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OPTIMAL LINEAR LQG CONTROL OVER LOSSY NETWORKS WITHOUT PACKET ACKNOWLEDGMENT

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ABSTRACT

This paper is concerned with control applications over lossy data networks. Sensor data is transmitted to an estimation-control unit over a network, and control commands are issued to subsystems over the same network. Sensor and control packets may be randomly lost according to a Bernoulli process. In this context, the discrete-time linear quadratic Gaussian (LQG) optimal control problem is considered.

It is known that in the scenario described above, and for protocols for which there is no acknowledgment of successful delivery of control packets (*e.g.* UDP-like protocols), the LQG optimal controller is in general nonlinear. However, the simplicity of a linear sub-optimal solution is attractive for a variety of applications. Accordingly, this paper characterizes the optimal linear static controller and compares its performance to the case when there is acknowledgment of delivery of packets (*e.g.* TCP-like protocols).

Key Words: Linear regulator, LQG control, maximum matrix principle, packet loss, UDP.

I. INTRODUCTION

Today, an increasing number of applications demands remote control of plants over unreliable networks. The recent evolution of sensor web technology [1] enables the development of wireless sensor networks that can be immediately used for estimation and control. In these systems issues of communication delay, data loss, and time-synchronization play critical roles. Communication and control become tightly coupled and these two issues cannot be addressed independently. The goal of this paper is to provide some partial answers to the question of how control loop performance is affected by communication constraints and what are the basic system-theoretic implications of using unreliable networks for control. This requires a generalization of classical control techniques that explicitly takes into account the stochastic nature of the communication channel.

We consider a generalized formulation of the linear quadratic Gaussian (LQG) optimal control problem by modeling the arrival of both observations and control packets as random processes whose parameters are related to the characteristics of the communication channel. Accordingly, two independent Bernoulli processes are considered, with parameters $\bar{\gamma}$ and $\bar{\nu}$, that govern packet losses between the sensors and the estimation-control unit, and between the latter and the actuation points.

In our analysis, we distinguish between two classes of protocols. The distinction resides simply in the availability of packet acknowledgments. Adopting

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the framework proposed by Imer *et al.* [2], we will refer to transmission control protocol (TCP)-like protocols if packet acknowledgments are available and to user datagram protocol (UDP)-like protocols otherwise.

We have shown in some previous work [3, 4], and [5] the existence of a critical domain of values for the parameters of the Bernoulli arrival processes, $\bar{\nu}$ and $\bar{\gamma}$, outside which a transition to instability occurs and the optimal controller fails to stabilize the system. In particular, we have shown that under TCP-like protocols the critical arrival probabilities for the control and observation channel are independent of each other. This is another consequence of the fact that the separation principle holds for these protocols. A more involved situation regards UDP-like protocols. In this case the critical arrival probabilities for the control and observation channels are coupled. The stability domain and the performance of the optimal controller degrade considerably as compared with TCP-like protocols as shown in Fig. 1.

We have also shown that in the TCP-like case the classic separation principle holds, and consequently the controller and estimator can be designed independently. Moreover, the optimal controller is a linear function of the state. In sharp contrast, in the UDP-like case, the optimal controller is in general nonlinear. In this case, a natural suboptimal solution is to use a static linear regulator, composed by a Kalman-like estimator and a state feedback controller as shown in Fig. 2. This is particularly attractive for sensor networks, where simplicity of implementation is highly desirable and complexity issues are a primary concern. Accordingly, in this paper, we focus on the performance of the UDP-like controller and compare it with the optimal one in the TCP-like case.

First, we formulate the problem of finding the optimal linear controller as a non-convex optimization problem. Then, we write, using Lagrange multipliers, a necessary condition for the optimum. Using a result of De Koning [6], we determine when such condition is also sufficient. We provide some numerical convergence results for the scalar case, and finally we show that the performance of the obtained solution is comparable to the one of the optimal controller in the TCP case.

In the past few years, networked embedded control systems have drawn considerable attention in the academic world. We will now try to set our work in the context of the existing literature. In [7] and [8], an estimator, *i.e.*, a Kalman filter, is placed at the sensor side of the link and no assumption is made on the statistical model of the data loss process. Smith *et al.* [9] focused on designing a suboptimal yet computationally efficient estimator for Markov Chain arrival processes.

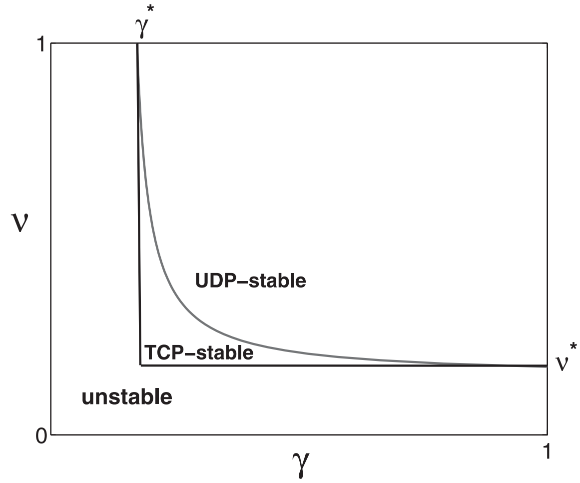

Fig. 1. Region of stability for UDP-like and TCP-like optimal control relative to measurement packet arrival probability $\bar{\gamma}$, and the control packet arrival probability $\bar{\nu}$.

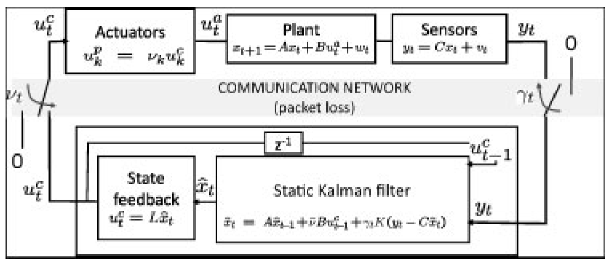

Fig. 2. **Overview of the system.** Architecture of the closed loop system over a communication network. The binary random variables ν_t and γ_t indicates whether packets are transmitted successfully.

In [10] the authors study the stability of Kalman Filter under general Markovian packet losses. In [11], the authors present a simple estimation scheme that is able to recover the fate of the control packet under UDP-like protocols by constraining the control signal sent to the plant. Drew *et al.* [12] analyze the problem of designing a controller over a wireless local area network (LAN). Control design has been investigated in the context of Cross Layer Design by Liu *et al.* [13]. Finally, in [14] and [15] the plant and the controller are modeled as deterministic time invariant discrete-time systems connected to zero-mean stochastic structured uncertainty, where the variance of the stochastic perturbation is a function of the Bernoulli parameters. Here, the controller design posed as an optimization problem to maximize mean-square stability of the closed loop system. While this method is suitable for the analysis of multiple input multiple output (MIMO) systems with many different controller and receiver compensation schemes [14], it does not include process and observation noise. The resulting controller is restricted to be time-invariant, hence suboptimal. Finally, within the context of UDP-like control, Epstein *et al.* [11] recently proposed to estimate not only the state of the system, but also a binary variable which indicates whether the previous control packet has been received or not. Such strategy improves closed loop performance at the price of a somewhat larger computational complexity.

The remainder of this paper is organized as follows. Section II provides the problem formulation. In Section III we summarize our previous results that are needed to understand the new contribution. In Section IV we consider the optimization problem leading to the optimal linear UDP-like controller and discusses a solution to a weaker, necessary solution for optimality. Section V shows the results and compares them to the optimal TCP-like controller (which is always linear). Finally, Section VI draws conclusions and outlines the agenda for future work.

II. PROBLEM FORMULATION

Consider the following linear stochastic system which models both observation and control packet losses:

$$x_{k+1} = Ax_k + Bu_k^a + w_k \tag{1}$$

$$u_k^a = \nu_k u_k^c \tag{2}$$

$$y_k = \gamma_k (Cx_k + v_k) \tag{3}$$

where u_k^a is the control input to the actuator, u_k^c is the desired control input computed by the controller, (x_0, w_k, v_k) are Gaussian, uncorrelated, white, with mean $(\bar{x}_0, 0, 0)$ and covariance (P_0, Q, R), respectively, and (γ_k, ν_k) are i.i.d. Bernoulli random variables with $P(\gamma_k = 1) = \bar{\gamma}$ and $P(\nu_k = 1) = \bar{\nu}$. The stochastic variable ν_k models the loss of packets between the controller and the actuator: if the packet is correctly delivered then $u_k^a = u_k^c$, otherwise if it is lost then the actuator does nothing, *i.e.*, $u_k^a = 0$. This zero-input compensation scheme is summarized by (2). This modeling choice is not unique: for example in the hold-input strategy, if the control packet u_k^c is lost, then the actuator could use the previous control value, *i.e.*, $u_k^a = u_{k-1}^a$. It has been shown that both strategies yield comparable performance [16]. In this work we will focus on zero-input strategy since it gives rise to simpler equations. The stochastic variable γ_k models the packet loss between the sensor and the controller: if the packet is delivered then $y_k = Cx_k + v_k$, otherwise if it is lost then the controller reads a zero, *i.e.*, $y_k = 0$. This observation model is summarized by (3). A different observation formalism was proposed in [17], where the missing observation was modeled as an observation for which the measurement noise had infinite covariance. It is possible to show that both models are equivalent [18], but the one considered in this paper has the advantage to lead to simpler analysis. This arises from the fact that when no packet is delivered, then the optimal estimator does not use the observation y_k at all, therefore its value is irrelevant.

We consider the following two information sets corresponding to the TCP-like and the UDP-like communication protocols respectively:

$$\mathscr{I}_k = \begin{cases} \mathscr{F}_k \triangleq \{\mathbf{y}^k, \boldsymbol{\gamma}^k, \boldsymbol{\nu}^{k-1}\}, & \text{TCP-like} \\ \mathscr{G}_k \triangleq \{\mathbf{y}^k, \boldsymbol{\gamma}^k\}, & \text{UDP-like} \end{cases} \tag{4}$$

where $\mathbf{y}^k \triangleq (y_k, y_{k-1}, \ldots, y_1)$, $\boldsymbol{\gamma}^k \triangleq (\gamma_k, \gamma_{k-1}, \ldots, \gamma_1)$, and $\boldsymbol{\nu}^k \triangleq (\nu_k, \nu_{k-1}, \ldots, \nu_1)$.

Consider also the following cost function:

$$J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0) \triangleq = \mathbb{E}\left[x_N' W_N x_N + \sum_{k=0}^{N-1} (x_k' W_k x_k + \nu_k u_k' U_k u_k) \times \middle| \mathbf{u}^{N-1}, \bar{x}_0, P_0 \right] \tag{5}$$

where $\mathbf{u}^{N-1} \triangleq (u_{N-1}, u_{N-2}, \ldots, u_1)$. Note that we are weighing the input only if it is successfully received at the plant. In fact, if it is not received, the plant applies zero input and therefore there is no energy expenditure.

We now look for a control input sequence $\mathbf{u}^{*N-1}$ as a function of the admissible information set $\mathscr{I}_k$, *i.e.*, $u_k = g_k(\mathscr{I}_k)$, that minimizes the functional defined in (5), *i.e.*,

$$J_N^*(\bar{x}_0, P_0) \triangleq \min_{\mathbf{u_k} = \mathbf{g}_k(\mathscr{I}_k)} J_N(\mathbf{u}^{N-1}, \bar{x}_0, P_0) \tag{6}$$

where $\mathscr{I}_k = \{\mathscr{F}_k, \mathscr{G}_k\}$ is one of the sets defined in (4). The set $\mathscr{F}$ corresponds to the information provided under an acknowledgment-based communication protocols (TCP-like) in which successful or unsuccessful packet delivery at the receiver is acknowledged to the sender within the same sampling period. The set $\mathscr{G}$ corresponds to the information available at the controller under communication protocols in which the sender receives no feedback about the delivery of the transmitted packet to the receiver (UDP-like). The UDP-like schemes are simpler to implement than the TCP-like schemes from a communication standpoint. However, in the UDP-like scheme the information set available to the controller is less rich.

III. PREVIOUS WORK

Before introducing new results, it is necessary to review recently published results [3–5], and [16] for both the TCP-like and the UDP-like case.

3.1 TCP-like case: estimator and controller design

The LQG control problem for the TCP-like case has been solved in full generality in [3].

Finite horizon LQG. The main results are summarized below:

- The separation principle holds under TCP-like communication, since the optimal estimator is independent of the control input u_k.
- The optimal estimator gain K_k is time-varying and stochastic since it depends on the past observation arrival sequence $\{\gamma_j\}_{j=1}^k$.
- The optimal LQG controller is a linear function of estimated state $\hat{x}_{k|k}$, *i.e.*, $u_k = L_k \hat{x}_{k|k}$.
- The final cost cannot be computed explicitly, since it depends on the realization of ν_t and γ_t, but can be analytically bounded.

Infinite horizon LQG. Consider the system (1)–(3) with the following additional hypothesis: $W_N = W_k = W$ and $U_k = U$. Moreover, let (A, B) and $(A, Q^{\frac{1}{2}})$ be controllable, and let (A, C) and $(A, W^{\frac{1}{2}})$ be observable. There exist critical arrival probabilities ν_c and γ_c , such that, for $\bar{\nu} > \nu_c$ and $\bar{\gamma} > \gamma_c$:

1. The infinite horizon optimal controller gain is constant:

$$\lim_{k\to\infty} L_k = L_\infty = -(B'S_\infty B + U)^{-1} \times B'S_\infty A \tag{7}$$

2. The infinite horizon optimal estimator gain K_k is stochastic and time-varying since it depends on the past observation arrival sequence $\{\gamma_j\}_{j=1}^k$.
3. The expected minimum cost can be bounded by two deterministic sequences:

$$\frac{1}{N} J_N^{min} \le \frac{1}{N} J_N^* \le \frac{1}{N} J_N^{max} \tag{8}$$

where J_N^{min}, J_N^{max} converge to the following values:

$$\begin{aligned} J_\infty^{max} &\triangleq \lim_{N\to+\infty} \frac{1}{N} J_N^{max} \\ &= \mathrm{trace}((A'S_\infty A + W - S_\infty)(\overline{P}_\infty - \\ &\quad +\bar{\gamma}\overline{P}_\infty C'(C\overline{P}_\infty C' + R)^{-1} C\overline{P}_\infty)) \\ &\quad +\mathrm{trace}(S_\infty Q) \\ J_\infty^{min} &\triangleq \lim_{N\to+\infty} \frac{1}{N} J_N^{min} \\ &= (1-\bar{\gamma})\mathrm{trace}((A'S_\infty A + W - S_\infty)\underline{P}_\infty) \\ &\quad +\mathrm{trace}(S_\infty Q) \end{aligned}$$

and the matrices S_∞, $\overline{P}_\infty$, $\underline{P}_\infty$ are the positive definite solutions of the following equations:

$$\begin{aligned} S_\infty &= A'S_\infty A + W - \bar{\nu}A'S_\infty B \\ &\quad \times (B'S_\infty B + U)^{-1} B'S_\infty A \\ \overline{P}_\infty &= A\overline{P}_\infty A' + Q - \bar{\gamma}A\overline{P}_\infty C' \\ &\quad \times (C\overline{P}_\infty C' + R)^{-1} C\overline{P}_\infty A' \\ \underline{P}_\infty &= (1-\bar{\gamma})A\underline{P}_\infty A' + Q. \end{aligned}$$

The critical probability ν_c can be numerically computed via the solution of a quasi-convex linear matrix inequality (LMI) optimization problem, as shown in [3]. Also the following analytical bounds are provided:

$$\begin{aligned} p_{min} &\le \nu_c, \gamma_c \le p_{max} \\ p_{min} &\triangleq 1 - \frac{1}{\max_i |\lambda_i^u(A)|^2} \\ p_{max} &\triangleq 1 - \frac{1}{\prod_i |\lambda_i^u(A)|^2} \end{aligned}$$

where $\lambda_i^u(A)$ are the unstable eigenvalues of A. Moreover, $\nu_c = p_{min}$ when B is square and invertible [19], and $\nu_c = p_{max}$ when B is rank one [15]. Dually, $\gamma_c = p_{min}$ when C is square and invertible, and $\gamma_c = p_{max}$ when C is rank one.

3.2 UDP-like case: estimator and controller design

As stated above, the LQG optimal control problem for the UDP-like case presents analytical complications. The lack of acknowledgment of the arrival of a control packet has dramatic effects on the controller design. Complete derivations for this case are presented in [4]. Here is a summary of them:

- The innovation step in the design of the estimator now explicitly depends on the input u_k;
- the **separation principle** is **not valid** anymore in this setting.
- the LQG optimal control feedback $u_k = g_k^*(\mathcal{G}_k)$ with horizon $N \ge 2$ that minimizes the functional

(5) under UDP-like communication is, in general, a **nonlinear** function of information set $\mathcal{G}_k$.
- In the special case in which the full state can be observed whenever the observation packet arrives, *i.e.*, C is invertible and $R=0$, the LQG controller is linear in the state, although the separation principle does not hold.

Our experience in the design of control systems over wireless sensor networks has taught us that it may be extremely difficult to design and implement a TCP-like protocol on such infrastructure. Therefore, there arises the need to design an easily computable controller that, although suboptimal, can guarantee "acceptable" performance in UDP-like scenarios. The rest of paper will deal with selecting such regulator within the class of linear static controllers.

IV. A LINEAR STATIC CONTROLLER FOR UDP-LIKE NETWORKED SYSTEMS

We want to find optimal static gains L, K for the LQG controller and estimator respectively. The estimator equations are:

$$\begin{aligned}\hat{x}_{k+1} &= A\hat{x}_k + \bar{\nu}Bu_k + \gamma_k K(y_k - \hat{y}_k)\\ u_k &= -L\hat{x}_k \qquad (9)\\ \hat{y}_k &= C\hat{x}_k. \qquad (10)\end{aligned}$$

After some simple algebra the close loop dynamics can be written as,

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A & -\nu_k BL \\ \gamma_k KC & A-\bar{\nu}BL-\gamma_k KC \end{bmatrix}\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} + \begin{bmatrix} w_k \\ \gamma_k K v_k \end{bmatrix}.$$

If we define the vector $z_k \triangleq [x_k^T \;\; \hat{x}_k^T]^T \in \mathbb{R}^{2n}$, then the previous equation can be written in a more compact form as

$$z_{k+1} = G_{\gamma_k,\nu_k}(K,L)z_k + d_k. \qquad (11)$$

Now let

$$P_k \triangleq \mathbb{E}\left[\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}[x_k^T \;\; \hat{x}_k^T]\right] = \begin{bmatrix} P_k^{11} & P_k^{12} \\ P_k^{12T} & P_k^{22} \end{bmatrix}$$

where P_k is the covariance of the vector z_k. Its evolution is given by

$$\begin{aligned}P_{k+1} &= \mathbb{E}[G_{\gamma_k,\nu_k}(K,L)z_k z_k^T G_{\gamma_k,\nu_k}^T(K,L)] + \mathbb{E}[d_k d_k^T]\\ &= \mathbb{E}_{\nu,\gamma}[G_{\gamma_k,\nu_k}(K,L)P_k G_{\gamma_k,\nu_k}^T(K,L)] + D(K)\\ &= \overline{G}(K,L,P_k) + D(K) \qquad (12)\end{aligned}$$

where

$$D(K) = \begin{bmatrix} Q & 0 \\ 0 & \bar{\gamma}KRK^T \end{bmatrix} \qquad (13)$$

$$\begin{aligned}\overline{G}(K,L,P) &= \bar{\gamma}\bar{\nu}G_{11}PG_{11}^T + \bar{\gamma}(1-\bar{\nu})G_{10}PG_{10}^T\\ &\quad +(1-\bar{\gamma})\bar{\nu}G_{01}PG_{01}^T\\ &\quad +(1-\bar{\gamma})(1-\bar{\nu})G_{00}PG_{00}^T\\ G_{11} &= \begin{bmatrix} A & -BL \\ KC & A-\bar{\nu}BL-KC \end{bmatrix}\\ G_{10} &= \begin{bmatrix} A & 0 \\ \bar{\gamma}KC & A-\bar{\nu}BL-\bar{\gamma}KC \end{bmatrix}\\ G_{01} &= \begin{bmatrix} A & -\bar{\nu}BL \\ 0 & A-\bar{\nu}BL \end{bmatrix}\\ G_{00} &= \begin{bmatrix} A & 0 \\ 0 & A-\bar{\nu}BL \end{bmatrix}.\end{aligned}$$

We next define the following cost:

$$\begin{aligned}c_k &= \mathbb{E}[x_k^T W x_k + \bar{\nu}u_k^T U u_k]\\ &= \text{Trace}\left(\begin{bmatrix} W & 0 \\ 0 & \bar{\nu}L^T U L \end{bmatrix} P_k\right)\\ &= \text{Trace}(N(L)P_k) \qquad (14)\end{aligned}$$

where

$$N(L) = \begin{bmatrix} W & 0 \\ 0 & \bar{\nu}L^T U L \end{bmatrix}. \qquad (15)$$

Clearly, if P_k converges to a finite value P_∞, then does the cost, *i.e.*, c_k converges to c_∞. Therefore, our objective is to minimize this cost function with respect to K, L. The optimization problem can be written as follows:

$$\begin{aligned}\text{Min}_{K,L} \quad & Tr(PN(L))\\ s.t. \quad & P = \overline{G}(K,L,P) + D(K), \qquad (16)\\ & P \geq 0.\end{aligned}$$

This is a non-convex optimization problem, and in the next section we will find necessary conditions for the existence of an optimum.

4.1 Necessary conditions

Using Lagrange multipliers the optimization problem can be rewritten as:

$$\begin{aligned}\text{Min}_{K,L,P,\Lambda} \quad & J = Tr(PN(L)) \\ & +Tr(\Lambda(\bar{G}(K,L,P) \\ & +D(K)) - P) \\ s.t. \quad & P \geq 0, \quad \Lambda \geq 0.\end{aligned} \tag{17}$$

According to the minimum matrix principle [20], necessary conditions for the optimum are:

$$\frac{\partial J}{\partial \Lambda}=0, \quad \frac{\partial J}{\partial P}=0, \quad \frac{\partial J}{\partial K}=0, \quad \frac{\partial J}{\partial L}=0. \tag{18}$$

The first two conditions above can be written respectively as:

$$P=\overline{G}(K,L,P)+D(K), \quad P \geq 0 \tag{19}$$

$$\Lambda=\underline{G}(K,L,\Lambda)+N(L), \quad \Lambda \geq 0 \tag{20}$$

where

$$\begin{aligned}\underline{G}(K,L,P) &= \bar{\gamma}\bar{\nu}G_{11}^T P G_{11} + \bar{\gamma}(1-\bar{\nu})G_{10}^T P G_{10} \\ &+(1-\bar{\gamma})\bar{\nu}G_{01}^T P G_{01} \\ &+(1-\bar{\gamma})(1-\bar{\nu})G_{00}^T P G_{00}\end{aligned}$$

Note that the operator $\underline{G}(K,L,P)$ is simply the dual of $\overline{G}(K,L,P)$. Let us consider the following partition of P and Λ and new matrices:

$$P=\begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \quad \Lambda=\begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_2 \end{bmatrix}$$

$$\overline{\Lambda}=\Lambda_1-\Lambda_2, \quad \underline{\Lambda}=\Lambda_2, \quad \overline{P}=P_1-P_2, \quad \underline{P}=P_2.$$

As shown in [21], the minimality assumption implies that:

$$\Lambda_{12}=-\underline{\Lambda}<0, \quad P_{12}=\underline{P}>0. \tag{21}$$

An immediate result is that $\lim_{k\to\infty} \mathbb{E}[(x_k-\hat{x}_k)\hat{x}_k^T]=P_{12}-P_2=0$, *i.e.*, the estimate is asymptotically uncorrelated with the error estimate, similarly to the standard Kalman filtering.

Equation (21) can be used to simplify (19) and (20). In fact we can use the change of variables:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}=\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}=\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \tag{23}$$

which gives:

$$\begin{aligned}\begin{bmatrix} \xi_{1,k+1} \\ \xi_{2,k+1} \end{bmatrix} &= \begin{bmatrix} A-\gamma_k KC & (\bar{\nu}-\nu_k)BL \\ \gamma_k KC & A-\bar{\nu}BL \end{bmatrix}\begin{bmatrix} \xi_{1,k} \\ \xi_{2,k} \end{bmatrix} \\ &+\begin{bmatrix} w_k-\gamma_k K v_k \\ \gamma_k K v_k \end{bmatrix} \\ \begin{bmatrix} \zeta_{1,k+1} \\ \zeta_{2,k+1} \end{bmatrix} &= \begin{bmatrix} A-\nu_k BL & -\nu_k BL \\ (\nu_k-\bar{\nu})BL & A-(\bar{\nu}-\nu_k)BL-\gamma_k KC \end{bmatrix} \\ &\cdot\begin{bmatrix} \zeta_{1,k} \\ \zeta_{2,k} \end{bmatrix}+\begin{bmatrix} w_k-\gamma_k K v_k \\ \gamma_k K v_k \end{bmatrix}.\end{aligned}$$

If we substitute the state space representation given by (22) into (19) and (23) into (20), then the matrices P and Λ become diagonal and are given by $P=\text{diag}(\overline{P},\underline{P})$ and $\Lambda=\text{diag}(\overline{\Lambda},\underline{\Lambda})$. By using this fact, then it is sufficient to compute the diagonal terms of (19) and (20) which are given by the following:

$$\begin{aligned}\overline{P} &= \mathbb{E}[(A-\gamma_k KC)\overline{P}(A-\gamma_k KC)^T] \\ &+\mathbb{E}[(\bar{\nu}-\nu_k)^2 BL\underline{P}L^T B^T]+Q+\bar{\gamma}KRK^T \\ \underline{P} &= \mathbb{E}[\gamma_k^2 KC\overline{P}C^T K^T] \\ &+\mathbb{E}[(A-\bar{\nu}BL)\underline{P}(A-\bar{\nu}BL)^T]+\bar{\gamma}KRK^T \\ \overline{\Lambda} &= \mathbb{E}[(A-\nu_k BL)^T\overline{\Lambda}(A-\nu_k BL)] \\ &+\mathbb{E}[(\bar{\nu}-\nu_k)^2 L^T B^T\underline{\Lambda}BL]+W+\bar{\nu}L^T UL \\ \underline{\Lambda} &= \mathbb{E}[\nu_k^2 L^T B^T\overline{\Lambda}BL] \\ &+\mathbb{E}[(A-(\bar{\nu}-\nu_k)BL-\gamma_k KC)^T\underline{\Lambda} \\ &\cdot(A-(\bar{\nu}-\nu_k)BL-\gamma_k KC)]+\bar{\gamma}L^T UL.\end{aligned}$$

The previous equations can be then written as follows

$$\begin{aligned}\overline{P} &= \bar{\gamma}(A-KC)\overline{P}(A-KC)^T+(1-\bar{\gamma})A\overline{P}A^T \\ &+\bar{\nu}(1-\bar{\nu})BL\underline{P}L^T B^T+Q+\bar{\gamma}KRK^T \\ &=\Phi_1(\overline{P},\underline{P},K,L)\end{aligned} \tag{24}$$

$$\underline{P} = (A-\bar{v}BL)\underline{P}(A-\bar{v}BL)^T + \bar{\gamma}K(C\overline{P}C^T+R)K^T = \Phi_2(\overline{P}, \underline{P}, K, L) \quad (25)$$

$$\overline{\Lambda} = \bar{v}(A-BL)^T\overline{\Lambda}(A-BL)+(1-\bar{v})A^T\overline{\Lambda}A + W+\bar{v}(L^T(U+(1-\bar{v})B^T\underline{\Lambda}B)L = \Phi_3(\overline{\Lambda}, \underline{\Lambda}, L) \quad (26)$$

$$\underline{\Lambda} = \bar{\gamma}(A-KC)^T\underline{\Lambda}(A-KC)+(1-\bar{\gamma})A^T\underline{\Lambda}A + \bar{v}L^T(B^T\overline{\Lambda}B+(1-\bar{v})B^T\underline{\Lambda}B+U)L = \Phi_4(\overline{\Lambda}, \underline{\Lambda}, K, L). \quad (27)$$

Similarly, if we use (21) into the last two partial derivatives of (18) written in the new state variable above, then it is clear that $\frac{\partial J}{\partial K}$ and $\frac{\partial J}{\partial L}$ are equivalent to the partial derivatives of the trace of the right hand sides of (24) and (26), respectively, which give the following constraints:

$$K = A\overline{P}C^T\left(C\overline{P}C^T + R\right)^\dagger = \Phi_5(\overline{P}) \quad (28)$$

$$L = (B^T\overline{\Lambda}B+(1-\bar{v})B^T\underline{\Lambda}B+U)^\dagger B^T\overline{\Lambda}A = \Phi_6(\overline{\Lambda}, \underline{\Lambda}) \quad (29)$$

where the symbol † represents the Moore-Penrose pseudoinverse. Note that if $\bar{v}=\bar{\gamma}=1$ and we substitute (28) into (24), and (29) into (26), we obtain the standard Algebraic Riccati equations for the Kalman filter and LQ optimal controller, respectively. Next section provides an iterative algorithm that converges to solution of the optimization problem if such a solution exists.

4.2 Iterative solution

As described above, the six coupled nonlinear equations (24)–(29) define a set of necessary conditions. A natural choice to try to find a fixed point is to use an iterative approach as the following:

$$\overline{P}_{k+1} = \Phi_1(\overline{P}_k, \underline{P}_k, K_k, L_k) \quad (30)$$

$$\underline{P}_{k+1} = \Phi_2(\overline{P}_k, \underline{P}_k, K_k, L_k) \quad (31)$$

$$\overline{\Lambda}_{k+1} = \Phi_3(\overline{\Lambda}_k, \underline{\Lambda}_k, L_k) \quad (32)$$

$$\underline{\Lambda}_{k+1} = \Phi_4(\overline{\Lambda}_k, \underline{\Lambda}_k, K_k, L_k) \quad (33)$$

$$K_k = \Phi_5(\overline{P}_k) \quad (34)$$

$$L_k = \Phi_6(\overline{\Lambda}_k, \underline{\Lambda}_k). \quad (35)$$

For ease of notation, if we substitute the last two equations for the gains K_k, L_k into the previous four, the iterative update can be written in a more compact form as follows:

$$(\overline{P}_{k+1}, \underline{P}_{k+1}, \overline{\Lambda}_{k+1}, \underline{\Lambda}_{k+1}) = \Phi(\overline{P}_k, \underline{P}_k, \overline{\Lambda}_k, \underline{\Lambda}_k). \quad (36)$$

It was shown by De Koning in [6] that, under some standard conditions on stabilizability and detectability of the open loop system, the necessary conditions given by (24)–(29) are also sufficient and the iterative solution given by (30)–(35) converges to the fixed point solution. We adapt his results to our scenario in the following theorem:

Theorem 1. Let us consider the closed loop control system defined by (1)–(2) and (9)–(10), where v_k and γ_k are Bernoulli random variables with mean $\bar{v}$ and $\bar{\gamma}$, respectively. Assume that (A, B), (A^T, C^T), $(A, W^{1/2})$ and $(A^T, Q^{1/2})$ are all stabilizable, and $U>0$, $R>0$. Then, the sequence defined by (30)–(35) starting from initial conditions $\overline{P}_0=\underline{P}_0=\overline{\Lambda}_0=\underline{\Lambda}_0=0$ converges to the unique solution of the optimization problem defined by (16), *i.e.*,

$$\lim_{k\to\infty} \Phi^k(0, 0, 0, 0) = (\overline{P}^*, \underline{P}^*, \overline{\Lambda}^*, \underline{\Lambda}^*)$$

if and only if the sequence defined by (30)–(35) where $W=Q=0$, $V=R=0$ and initial conditions $\overline{P}_0=\overline{\Lambda}_0=I$ and $\underline{P}_0=\underline{\Lambda}_0=0$ converge to zero, *i.e.*,

$$\lim_{k\to\infty} \Phi^k(I, 0, I, 0) = (0, 0, 0, 0).$$

The proof of the previous theorem is rather involved and requires the use of the homotopic continuation method to prove convergence, therefore it is omitted. We refer the interested reader to [6] and [22] for details.

V. DISCUSSION

In the previous section we provided necessary and sufficient conditions for the existence of an optimum, along with an iterative method to compute it. This section shows some numerical examples and applications of the proposed iterative algorithm.

For the sake of simplicity and to be able to provide a readable plot, consider a scalar version of the system of (1)–(3), with $B=C=Q=R=W=U=1$, $A=1.1$, $v=\gamma=0.8$. Fig. 3 shows a contour plot of the infinite horizon cost as a function of the controller and estimator gains, K and L respectively.

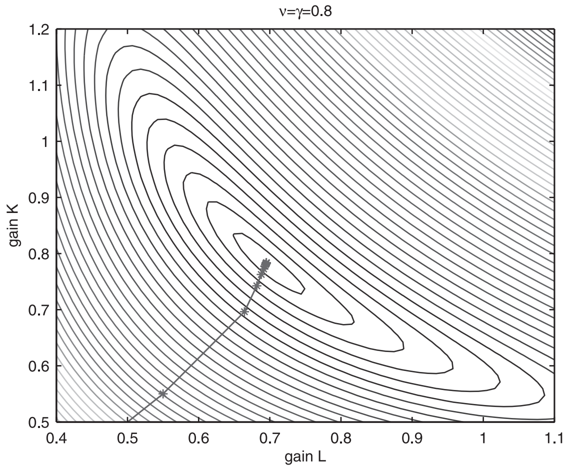

Fig. 3. Contour plot of infinite horizon cost for fixed control and estimation gains L and K. Solid line corresponds to the (L_k, K_k) points computed using the iterative solution of (30)–(35) with zero initial conditions.

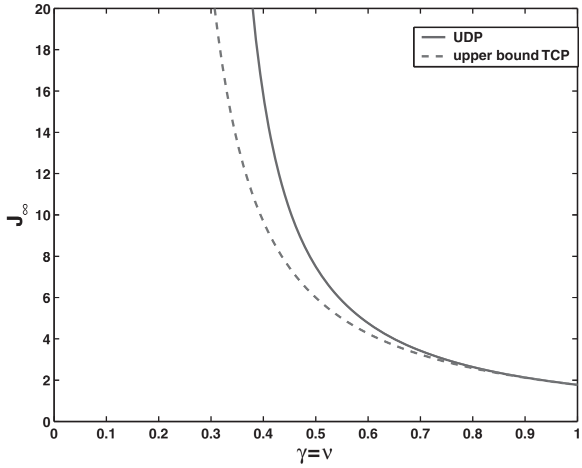

Fig. 4. Comparison of optimal linear LQG-like controllers under TCP-like and UDP-like communication protocols for the pendubot.

The infinite horizon cost function J is computed as $J(K, L) = \text{trace}(N(L)P)$ where P is given by the linear equation (19) for fixed values of K and L. Fig. 3 also shows the trace (L_k, K_k) obtained from (30)–(35) with zero initial conditions for the matrices $\overline{P}$, $\underline{P}$, $\overline{\Lambda}$, $\underline{\Lambda}$, thus providing a sanity check of iterative solution of the optimal LQG problem. Note that the cost function is non-convex, but that there is a unique minimum.

We now consider the LQG design for the pendubot, which is an example of a MIMO unstable system. The pendubot consists of two-link planar robot with torque actuation only on the first link, and we address the interested reader to [23] for more details and references. We are interested in designing a controller that stabilizes the pendubot in the up-right position. The state space representation of the system linearized about the unstable equilibrium point and discretized with sampling period $T_s = 0.005$ s is given by:

$$A = \begin{bmatrix} 1.001 & 0.005 & 0.000 & 0.000 \\ 0.35 & 1.001 & -0.135 & 0.000 \\ -0.001 & 0.000 & 1.001 & 0.005 \\ -0.375 & -0.001 & 0.590 & 1.001 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.001 \\ 0.540 \\ -0.002 \\ -1.066 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \quad U = 2$$

$$Q = qq^T, \quad q = \begin{bmatrix} 0.003 \\ 1.000 \\ -0.005 \\ -2.150 \end{bmatrix}$$

$$W = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix A has two stable and two unstable eigenvalues, $\text{eig}(A) = (1.061, 1.033, 0.968, 0.941)$. It is easy to verify that the pairs (A, B) and (A, Q) are controllable, (A, C) and (A, W) are observable, and $R > 0$, as required by the assumptions of the theorems presented in the previous sections.

Fig. 4 shows a comparison between the optimal linear TCP-like LQG controller and the optimal linear UDP-like controller derived above, for different values of $\bar{\nu} = \bar{\gamma}$. The figure suggests that for sufficiently high arrival rate, implementing an optimal controller over a TCP-like network does not provide a significant advantage. This is particularly useful to the designer, who can trade off high complexity in the network design with a little performance loss.

VI. CONCLUSION AND FUTURE WORK

In this paper we analyzed a generalized version of the LQG control problem in the case where both observation and control packets may be lost during transmission over a communication channel. This situation arises frequently in distributed systems where sensors, controllers and actuators reside in different physical locations and have to rely on data networks to exchange information. In this context controller design heavily depends on the communication protocol used. In fact, in TCP-like protocols, acknowledgments of successful transmissions of control packets are provided to the controller, while in UDP-like protocols, no such feedback is provided. In the first case, the separation principle holds and the optimal control is a linear function of the state. As a consequence, controller and estimator design problems are decoupled. UDP-like protocols present a much more complex problem. We have shown that the lack of acknowledgment of control packets results in the failure of the separation principle. Estimation and control are now intimately coupled. We have shown that the LQG optimal control is, in general, nonlinear in the estimated state. In the particular case, where we have access to full state information, the optimal controller is linear in the state. To fully exploit UDP-like protocols it is necessary to have a controller/estimator design methodology for the general case when there is measurement noise and under partial state observation. As UDP-like protocols are the only practical solution in many cases where the channel is too unreliable to guarantee successful delivery of acknowledgment, it would prove extremely valuable to determine the optimal time-invariant LQG controller. Among all possible choices we focused on the class of linear controllers, for their simplicity in implementation. After describing the optimization problem, we derived necessary and sufficient conditions for the existence of a unique solution. Another very interesting finding is that for practical purposes, control performance is not greatly affected by lack of optimality of the linear controller.

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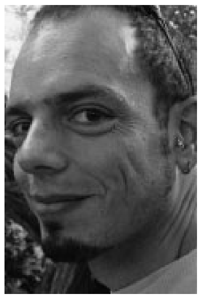

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