A Three-Stage Colonel Blotto Game: When to Provide More Information to an Adversary

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Abstract. In this paper, we formulate a three-player three-stage Colonel Blotto game, in which two players fight against a common adversary. We assume that the game is one of complete information, that is, the players have complete and consistent information on the underlying model of the game; further, each player observes the actions taken by all players up to the previous stage. The setting under consideration here is similar to the one considered in our recent work [8], but with a different information structure during the second stage of the game; this leads to a significantly different solution.

In the first stage, players can add additional battlefields. In the second stage, the players (except the adversary) are allowed to transfer resources among each other if it improves their expected payoffs, and simultaneously, the adversary decides on the amount of resource it allocates to the battle with each player subject to its resource constraint. At the third stage, the players and the adversary fight against each other with updated resource levels and battlefields. We compute the subgame-perfect Nash equilibrium for this game. Further, we show that when playing according to the equilibrium, there are parameter regions in which (i) there is a net positive transfer, (ii) there is absolutely no transfer, (iii) the adversary fights with only one player, and (iv) adding battlefields is beneficial to a player. In doing so, we also exhibit a counterintuitive property of Nash equilibrium in games: extra information to

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a player in the game does not necessarily lead to a better performance for that player. The result finds application in resource allocation problems for securing cyber-physical systems.

1 Introduction

The Colonel Blotto game is a two-player complete-information static non-cooperative game, in which two resource-constrained players fight against each other on a fixed number of battlefields. The players decide on the allocation of resources on each battlefield subject to their resource constraints. On each battlefield, the player deploying the maximum resource is declared the winner of that battlefield and accrues certain payoff. The goal of each player is to maximize the expected total number of battlefields that he/she wins.

The setup of the Colonel Blotto game is found naturally in several engineering and economic systems. Consider, for example, a data center with multiple servers under attack from a hacker. Each server can be viewed as a battlefield with the data center and the hacker viewed as the two players. Each player has limited computational resource to deploy – the data center deploys resource for securing the servers, and the hacker deploys resource for hacking the servers. The resulting game is captured by the Colonel Blotto game. Similarly, the competition between two research companies that are deploying their resources in different projects can also be analyzed within the framework of the Colonel Blotto game.

The Colonel Blotto game in which both players have equal resources and there are three battlefields was first solved in [3]. This result was later extended to the case of symmetric resources and arbitrary number of battlefields in [7]. In the same paper, the authors computed the Nash equilibrium for the case of asymmetric resources and two battlefields. However, Colonel Blotto game with asymmetric resources and three or more battlefields remained open until 2006, when Roberson established the existence of a Nash equilibrium in mixed strategies, and computed the (mixed) equilibrium strategies of the players in [14]. A similar setup was also considered in [13], in which the resource levels of both players were considered to be equal.

The work of Roberson sparked great interest in the field; numerous theoretical extensions of the game followed after 2006. In particular, [11] and [12] considered two-stage Colonel Blotto games. In [11], the authors identified situations in which adding battlefields during the first stage of the game is beneficial to the players. In [12], the authors considered a three-player Colonel Blotto game, in which the first two players fight against a common adversary. They have identified conditions under which forming a coalition could be beneficial to both players. However, they do not obtain a Nash equilibrium of the game.

Applications of the Colonel Blotto game has also received attention. References [4] and [5] studied phishing attacks and defense strategies over the internet. References [1] and [2] conducted experimental studies of the Colonel Blotto game with human subjects, and proposed a novel decision procedure, which the authors called *multi-dimensional reasoning*. Another interesting experimental paper is [10], where the authors study social interactions using a Facebook application called "Project Waterloo", which allows users to invite both friends and strangers to play Colonel Blotto against themselves.

Recently, we have formulated in [8] a three-stage Colonel Blotto game with hierarchical information structure, in which two players fight against a common adversary. In that paper, the problem formulation was as follows: At the first stage, the players may add battlefields. At the second stage, the game has a hierarchical information structure; the players may transfer some resources to each other, and the adversary has access to the amount of resource transferred. Based on this information, the adversary decides on its allocation of resources for the battles against the two players. At the third stage, the adversary fights two battles against the two players with the updated resource levels and battlefields. We further assumed that this is a game of complete information, that is, at any stage, all players including the adversary have access to all the information that has been generated in the past stage(s), and this is common knowledge.

This paper also considers a similar setup as in [8], but with a different information structure. In [8], we have assumed that the adversary has access to the information about the amount of resources that are transferred between the players during the second stage. In this paper, on the other hand, we assume that the adversary does not have access to that information. In other words, the transfer between the two players and the resource allocation of the adversary towards the two battles happen *simultaneously*[3]. This leads to a very different Nash equilibrium. One of the primary goal of this paper is to underscore the importance of information structure in the allocation of resources in a class of Colonel Blotto games. Furthermore, this study also provides insight on "what information about the formation of a strategic alliance should be made public" in such games. The information that is made public in a strategic alliance between two cyber-physical systems may have severe repercussions on the security and vulnerabilities of those systems if they are attacked by a strategic adversary.

1.1 Outline of the Paper

We formulate the three-stage three-player Colonel Blotto game problem and identify several outstanding issues in Section 2. Thereafter, we recall the Nash equilibrium and the equilibrium expected payoffs to the players in the classical static two-player Colonel Blotto game in Section 3. The discussion in this section is based on [14]. In Section 4, we compute the subgame-perfect Nash equilibrium of the game formulated in Section 2 for three specific cases. We also discuss and comment on the Nash equilibrium obtained in that section. We provide some concluding discussions and state the future directions that the research can take in Section 5.

Before we discuss the general setup of the game, we introduce a few notations in the next subsection.

[3] In decision problems, when decision makers act *simultaneously*, then it does not necessarily mean that they act at the same time instant; it simply means that a decision maker may not have access to the action of the other decision maker who may have acted in the past. The two cases require the same analysis.

1.2 Notations

For a natural number N, we use $[N]$ to denote the set $\{1, \dots, N\}$. $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote, respectively, the sets of all non-negative real numbers and non-negative integers. Let $\mathcal{X}_i, i \in [N]$ be non-empty sets. If $x_1 \in \mathcal{X}_1, \dots, x_N \in \mathcal{X}_N$ are elements, then $x_{1:N}$ denotes the sequence $\{x_1, \dots, x_N\}$. Similarly, $\mathcal{X}_{1:N}$ denotes the product space $\mathcal{X}_1 \times \dots \times \mathcal{X}_N$.

2 Problem Formulation

In this section, we formulate a three-player three-stage Colonel Blotto game. The first two players are fighting against an adversary, call it A, who is the third player in the game. Henceforth, we use Player 3 and A interchangeably to refer to the adversary. Each player is endowed with some resources at the beginning of the game. We use β_i and α, respectively, to denote the initial endowment of the resources of Player $i \in \{1, 2\}$ and the adversary. At the beginning of the game, for every Player $i \in \{1, 2\}$, there are $n_i \geq 3$ battlefields, each with payoff v_i, at which the battle between Player i and the adversary will take place.

During the first stage of the game, each of the first two players may add additional battlefields at some cost. During the second stage of the game, the first two players may exchange resources among themselves if it improves their payoffs, while the adversary decides on the allocation of resources to fight against the first two players. At the final stage, the players fight against the adversary with updated battlefields and resources. We consider here a game of complete and perfect information.

2.1 Information Structures and Strategies of the Players

At the first stage of the game, the players know all the parameters and the model of the game, and we assume that this is common knowledge. Player $i \in \{1, 2\}$ decides on $m_i \in \mathbb{Z}_+$, the number of battlefields he/she wants to add to the existing set of battlefields and pays a total cost of cm_i^2. The adversary does not take any action at the first stage.

At the second stage of the game, all players, including the adversary, observe the number of battlefields (m_1, m_2) that were added. At this stage, the first two players decide on the transfers: Player i chooses a function $t_{i,j} : \mathbb{Z}_+^2 \to [0, \beta_i]$ which takes the number of battlefields added by the players, (m_1, m_2), as input, and outputs the amount of resource he/she transfers to Player $j \neq i$, where $i, j \in \{1, 2\}$. These functions have to satisfy the following constraint:

$$t_{i,j}(m_1, m_2) \leq \beta_i, \quad \text{for all } m_1, m_2 \in \mathbb{Z}_+, j \neq i.$$

The adversary does not observe the transfers among the players, and decides on functions $\alpha_i : \mathbb{Z}_+^2 \to [0, \alpha]$ with the constraint

$$\alpha_1(m_1, m_2) + \alpha_2(m_1, m_2) \leq \alpha, \quad \text{for all } m_1, m_2 \in \mathbb{Z}_+^2.$$

We use r_i to denote the amount of resource available to Player $i \in \{1, 2\}$ after the redistribution of resources. This is given by

$$r_i := r_i(t_{1,2}, t_{2,1}) = \beta_i + (t_{j,i} - t_{i,j}) \qquad i \neq j, i, j \in \{1, 2\}.$$

For a given triple $\alpha_i, r_i \in \mathbb{R}_+$ and $m_i \in \mathbb{Z}_+$, let us define the sets

$$\mathcal{A}_i(\alpha_i, m_{1:2}) := \left\{ \{\alpha_{i,k}\}_{k=1}^{n_i+m_i} \subset \mathbb{R}_+ : \sum_{k=1}^{n_i+m_i} \alpha_{i,k} = \alpha_i(m_1, m_2) \right\},$$
$$\mathcal{B}_i(r_i, m_i) := \left\{ \{\beta_{i,k}\}_{k=1}^{n_i+m_i} \subset \mathbb{R}_+ : \sum_{k=1}^{n_i+m_i} \beta_{i,k} = r_i(t_{1,2}, t_{2,1}) \right\}.$$

At the final stage of the game, Player i and the adversary play the usual static two-player Colonel Blotto game on $n_i + m_i$ battlefields, with Player i having r_i and adversary having α_i amounts of resource. Thus, given the resource levels r_i of Player i, $i = 1, 2$, and α_i of the adversary, the action spaces of Player i and the adversary are, respectively, $\mathcal{B}_i(r_i, m_i)$ and $\mathcal{A}_i(\alpha_i, m_i)$. If $n_i \geq 3$, then there is no pure strategy Nash equilibrium of Player i and the adversary at the final stage. Thus, given the resource levels of the players, Player i and the adversary, respectively, decide on probability measures $\mu_i \in \wp(\mathcal{B}_i(r_i, m_i))$ and $\nu_i \in \wp(\mathcal{A}_i(\alpha_i, m_i))$ over their respective action spaces.

Henceforth, we use $\gamma^i := \{\gamma_1^i, \gamma_2^i, \gamma_3^i\}$ to denote the strategy of Player $i \in \{1, 2, A\}$, which is defined as follows:

$$\begin{aligned}
\gamma_1^i &:= m_i, \qquad \text{for } i \in \{1, 2\}, \\
\gamma_2^1(m_1, m_2) &:= \{t_{1,2}(m_1, m_2)\}, \quad \gamma_2^2(m_1, m_2) := \{t_{2,1}(m_1, m_2)\}, \\
\gamma_2^A(m_1, m_2) &:= \{\alpha_1(m_1, m_2), \alpha_2(m_1, m_2)\} \\
\gamma_3^i(m_1, m_2, t_{1,2}, t_{2,1}) &:= \{\mu_i\}, \quad i \in \{1, 2\}, \\
\gamma_3^A(m_1, m_2, t_{1,2}, t_{2,1}) &:= \{\nu_1, \nu_2\}.
\end{aligned}$$

Thus, each γ^i is a collection of functions; we denote the set of all such γ^is by Γ^i.

2.2 Payoff Functions of the Players

Consider the game between Player i and the adversary at the third stage of the game. Let us use $\beta_{i,k}$ and $\alpha_{i,k}$ to denote, respectively, the amounts of resource Player i and the adversary deploy on battlefield $k \in [n_i+m_i]$. On every battlefield $k \in [n_i + m_i]$, the player who deploys maximum amount of resource wins and receives a payoff v_i. In case of a tie, the players share the payoff equally[4]. We

[4] It should be noted that if players play according to the Nash equilibrium strategies on the battlefields, then the case of both players having equal resource on a battlefield has a measure zero. Therefore, in equilibrium, the tie breaking rule does not affect the equilibrium expected payoffs.

let $p_{i,k}(\beta_{i,k}, \alpha_{i,k})$ denote the payoff that Player i receives on the battlefield k, which we take to be given by

$$p_{i,k}(\beta_{i,k}, \alpha_{i,k}) = \begin{cases} v_i & \beta_{i,k} > \alpha_{i,k}, \\ \frac{v_i}{2} & \beta_{i,k} = \alpha_{i,k}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \in \{1, 2\}$ and $k \in [n_i + m_i]$. The payoff to the adversary on a battlefield k in the battle with Player i is given by

$$p^A_{i,k}(\beta_{i,k}, \alpha_{i,k}) = v_i - p_{i,k}(\beta_{i,k}, \alpha_{i,k}).$$

We take the expected payoff functionals of Player i and the adversary as

$$\pi_i(\gamma^{1:3}) = \mathbb{E}\left[\sum_{k=1}^{n_i+m_i} p_{i,k}(\beta_{i,k}, \alpha_{i,k})\right] - cm_i^2, \quad i \in \{1, 2\},$$
$$\pi_A(\gamma^{1:3}) = \mathbb{E}\left[\sum_{i=1}^{2} \sum_{k=1}^{n_i+m_i} p^A_{i,k}(\beta_{i,k}, \alpha_{i,k})\right],$$

where the expectation is taken with respect to the probability induced on the random variables $\{\beta_{i,k}, \alpha_{i,k}\}_{i,k}$ by the choice of strategies of the players in the game. The model of the game and the payoff functions are common knowledge among the players. The Colonel Blotto game formulated above is referred to as $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$.

We now define the Nash equilibrium of the game formulated above. The set of strategy profiles $\{\gamma^{1\star}, \gamma^{2\star}, \gamma^{A\star}\}$ is said to form a Nash equilibrium of the game if it satisfies

$$\pi_i(\gamma^{1:2\star}, \gamma^{A\star}) \geq \pi_i(\gamma^i, \gamma^{-i\star}, \gamma^{A\star}), \qquad i \in \{1, 2\}$$
$$\pi_A(\gamma^{1:2\star}, \gamma^{A\star}) \geq \pi_A(\gamma^{1:2\star}, \gamma^A)$$

for all possible $\gamma^i \in \mathit{\Gamma}^i$, $i \in \{1, 2, A\}$, where $\gamma^{1:2} := \{\gamma^1, \gamma^2\}$.

The set of all subgame-perfect Nash equilibria (SPNE) of a complete information game is a subset of all Nash equilibria of the game, and they can be obtained using a dynamic programming type argument (for precise definition, see [6]). In Section 4, we compute the SPNE of the game formulated above.

2.3 Research Questions and Solution Approach

At the outset, it is not clear what kind of solution we would expect in such a game. We are particularly interested in investigating the conditions on the parameters of the game, under which the following scenarios are possible:

1. There is a positive transfer from one player to another. Since this is a non-cooperative game, the transfer should *increase or maintain* the payoffs to both players - the player who transfers resources and the player who accepts the transfer.

2. There is no transfer among the players at the second stage.
3. The adversary allocates all its resource to fight only one player.
4. The players have an incentive to add new battlefields.

We first recall some relevant results on the two-players static Colonel Blotto game from [14]. Solving the general problem formulated above is somewhat difficult due to the discontinuity of the expected payoff functions in the endowments of the players in the static game. Therefore, we restrict our attention to a subset of all possible parameter regions in order to keep the analysis tractable. We compute the parameter regions which feature the scenarios listed above.

3 Relevant Results on the Static Two-Player Colonel Blotto Game

In this section, we recall the two-player Colonel Blotto game considered in [14]. The setting is that of two agents, and for clarity, we call them agents in this section. Agent $i \in \{1, 2\}$ is endowed with certain amount of resources, denoted by $r_i \in \mathbb{R}_+$. There are a total of n battlefields over which the agents fight. Define $\mathcal{R}_i := \{a \in \mathbb{R}_+^n : \sum_{k=1}^n a_k \leq r_i\}$ and let $\partial\mathcal{R}_i$ be the boundary of the region $\mathcal{R}_i$. The action space of Agent i is $\mathcal{R}_i$. Each agent decides on a mixed strategy over its action space, that is, a probability distribution over its action space, denoted by $\mu_i \in \wp(\mathcal{R}_i)$.

On each battlefield, the agent who deploys maximum resources wins, and accrues a payoff denoted by $v \in \mathbb{R}_+$[5]. In case both agents deploy equal amount of resources, then each accrue a payoff of $\frac{v}{2}$.

For a strategy of Agent i, μ_i, let $\mathrm{Pr}^k_{\#}\mu_i$ denote the marginal of μ_i on the k^{th} battlefield. Since any agent winning a battlefield is dependent only on the amount of resources deployed by both agents, for a given strategy tuple of the agents (μ_1, μ_2), the expected payoff to Agent i on battlefield $k \in [n]$ is dependent solely on the marginal distributions $(\mathrm{Pr}^k_{\#}\mu_1, \mathrm{Pr}^k_{\#}\mu_2)$.

For this game, we assume that all the parameters defined above is common knowledge among the agents. Let us denote this game by $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$. We now recall the following result from [14].

Theorem 1. *For the static Colonel Blotto game* $\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$ *with* $n \geq 3$, *there exists a Nash equilibrium* $(\mu_1^\star, \mu_2^\star)$ *with unique payoffs to each agent.*

The set of all Nash equilibria of the game $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$ is denoted by $\mathtt{NE}(\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$. Note that we do not claim uniqueness of Nash equilibrium of the game $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$. However, for any $i \in \{1, 2\}$, there exists a unique measure $\nu \in \wp([0, r_i])$ such that if $\mu_i^\star$ and $\tilde{\mu}_i^\star$ are two Nash equilibrium strategies of Agent i, then $\mathrm{Pr}^k_{\#}\mu_i^\star = \mathrm{Pr}^l_{\#}\tilde{\mu}_i^\star = \nu$ for all $l, k \in [n]$. In other words, the marginals on any two battlefields under any two equilibrium strategies for a agent are the same, and this marginal is unique.

[5] Typically, v is taken to be $\frac{1}{n}$ in the static Colonel Blotto game.

We have the following result on the expected payoffs of the agents when playing under Nash equilibrium strategies in the game $\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$.

Lemma 1. *Consider the static Colonel Blotto game* $\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)$ *with* $n \geq 3$*. Let* $P^i(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$ *denote the expected payoff to Agent* i *when both agents act according to Nash equilibrium strategies. If* r_1 *and* r_2 *are such that* $\frac{1}{n-1} \leq \frac{r_1}{r_2} \leq n-1$*, then the expected payoffs to the agents under Nash equilibrium strategies* $(\mu_1^\star, \mu_2^\star)$ *are*

$$P^1(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)) = \begin{cases} nv\left(\frac{2}{n} - \frac{2r_2}{n^2 r_1^2}\right) & \text{if } \frac{1}{n-1} \leq \frac{r_1}{r_2} < \frac{2}{n} \\ nv\left(\frac{r_1}{2r_2}\right) & \text{if } \frac{2}{n} \leq \frac{r_1}{r_2} \leq 1 \\ nv\left(1 - \frac{r_2}{2r_1}\right) & \text{if } 1 \leq \frac{r_1}{r_2} \leq \frac{n}{2} \\ nv\left(1 - \frac{2}{n} + \frac{2r_1}{n^2 r_2^2}\right) & \text{if } \frac{n}{2} < \frac{r_1}{r_2} < n-1 \end{cases},$$

$$P^2(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)) = nv - P^1(\boldsymbol{SCB}(\{1, r_1\}, \{2, r_2\}, n, v)).$$

If $r_1 = 0$*, then* $P^1(\boldsymbol{SCB}(\{1, 0\}, \{2, r_2\}, n, v)) = 0$*.*

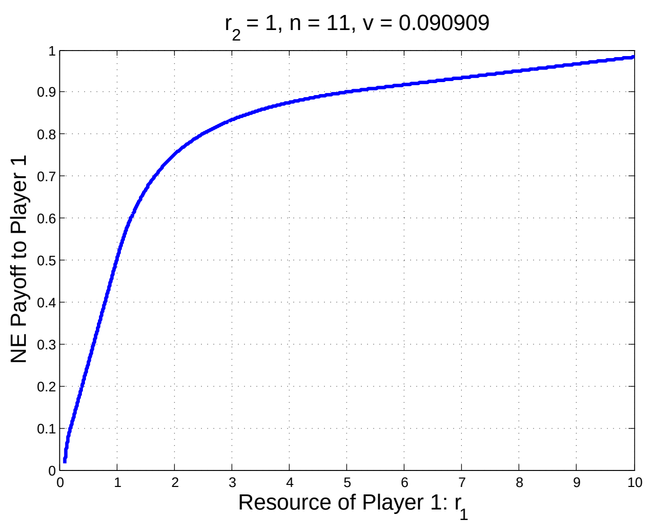

Fig. 1. For a fixed resource $r_2 = 1$ of Agent 2, the payoff to Agent 1 is a concave function of its endowment of resources r_1. Here, $n = 11$ and $v = \frac{1}{n}$.

Remark 1. Note that for fixed r_2, n and v, $r_1 \mapsto P^1(\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$ is a concave monotonically increasing function in the parameter region $\frac{1}{n-1} \leq \frac{r_1}{r_2} \leq n-1$. This is also illustrated in Figure 1 for a specific set of parameters. Furthermore, $r_1 \mapsto P^1(\mathbf{SCB}(\{1, r_1\}, \{2, r_2\}, n, v))$ is a non-decreasing function on $\mathbb{R}_+$ (note that here we do not restrict the range of r_1). This is a consequence of the result in [14]. □

This completes our revisit of the results for two-player static Colonel Blotto game from [14].

4 SPNE of the Game

We now consider the three-stage Colonel Blotto game formulated in the Section 2. To ease exposition, let us write $t := t_{1,2} - t_{2,1}$, which denotes the net transfer from Player 1 to Player 2. This can be negative if Player 2 transfers more resources than Player 1. We further define $r_1 := r_1(t) = \beta_1 - t$ and $r_2 := r_2(t) = \beta_2 + t$ to denote, respectively, the resource levels of Player 1 and Player 2 after the transfer is complete.

As stated previously, we are interested in computing the subgame-perfect Nash equilibrium for the three-stage game. At the final stage of the game, all players know the resource levels of all players and the resource allocation of the adversary for the battles against the other two players. All players also know the updated number of battlefields over which the battle is to be fought. Thus, the game at the final stage comprises two instances of the static Colonel Blotto game recalled in the previous section. This insight results in the following lemma.

Lemma 2. *At the final stage, Player $i \in \{1, 2\}$ and the adversary will play a static Colonel Blotto game $\boldsymbol{SCB}(\{1, r_i\}, \{A, \alpha_i\}, n_i + m_i, v_i)$. Thus, the SPNE strategy pair of Player i and the adversary at the third (last) stage is $(\mu_i^\star, \nu_i^\star) \in$ $\mathit{NE}(\boldsymbol{SCB}(\{1, r_i\}, \{A, \alpha_i\}, n_i + m_i, v_i))$.*

In the light of the lemma above, to compute the SPNE of the game, we need to compute (i) at the second stage, the allocation functions of the adversary $\{\alpha_1^\star, \ldots, \alpha_N^\star\}$, the transfer functions $\{t_{1,2}^\star\}$ and $\{t_{2,1}^\star\}$ of the first two players, and (ii) at the first stage, the battlefields added by the first two players $m_1^\star$ and $m_2^\star$.

As noted in the previous section, the expected payoff functions of the players in the static Colonel Blotto game are computed in four different parameter regions. Thus, for the game at hand, we have a total of 64 different cases to consider. To ease the exposition, we consider here only four of these cases. These cases comprise games in which, when players act according to Nash equilibrium at the first stage (so that m_1, m_2 are fixed and common knowledge), the ratio of the adversary's allocation of resource for the battle with Player i and Player i's resources after the transfer is complete lie in the interval $(\frac{2}{n_i+m_i}, \frac{n_i+m_i}{2})$. This simplification leads us to the following four cases:

1. $2/n_1 < \alpha_1/r_1 < 1$ and $2/n2 < \alpha_2/r_2 < 1$
2. $2/n_1 < \alpha_1/r_1 < 1$ and $2/n_2 < r_2/\alpha_2 < 1$
3. $2/n_1 < r_1/\alpha_1 < 1$ and $2/n_2 < r_2/\alpha_2 < 1$
4. $2/n_1 < r_1/\alpha_1 < 1$ and $2/n_2 < \alpha_2/r_2 < 1$

It should be noted that the second and third cases are essentially the same with the only indices of the first two players interchanged. Thus, we only focus on three cases, Cases 1, 2 and 4, with the understanding that the result for Case 3 can be obtained from the result of Case 2.

In the next subsection, we compute the reaction curves of the players (also called best response strategies) for the game at the second stage. Thereafter, we compute the SPNE of the game in the sequel for all the three cases.

4.1 Reaction Functions of the Players

In this subsection, we compute the best response strategies of the players in the game.

Preliminary Notations: We use the following notations to describe the allocation strategy of the adversary for various cases:

$$a_1(m_1,m_2,t) := \frac{\alpha}{1+\sqrt{\frac{(n_2+m_2)v_2(\beta_2+t)}{(n_1+m_1)v_1(\beta_1-t)}}},$$

$$\lambda_1(m_1,m_2,t) := \sqrt{\frac{(n_2+m_2)v_2(\beta_1-t)(\beta_2+t)}{(n_1+m_1)v_1}},$$

$$d(m_1,m_2,t) := \begin{cases} \alpha & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} > \frac{(n_2+m_2)v_2}{\beta_2+t} \\ 0 & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} < \frac{(n_2+m_2)v_2}{\beta_2+t} \\ \alpha \; w.p. \; p\in(0,1) & \text{if } \frac{(n_1+m_1)v_1}{\beta_1-t} = \frac{(n_2+m_2)v_2}{\beta_2+t} \end{cases}.$$

Note that in the definition of $d(m_1,m_2,t)$, the value of probability p can take any value in the interval $(0,1)$. The next lemma computes the reaction curves of the players at the second stage of the game.

Lemma 3. *Consider a game* $\mathbf{CB}(\underline{n},\underline{\beta},\alpha,\underline{v},c)$*. For a* $t\in[-\beta_2,\beta_1]$*, let* $r_1=\beta_1-t$ *and* $r_2=\beta_2+t$*. Fix* $m_1,m_2\in\mathbb{Z}_+$*. The reaction curves of the players at the second stage are given by the following expressions in various cases:*

1. *If* $\frac{2}{n_1+m_1} < \frac{\alpha}{\beta_1-t} < 1$ *and* $\frac{2}{n_2} < \frac{\alpha}{\beta_2+t} < 1$*, then*
$$\alpha_1^*(m_1,m_2,t) = d(m_1,m_2,t).$$
2. *If* $\frac{2}{n_i+m_i} < \frac{r_i}{a_i(m_1,m_2,t)} < 1, \; i=1,2,$ *then*
$$\alpha_1^*(m_1,m_2,t) = a_1(m_1,m_2,t).$$
3. *If* $\frac{2}{n_1+m_1} < \frac{\alpha-\lambda_1(m_1,m_2,t)}{\beta_1-t} < 1$ *and* $\frac{2}{n_2+m_2} < \frac{\beta_2+t}{\lambda_1(m_1,m_2,t)} < 1$*, then*
$$\alpha_1^*(m_1,m_2,t) = \alpha-\lambda_1(m_1,m_2,t).$$

In all cases, if α_1 *is a constant (or dependent only on* m_1,m_2*), then*

$$t^*_{1,2}(m_1,m_2,\alpha_1) = \begin{cases} 0 & \text{if } \alpha_1 > 0, \\ t_{1,2}\in[0,\beta_1] & \text{if } \alpha_1 = 0. \end{cases}$$

$$t^*_{2,1}(m_1,m_2,\alpha_1) = \begin{cases} 0 & \text{if } \alpha_1 < \alpha, \\ t_{2,1}\in[0,\beta_2] & \text{if } \alpha_1 = \alpha. \end{cases}$$

Proof: The proof is available in [9, Lemma 4, p. 13], but we recall it here for the convenience of the reader.

Since Player i and the adversary are going to play a static Colonel Blotto game $\mathbf{SCB}(\{i, r_i\}, \{A, \alpha_i\}, n_i + m_i, v_i)$ at the final stage of the game, the expected payoff functions to the players are given by the result of Lemma 1 (that are dependent on the ratio r_i/α_i).

The reaction function for the adversary is the best response strategy of the adversary given the strategy of the other two players. Towards this end, fix $m_{1:2}$ and t and define $e_i := (n_i + m_i)v_i$ for $i = 1, 2$. The expected payoff function to the adversary as a function of the adversary's allocation α_1 to the battle with Player 1 for the three cases, respectively, are

$$\begin{aligned}
\text{Case 1: } \pi_A(\alpha_1) &= \frac{e_1\alpha_1}{2(\beta_1 - t)} + \frac{e_2(\alpha - \alpha_1)}{2(\beta_2 + t)}, \\
\text{Case 2: } \pi_A(\alpha_1) &= e_1\left(1 - \frac{(\beta_1 - t)}{2\alpha_1}\right) \\
&\quad + e_2\left(1 - \frac{(\beta_2 + t)}{2(\alpha - \alpha_1)}\right), \\
\text{Case 3: } \pi_A(\alpha_1) &= \frac{e_1\alpha_1}{2(\beta_1 - t)} + e_2\left(1 - \frac{(\beta_2 + t)}{2(\alpha - \alpha_1)}\right).
\end{aligned}$$

In Cases 2 and 3, the payoff to the adversary π_A is a concave function of α_1, since the second derivative of π_A with respect to α_1 is strictly negative. One can set the first derivative of π_A to zero to get the optimal value of α_1 as a function of m_1, m_2, and t. The fact that $d(m_1, m_2, t)$ maximizes the payoff π_A in Case 1 can be verified easily. This completes the proof of the lemma. ■

Having now computed the reaction functions of the players at the second stage of the game, we now compute the SPNE strategies of the players below.

4.2 The Case of Weakest Adversary

We now turn our attention to computing SPNE of the game for Case 1, in which the adversary has the least amount of resources among all players.

Preliminary Notation for Theorem 2 Let $\bar{m}_1 = \arg\max_{m_1 \in \mathbb{Z}_+} m_1 v_1 - c m_1^2$ and $\bar{m}_2 = \arg\max_{m_2 \in \mathbb{Z}_+} m_2 v_2 - c m_2^2$. Define

$$\begin{aligned}
\bar{t}_{1,2}(m_1, m_2) &= \frac{(n_2 + m_2)v_2\beta_1 - (n_1 + m_1)v_1\beta_2}{(n_1 + m_1)v_1 + (n_2 + m_2)v_2}, \\
\bar{t}_{2,1}(m_1, m_2) &= \frac{(n_1 + m_1)v_1\beta_2 - (n_2 + m_2)v_2\beta_1}{(n_1 + m_1)v_1 + (n_2 + m_2)v_2}, \\
\zeta_1 = \bar{t}_{2,1}(0, \bar{m}_2) &\qquad \zeta_2 = \bar{t}_{1,2}(\bar{m}_1, 0).
\end{aligned}$$

Theorem 2. *Consider a game* $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$ *with* $\alpha < \min\{\beta_1, \beta_2\}$ *and* $\frac{2}{n_i} < \frac{\alpha}{\beta_i}$ *for both* $i \in \{1, 2\}$*. If the parameters of the game satisfy either*

$$\begin{aligned} & \frac{(n_1 + \bar{m}_1)v_1}{\beta_1} < \frac{n_2 v_2}{\beta_2}, \quad \left(1 - \frac{\alpha}{2(\beta_2 + \zeta_2)}\right) v_2 < c, \\ & \frac{2}{n_1 + \bar{m}_1} < \frac{\alpha}{\beta_1 - \zeta_2} < 1, \quad \frac{2}{n_2} < \frac{\alpha}{\beta_2 + \zeta_2} < 1, \\ \textit{or} \quad & \frac{n_1 v_1}{\beta_1} > \frac{(n_2 + \bar{m}_2)v_2}{\beta_2}, \quad \left(1 - \frac{\alpha}{2(\beta_1 + \zeta_1)}\right) v_1 < c, \\ & \frac{2}{n_2 + \bar{m}_2} < \frac{\alpha}{\beta_2 - \zeta_1} < 1, \quad \frac{2}{n_1} < \frac{\alpha}{\beta_1 + \zeta_1} < 1, \end{aligned}$$

then there is a family of SPNEs for this game, given by

$$\begin{aligned} \alpha_1^\star(m_1, m_2) &= d(m_1, m_2, 0), \\ t_{1,2}^\star(m_1, m_2) &= \begin{cases} t \in [0, \bar{t}_{1,2}(m_1, m_2)) & \textit{if } \frac{(n_1+m_1)v_1}{\beta_1} < \frac{(n_2+m_2)v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases} \\ t_{2,1}^\star(m_1, m_2) &= \begin{cases} t \in [0, \bar{t}_{2,1}(m_1, m_2)) & \textit{if } \frac{(n_1+m_1)v_1}{\beta_1} > \frac{(n_2+m_2)v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases} \end{aligned}$$

$$m_1^\star = \begin{cases} \bar{m}_1 & \textit{if } \frac{(n_1+\bar{m}_1)v_1}{\beta_1} < \frac{n_2 v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases}, \qquad m_2^\star = \begin{cases} \bar{m}_2 & \textit{if } \frac{n_1 v_1}{\beta_1} > \frac{(n_2+\bar{m}_2)v_2}{\beta_2} \\ 0 & \textit{otherwise} \end{cases}.$$

Proof: The reaction curves of the players are given as in Lemma 3. It is easy to see that for given m_1 and m_2, the (family of) Nash equilibria stated above are the best response strategies of each other. Now, maximizing the cost functional of Players 1 and 2 over m_1 and m_2 given $\alpha_1^\star, t_{1,2}^\star$ and $t_{2,1}^\star$, we get the result. The sufficient conditions on the parameters ensure that Players 1 and 2 and the adversary's allocation have appropriate ratios if all players act according to the SPNE. ■

Remark 2. Along the equilibrium path, one player has an incentive to add battlefields and transfer some (or none) of its resource to the other player. □

Remark 3. In the theorem above, if $v_1 < c$, then $\bar{m}_1 = 0$. Similarly, if $v_2 < c$, then $\bar{m}_2 = 0$. □

4.3 Other Cases

We now consider other scenarios, where the adversary may have comparable or large endowment of resources as compared to any other player in the game.

Preliminary Notation for Theorem 3

$$s_i := \sqrt{v_i\beta_i}\left(\sqrt{n_j v_j \beta_j}\right), \quad i,j \in \{1,2\}, i \neq j,$$
$$c_1 := v_1\left(1-\frac{\alpha}{2\beta_1}\right) + \left(\sqrt{n_1+1}-\sqrt{n_1}\right)\frac{\sqrt{n_2 v_2 \beta_2 v_1}}{2\sqrt{\beta_1}}$$
$$c_2 := \left(\sqrt{n_2+1}-\sqrt{n_2}\right)\frac{\sqrt{v_2\beta_2 n_1 v_1}}{2\sqrt{\beta_1}}.$$

Theorem 3. *Consider a game* $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c) \in$. *The SPNE of the game is given as:*

1. *If* $\frac{2}{n_i+m_i} < \frac{\beta_i}{a_i(m_1,m_2,0)} < 1, \ i = 1, 2$ *and*
$$c > \frac{1}{2\alpha} \max_{i \in \{1,2\}} \left(v_i\beta_i + \left(\sqrt{n_i+1}-\sqrt{n_i}\right)s_i\right),$$
then $\alpha_1^\star(m_1, m_2) = a_1(m_1, m_2, 0)$.
2. *If* $\alpha > \lambda_1(m_1, m_2, 0)$, $\frac{2}{n_1+m_1} < \frac{\alpha - \lambda_1(m_1,m_2,0)}{\beta_1} < 1$, $\frac{2}{n_2+m_2} < \frac{\beta_2}{\lambda_1(m_1,m_2,0)} < 1$, *and* $c > \max\{c_1, c_2\}$, *then*
$$\alpha_1^\star(m_1, m_2) = \alpha - \lambda_1(m_1, m_2, 0).$$

In both cases, $t_{1,2}^\star(m_1, m_2) = t_{2,1}^\star(m_1, m_2) = 0$ *and* $m_1^\star = m_2^\star = 0$.

Proof: Given the best response strategies of the players in Lemma 3, one can just check that the given strategies indeed form a SPNE of the game. Furthermore, the sufficient conditions on c merely ensures that adding any battlefield gives a lower payoff to the first two players. ■

Remark 4. In the statement of both cases in Theorem 3 above, the sufficient conditions on c are not hard constraints. If the value of c is small, then adding battlefields may be beneficial to one or both players. The Nash equilibrium at the second stage of the game remains unchanged (as long as the restrictions on the parameters are met). □

4.4 Discussions on Equilibrium Strategies

In Theorem 2, we see that the amount of resource one player transfers to another could take any value in a set. This is due to the fact that the adversary does not attack the player who makes the transfer (hence, his payoff is not affected by making the transfer) if everyone plays according to the equilibrium.

Figure 2 shows that for a specific set of parameters, a transfer takes place from one player to another in certain regions of β_1 and β_2. It is interesting to note that there is a transfer from Player 1 to Player 2 even when the resource level of Player 1 is significantly small as compared to the resource level of Player 2.

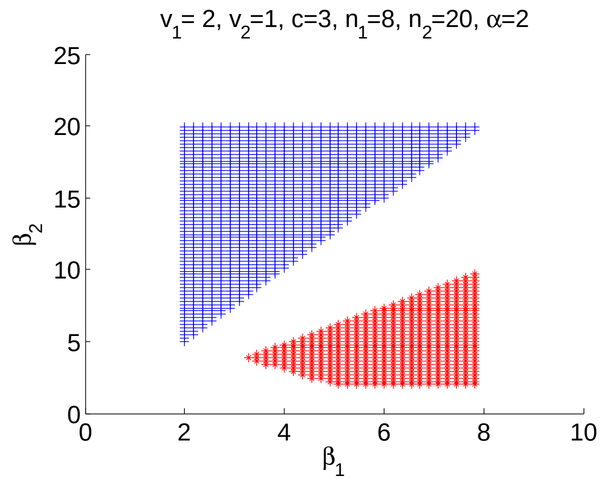

Fig. 2. For fixed parameters $v_1 = 2, v_2 = 1$, $c = 3$, $n_1 = 8, n_2 = 20$, and $\alpha = 2$, Player 1 transfers to Player 2 in the red region, whereas Player 2 transfers to Player 1 in the blue region. There is no addition of battlefield by any player (see also Remark 3) in the colored region. In the white region, transfer may or may not occur. See Theorem 2 for a complete characterization. This figure is taken from [8].

On the other hand, in Theorem 3, where the adversary has comparable or more resources than other players, the SPNE is unique, and there is no transfer among the first two players. There are two reasons why we see no transfer among the players as SPNE strategies in both cases. The first reason is because the adversary divides its resources into two positive parts, and allocates each of the two parts to the battle with one of the other two players. Since both players are fighting against the adversary, the best response strategies of the first two players are to not transfer their resources to the other player (see Lemma 3). The second reason, which is more subtle, is that the adversary does not observe the value of the transfer among the other two players (or in other words, all players act simultaneously in the second stage). If we allow the adversary to access information on the amount of resource transferred between the players, then the SPNE may feature a transfer *even if the adversary allocates positive resources to fight both players*. A few such cases are investigated in [12] and our earlier work [8]. However, [12] does not compute the Nash equilibrium strategies (or SPNE) of the players under such a setting.

In all cases, if the cost for adding battlefields is sufficiently high, then the first two players do not add any battlefield.

We now outline the differences in the behavior of the players in this game as compared to the one studied in [8]. For the case when the adversary is weakest (that is, have the least amount of resources among all players), the players act according to the same behavior as proved in [8]. This is because the adversary deploys all its resource to fight against only one player in this case. So, whether or not a transfer occurs, the behavior of the adversary remains unchanged. Thus, giving the adversary access to the information about the transfer does not result in any change in its behavior.

To compare the results in this paper with that in [8] for other cases, when the adversary has comparable or more resources than other players, we will recall the result for [8] for those cases. However, to ease exposition, we introduce the following definition.

Definition 1. *Consider the three-stage Colonel Blotto game formulated in Section 2. We say that the information structure of the three-stage game is* **N** *if at the second stage, the adversary does not observe the transfer between the first two players. We say, on the other hand, that the information structure of the three-stage game is* **T***, if at the end of the second stage, the adversary has access to the transfer between the players.* □

We now reproduce the result from [8] below for the case when adversary has comparable or more resources as compared to other players.

Preliminary Notation for Theorem 4

$$\begin{aligned}
\bar{t}_1(m_1, m_2) &:= \frac{(\beta_1 - \beta_2)}{2} - \frac{(\beta_1 + \beta_2)}{2}\sqrt{\frac{(n_1+m_1)v_1}{(n_1+m_1)v_1 + (n_2+m_2)v_2}}, \\
w_1(m_1, m_2) &:= (n_1+m_1)v_1 + \sqrt{(n_1+m_1)v_1((n_1+m_1)v_1 + (n_2+m_2)v_2)}, \\
\bar{m}_1 &:= \arg\max_{m_1 \in \mathbb{Z}_+} m_1 v_1 \left(1 - \frac{\alpha}{2(\beta_1+\beta_2)}\right) - c m_1^2, \\
\zeta_1(m_1, m_2) &:= \frac{4(n_1+m_1)v_1\alpha^2}{(n_2+m_2)v_2(\beta_1+\beta_2)^2}.
\end{aligned}$$

Theorem 4 **([8]).** *Consider a game* $\mathbf{CB}(\underline{n}, \underline{\beta}, \alpha, \underline{v}, c)$ *with information structure* **T** *in which the adversary has access to the information about the transfer of resources among the first two players at the second stage of the game. The SPNE of the game is given as:*

1. *Assume* $c > \frac{\beta_1+\beta_2}{4\alpha}\max\{w_1(1,0) - w_1(0,0), v_2\}$ *and let* $\bar{t}_1 := \bar{t}_1(m_1, m_2)$*. If* $\frac{2}{n_i+m_i} < \frac{r_i(t)}{a_i(m_1,m_2,t)} < 1$, $i = 1, 2$*, then*

$$\begin{aligned}
\alpha_1^\star(m_1, m_2, t) &= a_1(m_1, m_2, t), \\
t_{1,2}^\star(m_1, m_2) &= \begin{cases} \bar{t}_1 \ \textit{if} \ \frac{\beta_1-\beta_2}{2\beta_1\beta_2} > \sqrt{\frac{(n_1+m_1)v_1}{(n_2+m_2)v_2}} \\ 0 \ \ \textit{otherwise} \end{cases} \\
t_{2,1}^\star(m_1, m_2) &= 0, \qquad m_1^\star = m_2^\star = 0.
\end{aligned}$$

2. *If* $c > \frac{(\beta_1+\beta_2)v_2}{4\alpha}$, $\frac{2}{n_1+m_1} < \frac{\alpha-\lambda_1(m_1,m_2,t)}{(\beta_1-t)} < 1$, *and* $\frac{2}{n_2+m_2} < \frac{(\beta_2+t)}{\lambda_1(m_1,m_2,t)} < 1$, *then*

$$\alpha_1^\star(m_1,m_2,t) = \alpha - \lambda_1(m_1,m_2,t),$$

$$t_{1,2}^\star(m_1,m_2) = \begin{cases} \frac{\beta_1-\zeta_1(m_1,m_2)\beta_2}{\zeta_1(m_1,m_2)+1} & \text{if } \frac{\beta_1+\beta_2}{2\alpha} > \sqrt{\frac{(n_1+m_1)v_1\beta_2}{(n_2+m_2)v_2\beta_1}} \\ 0 & \text{otherwise.} \end{cases}$$

$$t_{2,1}^\star(m_1,m_2) = 0, \qquad m_1^\star = \bar{m}_1, \qquad m_2^\star = 0.$$

■

An interesting distinction in the behavior of the players between the games with two different information structure is as follows: In the game with information structure **N**, the players do not transfer resources among themselves. In contrast, the game with information structure **T** features a transfer. The reason for this behavior is the following. With information structure **T**, the adversary, after observing the transfer, allocates more resource to fight against Player 2 as compared to what it allocates in the game with information structure **N**. Thus, in game with information structure **T**, the transfer makes *both* Players 1 and 2 better off[6], while the adversary loses in terms of the expected payoff[7][8].

Remark 5. The analysis above exposes a very counterintuitive feature of games. One may be led into thinking that the extra information about the transfer to the adversary should make him better off, but this, clearly, is not the case in the game with information structure **T**. *In games, extra information to a player does not necessarily result in a better performance for that player!* □

5 Conclusion

We formulated a three-stage three-player Colonel Blotto (non-cooperative) game in which the first two players fight against a common adversary. The first two players could add battlefields at some cost and they can form a coalition and transfer resources among each other if it improves their expected payoffs. We computed subgame-perfect Nash equilibria of the game. We found that if the adversary is weakest, that is, has the least endowment of resources, then it

[6] Note here that since this is a non-cooperative game, if the transfer does not improve the expected payoffs to both Players 1 and 2, then either the receiving player will not accept the transfer, or the donating player will not initiate a transfer. The fact that a positive transfer is a Nash equilibrium implies that the transfer *increases or maintains* the expected payoffs to both players.

[7] We assume that the parameters of the game are such that the sufficient conditions on parameters are satisfied, enabling us to make this comparison.

[8] Since the Colonel Blotto game is a constant-sum game, the sum of total expected payoffs for all the players (including the adversary) is a constant. Thus, if Players 1 and 2 increase their expected payoffs, then it decreases the expected payoff to the adversary.

attacks only one of the two players (when playing under Nash equilibrium). The player who does not suffer an attack can transfer some of its resources to the other player. If the adversary has comparable or more resources than the other players, then there is no transfer of resources among the first two players when playing under Nash equilibrium. In all cases, additional battlefields are created by the first two players if the cost for adding them is sufficiently low.

The result gives a qualitative picture of how players should behave in order to secure cyber-physical systems. In case the cyber-physical systems under attack have significantly more resources (computational or physical) as compared to the attacker, then it is in their best interest to share their resources to secure themselves. On the other hand, if the adversary is as mighty as the systems, then it is in the best interests for the systems to use all their resources to secure themselves.

Furthermore, we see that adding battlefields could result in a better payoff. Consider, for example, a data center which acts to reduce the threat of data compromise. If adding additional servers for storing data is cheap, then it is in its best interest to keep small amount of data in different servers. In doing so, even if certain number of data servers are compromised, the amount of compromised data will be less.

In the future, we would like to extend the analysis to a general N-player game. Incomplete information static Colonel Blotto game is also an important problem that requires further investigation, in which the existence of a Nash equilibrium has not been established yet.

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