \frac{8p\sigma L^2 e^{cT}}{2\alpha_1 c} + \frac{1}{2}\mathbb{E}\left[\left(\sup_{t\le T} e^{ct}V(X_t,t)\right)^p\right]^{\frac{1}{p}},$$

where we used the assumption on V and (3). Rearranging the last line gives:

$$\mathbb{E}\left[\left(\sup_{t\le T} e^{ct}V(x_t,t)\right)^p\right]^{\frac{1}{p}} \le 2U_T + \frac{8p\sigma L^2 e^{cT}}{\alpha_1 c}. \tag{8}$$

Finally, we use (8) to control the p-th moments of $V(x_t,t)$:

$$\mathbb{E}\left[(V(X_T,T))^p\right]^{\frac{1}{p}} \le e^{-cT}\mathbb{E}\left[\left(\sup_{t\le T} e^{ct}V(x_t,t)\right)^p\right]^{\frac{1}{p}}$$
$$\le \underbrace{2e^{-cT}V(x_0,0) + \frac{\alpha_2\sigma L}{\alpha_3}\left(d + \frac{8Lp}{\alpha_1}\right)}_{T5}.$$

Using this p-th moment bound, via Markov's inequality we have that for $\epsilon > 0$:

$$\mathbb{P}\left(V(X_T,T) > \epsilon\right) \le \frac{\mathbb{E}\left[(V(X_T,T))^p\right]}{\epsilon^p} \le \left(\frac{T5}{\epsilon}\right)^p.$$

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Choosing $\epsilon = e(T5)$, and $p = \log\frac{1}{\delta}$ for $\delta \in (0,1)$ gives the desired result. $\square$

In Theorem 1 we constructed confidence sets in terms of level-sets of the Lyapunov function. However, in many settings it may be useful to construct balls around the minimum of the Lyapunov function in the Euclidean metric. In the following corollary to Theorem 1 we show that this distance is sub-Gaussian at every time instant $t \geq 0$.

Corollary 1. *Given Assumptions 1-2, initial condition x_0, and for a time $T \geq 0$, along trajectories of (2), for $\delta \in (0, e^{-\frac{1}{2}})$:*

$$\mathbb{P}\left(\|X_T\| > \epsilon_2(\delta)\right) \leq \delta$$

where:

$$\epsilon_2(\delta) = \sqrt{2\frac{\alpha_2 e^{-\frac{\alpha_3}{\alpha_2}T+1}}{\alpha_1}\|X_0\|^2 + \frac{\alpha_2\sigma L e}{\alpha_3\alpha_1}\left(d + \frac{8L\log\frac{1}{\delta}}{\alpha_1}\right)}.$$

The proof of Corollary 1 follows by invoking the upper and lower bounds on the Lyapunov function V in Assumption 1, and we omit it for brevity.

We remark that the bounds presented in Theorem 1 and Corollary 1 result from controlling higher-order moments of the continuous-time process. As such, they are tighter than error bounds developed under similar assumptions on the dynamics (yet weaker assumptions on the noise) in [9], and stronger than classic results in the stochastic approximation literature where commonly only the second moment is bounded [3].

IV. A SUBGAUSSIAN MOMENT BOUND FOR THE DISCRETIZED PROCESS

To extend these results to the discretized process, which is often more useful in practice, we introduce the interpolated process $\bar{X}_t$ for which $\bar{X}_0 = X_0$ and which evolves for all $t \in (ih, (i+1)h]$ for $i \in \mathbb{N}$:

$$d\bar{X}_t = f(\bar{X}_{ih}, ih)dt + Q_t dB_t \tag{9}$$

where the process noise Q_t and Brownian motion are chosen to be the same as in (2). We note that this means that $\hat{\Sigma}_i = \int_{ih}^{(i+1)h}\Sigma_s ds$. To bound the approximation error between the interpolated process and the true process we require a further assumption on the dynamics:

Assumption 3 (Stronger assumption on the smoothness of the dynamics). The dynamics $f(x,t)$ is jointly Lipschitz continuous in x and t:

$$\|f(x,t) - f(x',t')\| \leq L_f\|x - x'\| + L_T|t - t'|$$

Given this assumption we present the following theorem on the iterates of the discretized process:

Theorem 2. *Given Assumptions 1-3, initial condition X_0, and for a time $K \geq 0$, if $h < \min\left(\frac{m\alpha_1}{4\sqrt{2}LL_f^2}, \frac{1}{m}, 1\right)$ then for all $k \geq 1$ and $\delta \in (0,1)$ the iterates of (1) satisfy:*

$$\mathbb{P}\left(\|X_k\| > \epsilon_3(\delta)\right) \leq \delta$$

where:

$$\epsilon_3(\delta) = K\sqrt{\left(1 - \frac{mh}{4}\right)^{k-1}\|X_0\|^2 + \frac{L_T^2 + \sigma d + \sigma(1+dh)\log\frac{1}{\delta}}{\alpha_1 m}},$$

for K is a constant depending only on the smoothness parameters $\alpha_1, \alpha_2, \alpha_3, L$ and L_f and $m = \alpha_3/2\alpha_2$.

Before presenting the proof of Theorem 2, we comment about the bound. We first note that it guarantees that the individual iterates of the process are sub-Gaussian around the minimum of the Lyapunov function and that it inherits the desirable properties of the bounds in the previous section. The proof proceeds by bounding the distance between an interpolated version of the SDE (9) (which is equivalent to the discrete system (1)) and the limiting SDE that we studied in the previous section. Key to the proof is using the Lyapunov function as a 'metric' and invoking the exponential decay of the Lyapunov function and our results from Section III.

Proof. To begin the proof we first note that:

$$\|\bar{X}_t\|^2 \leq \frac{2}{\alpha_1}V(\bar{X}_t - X_t, t) + 2\|X_t\|^2. \tag{10}$$

Through this decomposition, we note that we simply have to analyze how $V(\bar{X}_t - X_t, t)$ evolves along trajectories of $\bar{X}_t - X_t$. By construction, we remark that $\bar{X}_0 = X_0$ and for all $t \in (ih, (i+1)h]$ for $i \in \mathbb{N}$:

$$d\left(\bar{X}_t - X_t\right) = \left(f(\bar{X}_{ih}, ih) - f(X_t, t)\right)dt,$$

since the randomness contributed to both processes is identical and cancels. It follows that:

$$\begin{aligned}
&\dot{V}(\bar{X}_t - X_t, t) \\
&= \underbrace{\nabla_x V(\bar{X}_t - X_t, t)^T f(\bar{X}_t - X_t, t) + \frac{dV(\bar{X}_t - X_t, t)}{dt}}_{:=T_1} \\
&\quad + \underbrace{\nabla_x V(\bar{X}_t - X_t, t)^T\left(f(\bar{X}_{ih}, ih) - f(\bar{X}_t - X_t, t)\right)}_{:=T_2} \\
&\quad \underbrace{-\nabla_x V(\bar{X}_t - X_t, t)^T f(X_t, t)}_{:=T_3}.
\end{aligned}$$

We proceed by constructing upper bounds on terms T_1, T_2, and T_3. By assumption, $T_1 \leq -\frac{\alpha_3}{\alpha_2}V(\bar{X}_t - X_t, t)$. To bound T_2 we make use of the assumed Lipschitz continuity of both $\nabla_x V$ and f in X and Young's inequality with constants ϵ_1 and ϵ_2:

$$\begin{aligned}
T_2 &\leq \|\nabla_x V(\bar{X}_t - X_t, t)\|\|f(\bar{X}_{ih}, ih) - f(\bar{X}_t - X_t, t)\| \\
&\leq LL_f\|\bar{X}_t - X_t\|\|\bar{X}_{ih} - \bar{X}_t + X_t\| \\
&\qquad + LL_T(t - ih)\|\bar{X}_t - X_t\| \\
&\leq \frac{\epsilon_1 LL_f + \epsilon_2 LL_T}{2\alpha_1}V(\bar{X}_t - X_t, t) + \frac{LL_f}{\epsilon_1}\|\bar{X}_{ih} - \bar{X}_t\|^2 \\
&\qquad + \frac{LL_f}{\epsilon_1}\|X_t\|^2 + \frac{LL_T}{2\epsilon_2}(t - ih)^2.
\end{aligned}$$

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Finally, for T_3 we once again use Young's inequality:

$$T_3 \leq \frac{\epsilon_1 LL_f}{2\alpha_1} V(\bar{X}_t - X_t, t) + \frac{LL_f}{2\epsilon_1}\|X_t\|^2.$$

Combining these upper bounds gives:

$$\begin{aligned}
&\dot{V}(\bar{X}_t - X_t, t) \\
&\leq \left(-\frac{\alpha_3}{\alpha_2} + \frac{3\epsilon_1 LL_f + \epsilon_2 LL_T}{2\alpha_1}\right) V(\bar{X}_t - X_t, t) \\
&\quad + \frac{LL_f}{\epsilon_1}\left(\frac{3}{2}\|X_t\|^2 + \|\bar{X}_{ih} - \bar{X}_t\|^2\right) + \frac{LL_T}{2\epsilon_2}(t - ih)^2.
\end{aligned}$$

To complete this upper bound we expand upon $\|\bar{X}_{ih} - \bar{X}_t\|^2$:

$$\begin{aligned}
\|\bar{X}_{ih} - \bar{X}_t\|^2 &= \left\|\int_{ih}^t f(\bar{X}_{ih}, ih)ds + \underbrace{\int_{ih}^t Q_s dBs}_{:=W_t}\right\|^2 \\
&\leq \left(2(t - ih)^2 L_f^2\right)\|\bar{X}_{ih}\|^2 + 2\|W_t\|^2,
\end{aligned}$$

where W_t is a d-dimensional Gaussian process with mean 0 and covariance $\int_{ih}^t \Sigma_s ds \preceq (t - ih)\sigma I_d$ for I_d the identity matrix in $\mathbb{R}^{dxd}$. Using this decomposition, we find that:

$$\begin{aligned}
&\dot{V}(\bar{X}_t - X_t, t) \\
&\leq \left(-\frac{\alpha_3}{\alpha_2} + \frac{3\epsilon_1 LL_f + \epsilon_2 LL_T}{2\alpha_1}\right) V(\bar{X}_t - X_t, t) \\
&\quad + \underbrace{\frac{LL_f}{\epsilon_1}}_{:=C_1}\left(\frac{3}{2}\|X_t\|^2 + 2(t - ih)^2 L_f^2\|\bar{X}_{ih}\|^2\right) \\
&\quad + \frac{LL_f}{\epsilon_1}\|W_t\|^2 + \underbrace{\frac{LL_T}{2\epsilon_2}(t - ih)^2}_{:=C_2}.
\end{aligned}$$

Choosing, for simplicity ϵ_1 and ϵ_2 as:

$$\epsilon_1 = \frac{\alpha_1\alpha_3}{4\alpha_2 LL_f} \qquad \epsilon_2 = \frac{\alpha_1\alpha_3}{4\alpha_2 LL_T},$$

and letting $m = \frac{\alpha_3}{2\alpha_2}$, we have by the fundamental theorem of calculus that for $t \in (ih, (i+1)h]$ for $i \in \mathbb{N}$:

$$\begin{aligned}
&V(\bar{X}_t - X_t, t) \\
&\leq \left(e^{-m(t-ih)} + \frac{4(t - ih)^3 C_1 L_f^2}{\alpha_1}\right) V(X_{ih} - \bar{X}_{ih}, ih) \\
&\quad + 4(t - ih)^3 C_1 L_f^2\|X_{ih}\|^2 + \frac{3C_1}{2}\int_{ih}^t \|X_s\|^2 e^{-m(t-s)}ds \\
&\quad + C_1\int_{ih}^t e^{-m(t-s)}\|W_s\|^2 ds + (t - ih)C_2,
\end{aligned}$$

where we have used the fact that $\int_{ih}^t e^{-m(t-s)}ds \leq (t - ih)$ and $\|\bar{X}_{ih}\|^2 \leq 2\|\bar{X}_{ih} - X_{ih}\|^2 + 2\|\bar{X}_{ih}\|^2$ to further simplify the bound. Choosing $t = (i+1)h$, taking the L_p norm of both sides, and using the Minkowski inequality gives:

$$\begin{aligned}
&\mathbb{E}\left[V(\bar{X}_t - X_t, t)^p\right]^{\frac{1}{p}} \\
&\leq \left(e^{-mh} + \frac{4h^3 C_1 L_f^2}{\alpha_1}\right)\mathbb{E}\left[V(X_{ih} - \bar{X}_{ih}, ih)^p\right]^{\frac{1}{p}} \\
&\quad + 4h^3 C_1 L_f^2 \mathbb{E}\left[\|X_{ih}\|^{2p}\right]^{\frac{1}{p}} + hC_2 \\
&\quad + \frac{3}{2}C_1 \underbrace{\mathbb{E}\left[\left(\int_{ih}^{(i+1)h} e^{-m((i+1)h-s)}\|X_s\|^2 ds\right)^p\right]^{\frac{1}{p}}}_{:=T_4} \\
&\quad + C_1 \underbrace{\mathbb{E}\left[\left(\int_{ih}^{(i+1)h} e^{-m((i+1)h-s)}\|W_s\|^2 ds\right)^p\right]^{\frac{1}{p}}}_{:=T_5}.
\end{aligned}$$

Using Minkowski's integral inequality, we can further upper bound T_4 and T_5 as:

$$\begin{aligned}
T_4 &\leq \int_{ih}^{(i+1)h} e^{-m((i+1)h-s)}\mathbb{E}\left[\|X_s\|^{2p}\right]^{\frac{1}{p}} ds \\
T_5 &\leq \int_{ih}^{(i+1)h} e^{-m((i+1)h-s)}\mathbb{E}\left[\|W_s\|^{2p}\right]^{\frac{1}{p}} ds
\end{aligned}$$

From the proof of Corollary 1, we know that:

$$\begin{aligned}
\mathbb{E}\left[\|X_s\|^{2p}\right]^{\frac{1}{p}} &= \left(\mathbb{E}\left[\|X_s\|^{2p}\right]^{\frac{1}{2p}}\right)^2 \\
&\leq \frac{2\alpha_2 e^{-2ms}}{\alpha_1}\|X_0\|^2 + \underbrace{\frac{\sigma L}{2m\alpha_1}\left(d + \frac{4Lp}{\alpha_1}\right)}_{:=D},
\end{aligned}$$

which implies that:

$$\begin{aligned}
T_4 &\leq \int_{ih}^{(i+1)h} \frac{2\alpha_2}{\alpha_1} e^{-(m(i+1)h + ms))}\|X_0\|^2 ds + hD \\
&\leq \frac{2\alpha_2 e^{-mh}(1 - e^{-mh})e^{-2mhi}}{\alpha_1 m}\|X_0\|^2 + hD.
\end{aligned}$$

To upper bound T_5, we use the fact that since W_s is Gaussian with mean 0 and covariance $\int_{ih}^t \Sigma_s ds \preceq (t - ih)\sigma I_d$, $\|W_s\|$ is a $\sqrt{8d}h\sigma$ norm-sub-Gaussian random variable [22], which satisfies: $\mathbb{E}\left[\|W_s\|^p\right]^{\frac{1}{p}} \leq \sqrt{8dp}h\sigma$. Thus we have that:

$$\begin{aligned}
&\mathbb{E}\left[V(\bar{X}_{(i+1)h} - X_{(i+1)h}, (i+1)h)^p\right]^{\frac{1}{p}} \\
&\leq \left(e^{-mh} + \frac{4h^3 C_1 L_f^2}{\alpha_1}\right)\mathbb{E}\left[V(X_{ih} - \bar{X}_{ih}, ih)^p\right]^{\frac{1}{p}} \\
&\quad + C_1 \frac{6\alpha_2 e^{-mh}(1 - e^{-mh})e^{-2mhi}}{\alpha_1 m}\|X_0\|^2 \\
&\qquad + \frac{8\alpha_2 h^3 L_f^2 C_1}{\alpha_1} e^{-2mhi}\|X_0\|^2 \\
&\qquad + hC_1\left(8dph\sigma^2 + \left(4h^2 L_f^2 + \frac{3}{2}\right)D\right) + hC_2.
\end{aligned}$$

Since for $h < 1/m$, we have $e^{-mh} < 1 - \frac{mh}{2}$, choosing $h < \min\left(\frac{m\alpha_1}{4\sqrt{2}LL_f^2}, \frac{1}{m}, 1\right)$ guarantees that $k :=$

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$\left(e^{-mh} + \frac{4h^3C_1L_f^2}{\alpha_1}\right) < 1 - \frac{mh}{4} < 1$, which in turn yields a recursion:

$$\begin{aligned}&\mathbb{E}\left[V(\bar{X}_{(i+1)h} - X_{(i+1)h}, (i+1)h)^p\right]^{\frac{1}{p}} \\ &\leq k\mathbb{E}\left[V(X_{ih} - \bar{X}_{ih}, ih)^p\right]^{\frac{1}{p}} \\ &\quad + C_1\frac{6\alpha_2 e^{-mh}(1-e^{-mh})e^{-2mhi}}{\alpha_1 m}\|X_0\|^2 \\ &\quad + \frac{8\alpha_2 h^3 L_f^2 C_1}{\alpha_1}e^{-2mhi}\|X_0\|^2 \\ &\quad + hC_1\left(8dph\sigma^2 + \left(4h^2L_f^2 + \frac{3}{2}\right)D\right) + hC_2.\end{aligned}$$

Using the fact that $X_0 = \bar{X}_0$ we must have that:

$$\begin{aligned}&\mathbb{E}\left[V(\bar{X}_{(i+1)h} - X_{(i+1)h}, (i+1)h)^p\right]^{\frac{1}{p}} \\ &\leq \left(\frac{6\alpha_2 e^{-mh}(1-e^{-mh})}{\alpha_1 m} + \frac{8\alpha_2 h^3 L_f^2}{\alpha_1}\right)C_1T_6\|X_0\|^2 \\ &\quad + \left(hC_1\left(8dph\sigma^2 + \left(4h^2L_f^2 + \frac{3}{2}\right)D\right) + hC_2\right)\sum_{j=1}^{i+1}k^j,\end{aligned}$$

where $T_6 = k^{2i}\sum_{j=0}^{i}k^{-j}$ because $e^{-mh} < k$. Since T_6 is a geometric series, with $k < 1 - \frac{mh}{4}$ for the possible choices of h, simple computation finds that $T_6 \leq 4\left(1-\frac{mh}{4}\right)^i/mh$. Since $1 - e^{-mh} < mh$, we further simplify to find:

$$\begin{aligned}&\mathbb{E}\left[V(\bar{X}_{(i+1)h} - X_{(i+1)h}, (i+1)h)^p\right]^{\frac{1}{p}} \\ &\leq 4C_1\frac{6\alpha_2 e^{-mh} + 8\alpha_2 h^2 L_f^2}{\alpha_1 m}\left(1-\frac{mh}{4}\right)^i\|X_0\|^2 \\ &\quad + \frac{4C_1}{m}\left(8dph\sigma^2 + \left(4h^2L_f^2 + \frac{3}{2}\right)D\right) + \frac{4C_2}{m}.\end{aligned}$$

Now, recalling that we are interested in bounding $\mathbb{E}\left[\|\bar{X}_t\|^{2p}\right]^{\frac{1}{p}}$, and that $t = (i+1)h$, we make use of (10), Minkowski's inequality, and our upper bounds, to find that:

$$\begin{aligned}&\mathbb{E}\left[\|\bar{X}_t\|^{2p}\right]^{\frac{1}{p}} \\ &\leq 2D + \frac{4\alpha_2}{\alpha_1}e^{-mh(i+1)}\|X_0\|^2 \\ &\quad + 8C_1\frac{6\alpha_2 e^{-mh} + 8\alpha_2 h^2 L_f^2}{\alpha_1^2 m}\left(1-\frac{mh}{4}\right)^i\|X_0\|^2 \\ &\quad + \frac{8C_1}{\alpha_1 m}\left(8dph\sigma + \left(4h^2L_f^2 + \frac{3}{2}\right)D\right) + \frac{8C_2}{\alpha_1 m} \\ &\leq K\left[\left(1-\frac{mh}{4}\right)^i\|X_0\|^2 + \frac{L_T^2 + \sigma d + \sigma(1+dh)p}{\alpha_1 m}\right].\end{aligned}$$

where K depends only on $\alpha_1, \alpha_2, \alpha_3, L$ and L_f. The remainder of the proof follows exactly as in the proof of Theorem 1. □

V. CONCLUSION

In this paper we derived new high-probability bounds around iterates of controlled stochastic nonlinear systems using properties of the deterministic drift. We remark that the bounds we derived are tighter than other recently derived bounds for tracking albeit under slightly stronger assumptions on the noise process than in [9]. Further, the scaling with the dimension of the state space is sub-linear which allows our bounds to scale effectively to high dimensions where deterministic bounds from e.g Reachability-based methods may be impossible to compute due to the curse of dimensionality [1].

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