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Event–Selected Vector Field Discontinuities Yield Piecewise–Differentiable Flows*

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Abstract. We study a class of discontinuous vector fields brought to our attention by multilegged animal locomotion. Such vector fields arise not only in biomechanics, but also in robotics, neuroscience, and electrical engineering, to name a few domains of application. Under the conditions that (i) the vector field's discontinuities are locally confined to a finite number of smooth submanifolds and (ii) the vector field is transverse to these surfaces in an appropriate sense, it is known that the vector field yields a well–defined flow that is Lipschitz continuous. We extend these results by showing this flow is piecewise–differentiable, so that it admits a first–order approximation (known as a Bouligand derivative) that is piecewise–linear and continuous at every point. We exploit this first–order approximation to infer existence of piecewise–differentiable impact maps (including Poincaré maps for periodic orbits), show that the flow is locally conjugate (via a piecewise–differentiable homeomorphism) to a flowbox, and assess the effect of perturbations (both infinitesimal and non–infinitesimal) on the flow. We use these results to give a sufficient condition for the exponential stability of a periodic orbit passing through a point of multiply intersecting events and apply the theory in illustrative examples to demonstrate synchronization in first– and second–order phase oscillator models abstracted from the legged locomotion application domain that motivated our interest in this class of models.

Key words. animal locomotion, biomechanics, differential equation with discontinuous right–hand side, hybrid dynamical system, neural biology, nonsmooth dynamical system, robotics, saltation matrix

AMS subject classifications. 34A36, 34A37, 34A38, 34A12, 34C15, 92C10, 92C20

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1. Introduction.

Parsimonious dynamical models for diverse physical phenomena are governed by vector fields that are smooth except along a finite number of surfaces of discontinuity. Examples include integrate–and–fire neurons that undergo a discontinuous change in

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membrane voltage during a threshold crossing [9,29,32]; electrical power systems that undergo discontinuous changes in network topology triggered by excessive voltages or currents [26]; legged locomotors that encounter discontinuities in velocities or forces due to intermittent interaction of viscoelastic limbs with terrain [4, 20, 28]. In each of these examples, behaviors of interest—*phase locking* [32] or *local synchronization* [29]; *voltage collapse* phenomena [17] [26, section II-A.2]; simultaneous touchdown of two or more legs [4, 20, 28]—occur at or near the intersection of multiple surfaces of discontinuity. Although analytical tools exist to study orbits that pass transversally through nonintersecting switching surfaces (e.g., to assess stability [3, 21], compute first–order variations [27, 52], and reduce dimensionality [13]), hybrid systems that admit simultaneous discrete transitions generally exhibit "branching;" i.e., the hybrid flow depends discontinuously[1] on initial conditions [47, Definition 3.11]. For instance, in the mechanical setting, the flow of a Lagrangian dynamical system subject to unilateral constraints is generically discontinuous near simultaneous–impact events [7, section 7]. In the case where a vector field is discontinuous across two transversally intersecting surfaces, others have established continuity of the flow and computed first–order approximations [9, 15, 16, 30]. Techniques applicable to arbitrary numbers of surfaces have been derived for the case of pure phase oscillators with perpendicular transition surfaces [39].

By restricting our attention to a subclass of systems that do not exhibit *sliding modes* [51] or *branching* [47, Definition 3.11], we extend previous work in this area in two ways. First, we accommodate an arbitrary number of nonlinear transition surfaces that are not required to be transverse, and we show that the flow admits approximations of order higher than one. Second, and more significantly, we show that the Lipschitz continuous flows and impact maps are *piecewise–differentiable* and leverage this fact to extend a suite of analytical and computational techniques from classical (smooth) dynamical systems theory to the present (nonsmooth) setting. The definition of piecewise–differentiability we employ (introduced in [42, 43, 45]) implies that although the flow is not classically differentiable, nevertheless it admits a first–order approximation (the so–called *Bouligand derivative* or *B–derivative* [45, Chapter 3]) that is piecewise–linear and continuous at every point. Establishing the existence of this first–order approximation in the framework of [45] is powerful; we exploit it to infer existence of piecewise–differentiable impact maps (including Poincaré maps for periodic orbits); demonstrate local conjugacy (via a piecewise–differentiable homeomorphism) between the flow and a *flowbox*; assess the effect of perturbations on the flow; and derive a straightforward procedure to compute the B–derivative. We use these results to give a sufficient condition for the exponential stability of a periodic orbit passing through a point of multiply intersecting event surfaces and apply the theory in illustrative examples to demonstrate synchronization in abstracted first– and second–order phase oscillator models.

1.1. Relation to prior work.
Existence, uniqueness, and Lipschitz continuity of the flow associated with the class of discontinuous vector fields considered in this paper have been established previously; see, for instance, [18, Chapter 2, section 8, corollary to Theorem 3]. Furthermore, previous work has established that this flow admits a first–order approximation that may be computed by introducing discontinuous updates to the classical variational equa-

[1] We note that hybrid state spaces do not possess a natural metric, and continuity of the flow depends on the chosen metric; this issue is discussed in detail elsewhere [12, sec. V-A].

tion via so–called *saltation matrices*; see section 7.1 for an overview and [3, 30] for the original constructions involving 1 and 2 surfaces of discontinuity, respectively. Our contribution in Theorem 4 (local flow) is a proof that this flow enjoys a richer piecewise–differentiable structure than had been previously established. Specifically, we prove that the nonsmooth flow is *piecewise–*C^r (PC^r) in a sense defined by the nonsmooth analysis community; see section 3.2 for an overview and [2] for the original definition. This result enables us to refine and extend several results familiar to the nonsmooth dynamics community, and to obtain a new conjugacy result that, to the best of our knowledge, has not appeared previously.

1.2. Virtues of PC^r flow. Since the flow associated with the class of discontinuous vector fields considered in this paper is Lipschitz continuous, the flow is automatically "piecewise–differentiable" in a certain sense: Rademacher's theorem [48, Theorem 5.2] implies that the flow is differentiable almost everywhere; i.e., the subset of the domain where the flow fails to be classically differentiable has (Lebesgue) measure zero. The existence of a derivative almost everywhere has limited value since, for an arbitrary Lipschitz continuous function, this derivative need not satisfy basic calculus operations like the chain rule. In Theorem 4 (local flow), we prove that the flow is in fact *piecewise–*C^r (PC^r) [45, section 4.1] and hence admits a first–order approximation (the B–derivative) that satisfies the chain rule [45, Theorem 3.1.1] and other operations familiar from calculus. To the best of our knowledge, this paper represents the first application of PC^r calculus to discontinuous dynamical systems. This benefits the field by introducing new analytical and computational tools that can be employed, for instance, to assess stability (as in section 7.2), optimality, or controllability [11, section 4.6]. We discuss the practical implications of introducing these new tools from nonsmooth analysis in section 9.

1.3. Organization. We begin in section 2 by motivating and illustrating the results developed in this paper using models abstracted from the legged locomotion application domain. Following a brief review of relevant technical background in section 3, we define the discontinuous but piecewise–smooth vector fields of interest and show that they yield continuous B–differentiable flows in section 4. In section 5 we demonstrate that such flows are locally conjugate (via piecewise–differentiable homeomorphisms) to a classical flowbox, leading to results in section 6 establishing the persistence of such flows under small perturbations. Section 7 develops stability results, and their application to simple oscillator models is given in section 8. The paper concludes with a brief summary in section 9 suggesting the relevance of these results to biological and engineered systems of practical interest.

2. Relevance for legged locomotion.

Consider the notional model for legged locomotion illustrated in Figure 1 near a configuration wherein two limbs impact terrain at or near the same instant in time. Although simultaneous–impact configurations occupy a measure–zero subset of state space, they are nevertheless frequently encountered in biomechanics and robotics, since legged animals and robots with four, six, and more limbs exhibit gaits with near–simultaneous touchdown of two or more legs [4, 20, 28]. Away from footfall events, the equations–of–motion governing the model's motion are smooth; at footfall events, the equations–of–motion are generally discontinuous. Thus we are led in this application domain to consider nonsmooth dynamical systems whose discontinuities are confined to smooth submanifolds of state space, and to focus our attention on trajectories that pass through points where two or more of these surfaces intersect.

We consider concrete models for the dynamics of legged locomotion near the simultaneous–touchdown configuration illustrated in Figure 1 at two levels of abstraction. First we focus on the effect of impulses arising from plastic impact between limbs and terrain, arriving in section 2.1 at a first–order[2] model. Subsequently we neglect impulses and focus on the effect of viscoelasticity in the limbs, arriving in section 2.2 at a second–order model. The first–order model in section 2.1 is sufficiently simple that we are able to derive a closed–form expression for the first–order approximation of its flow in section 2.3. In contrast, the second–order model in section 2.2 is sufficiently complex to preclude an analogous closed–form treatment, motivating us to develop more general analytical and computational tools in sections 4–7. These new techniques enable us in section 8 to assess stability in phase oscillator models abstracted from the locomotion models described below.

2.1. First–order locomotion model. We begin by imposing an instantaneous plastic collision law when either of the limbs illustrated in Figure 1 impact the ground plane. Before impact, for simplicity, we assume the two limbs remain at their rest length, and hence the vertical position z_j of each limb $j \in \{1, 2\}$ is constrained relative to the vertical position z and rotation θ of the body, and we neglect the horizontal motion; this reduces the number of mechanical degrees of freedom (DOF) from five to two. After impact with the ground, each limb $j \in \{1, 2\}$ is constrained at height $z_j = 0$, whence there remain two DOF. We choose to analyze the system's dynamics in two generalized coordinates that simplify subsequent calculations:

$$q_1 = z - \sin\theta + z_1, \quad q_2 = z + \sin\theta + z_2. \tag{1}$$

For purposes of illustration, we restrict our attention to preimpact initial conditions wherein the rotational velocity is zero and the vertical velocity is fixed at a particular $\dot{z} < 0$; this reduces the dimension of the unspecified initial conditions to two (namely, θ and z), enabling faithful representation in the illustration in Figure 1. When limb height z_j reaches zero with negative velocity $\dot{z}_j = \dot{z} < 0$, plastic impact instantaneously resets the limb velocity $\dot{z}_j$ to zero. Thus the first–order system dynamics are given by

$$\dot{q}_j = \begin{cases} \dot{z} + \dot{z}_j, & z_j > 0, \\ \dot{z}, & z_j = 0. \end{cases} \tag{2}$$

Note that the dynamics in (2) are discontinuous across the perpendicular event surfaces $H_j = \{z_j = 0\}$, $j \in \{1, 2\}$, and the trajectory of interest indicated in Figure 1 passes transversely through the intersection $H_1 \cap H_2$. We will analyze this model's behavior exhaustively in section 2.3.

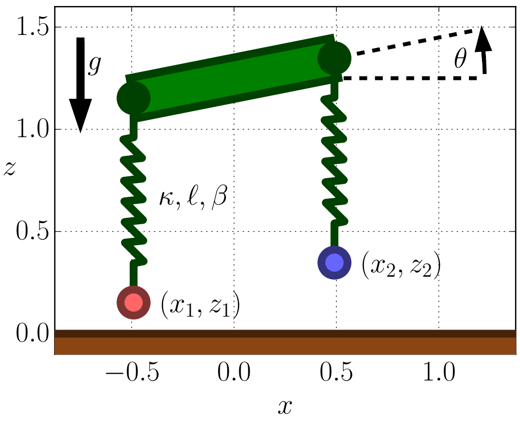
2.2. Second–order locomotion model. We now neglect impulsive effects and focus on the effect of discontinuous forces in the locomotion model illustrated in Figure 1. When a limb $j \in \{1, 2\}$ is in contact with the ground, viscoelastic forces developed in the limb are

[2]Keeping with convention [5, section 15], we refer to a model of a mechanical system as *first–order* if the model's state contains only positions, and *second–order* if the model's state contains both positions and velocities. This sense of "first–/second–order" should not be confused with the sense intended when we discuss existence of first– and higher–order approximations of piecewise–differentiable functions in what follows.

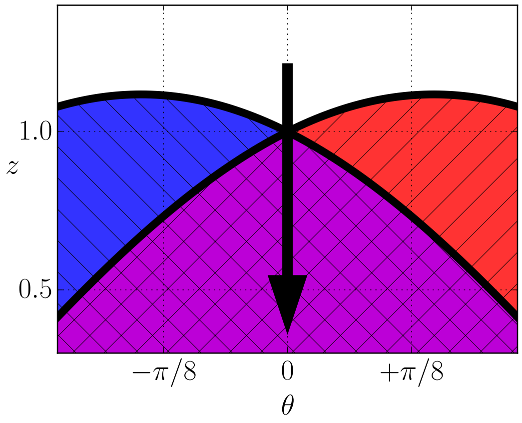
applied to the body at the limb attachment point. The second–order equations–of–motion for the intrinsic generalized coordinates in (1) are[3]

$$\ddot{q}_j = \begin{cases} -2g, & z_j > 0, \\ -g + 2\kappa(\ell_j - \ell) - 2\beta\dot{\ell}_j, & z_j = 0, \end{cases} \tag{3}$$

where $\ell_j = z - (-1)^j \sin\theta - z_j$ denotes the length of limb $j \in \{1,2\}$ and $\dot{\ell}_j = \dot{z} - (-1)^j \dot{\theta}\cos\theta - \dot{z}_j = \dot{z} < 0$ denotes the instantaneous rate of extension/contraction of limb $j \in \{1,2\}$ after impact (recall that $\dot{\theta}$ was initialized to zero for purposes of illustration, and $\dot{z}_j = 0$ after impact). Note that the dynamics in (3) are discontinuous across the perpendicular event surfaces $H_j = \{z_j = 0\}$, $j \in \{1,2\}$, and the trajectory of interest passes transversely through the intersection $H_1 \cap H_2$.



(a) Saggital–plane model for legged locomotion.



(b) Trajectory undergoing simultaneous footfalls.

Figure 1. *Notional model for saggital–plane quadrupedal locomotion (for clarity, only two legs are illustrated).* (a) *The model is comprised of a massive body with three DOF (horizontal position x, height z, and pitch rotation angle θ) and two limbs joining body hip joints to point–mass feet each with two DOF (horizontal position x_j and height z_j, $j \in \{1,2\}$).* (b) *A trajectory of interest (representing, e.g., a trot gait for the quadrupedal model; illustrated by a downward–pointing arrow) passes transversely through the intersection of surfaces corresponding to configurations where the feet impact the ground. The model's dynamics are generally discontinuous across these surfaces, as the velocity and forces change discontinuously during footfalls.*

2.3. First–order approximation of a nonsmooth flow.

To motivate and illustrate the results contained in this paper, we begin by studying the first–order locomotion dynamics from section 2.1, whose flow can be written in closed form. Letting $x = (q_1, q_2)$ and noting that $z - z_j = \ell$ before impact for $j \in \{1,2\}$, the time derivative of x is given by a discontinuous vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ defined by

$$\forall x \in \mathbb{R}^2 : F(x) = \dot{z}\eta(x), \tag{4}$$

[3] For simplicity and clarity of the exposition, all masses and moments of inertia are taken to be unity.

where $\dot{z} < 0$ is a constant and $\eta : \mathbb{R}^2 \to \{1, 2\}$ is defined by

$$\forall x \in \mathbb{R}^2,\ j \in \{1,2\} : \eta_j(x) = \begin{cases} 2, & x_j > \ell, \\ 1, & x_j \le \ell. \end{cases} \tag{5}$$

The vector field and some of its integral curves are illustrated in Figure 2a.

Standard results (see, for instance, [18, Chapter 2, section 8, corollary to Theorem 3]) imply that there exists a Lipschitz continuous global flow $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^2$ associated with this vector field; i.e., for every $x \in \mathbb{R}^2$ the restriction $\phi|_{\mathbb{R}\times\{x\}} : \mathbb{R} \to \mathbb{R}^2$ is absolutely continuous and

$$\phi(t, x) = x + \int_0^t F(\phi(s, x))ds. \tag{6}$$

Let $\chi_0, \chi_0^{-1} : \mathbb{R} \to \mathbb{R}$ be the piecewise–linear homeomorphisms defined by

$$\forall s \in \mathbb{R} : \chi_0(s) = \begin{cases} s, & s > 0, \\ 2s, & s \le 0, \end{cases} \quad \forall \widetilde{s} \in \mathbb{R} : \chi_0^{-1}(\widetilde{s}) = \begin{cases} \widetilde{s}, & \widetilde{s} > 0, \\ \frac{1}{2}\widetilde{s}, & \widetilde{s} \le 0, \end{cases} \tag{7}$$

and let $\chi, \chi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ be the piecewise–linear homeomorphisms defined by

$$\forall x \in \mathbb{R}^2 : \chi(x) = (\chi_0(x_1), \chi_0(x_2)),\ \forall \widetilde{x} \in \mathbb{R}^2 : \chi^{-1}(\widetilde{x}) = \left(\chi_0^{-1}(\widetilde{x}_1), \chi_0^{-1}(\widetilde{x}_2)\right). \tag{8}$$

Note that $\chi_0 \circ \chi_0^{-1} = \mathrm{id}_{\mathbb{R}}$ and hence $\chi \circ \chi^{-1} = \mathrm{id}_{\mathbb{R}^2}$. Away from points where F is discontinuous (and hence χ is nonsmooth), there is no ambiguity in the definition of the "pushforward" $\widetilde{F} := D\chi \circ F \circ \chi^{-1} : \mathbb{R}^2 \to T\mathbb{R}^2$. In fact, the vector field $\widetilde{F}$ is constant,

$$\forall \widetilde{x} \in \mathbb{R}^2 : \widetilde{F}(\widetilde{x}) = -2 \cdot \mathbb{1}, \tag{9}$$

where $\mathbb{1} = [1, 1]^T \in \mathbb{R}^2$ is the vector of ones, and hence the flow $\widetilde{\phi} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ associated with $\widetilde{F}$ has the simple form

$$\forall (t, \widetilde{x}) \in \mathbb{R} \times \mathbb{R}^2 : \widetilde{\phi}(t, \widetilde{x}) = \widetilde{x} - 2t\mathbb{1}. \tag{10}$$

Since the homeomorphism χ provides conjugacy between the flows, we have

$$\forall (t, x) \in \mathcal{F} : \chi \circ \phi(t, x) = \widetilde{\phi}(t, \chi(x)) = \chi(x) - 2t\mathbb{1}; \tag{11}$$

this relationship is illustrated in Figure 2. If $t \in \mathbb{R}$ and $x, w \in \mathbb{R}^2$ are such that $x, x + w, x - w$ lie in the first quadrant and $\phi(t, x), \phi(t, x + w), \phi(t, x - w)$ lie in the third quadrant as in Figure 2, the conjugacy in (11) can be used to compute a first–order approximation of the flow ϕ, since

$$\begin{aligned} \phi(t, x + sw) &= \chi^{-1}\left(\chi(x + sw) - 2t\mathbb{1}\right) \\ &= \frac{1}{2}\left((x + sw) - 2t\mathbb{1}\right) \\ &= \frac{1}{2}(x + sw) - t\mathbb{1}, \end{aligned} \tag{12}$$

and hence

$$\lim_{s\to 0^+} \frac{1}{s}\left(\phi(t, x + sw) - \phi(t, x)\right) = \frac{1}{2}w. \tag{13}$$

Remarkably, the first–order approximation of the flow turns out to be linear, so the nonsmooth flow ϕ is C^1 with respect to state along the trajectory initialized at x (note that the time derivative of ϕ is F, which is discontinuous along this trajectory). Furthermore, the first–order effect of the flow is a contraction with rate $1/2$. These happy accidents will not arise in general—if F were perturbed in the second or fourth quadrant of Figure 2a, its flow would no longer be classically differentiable—so we seek a general technique for deriving first– and higher–order approximations of the flow; the remainder of the paper is devoted to developing analytical and computational tools suited to this aim.

3. Preliminaries.

The mathematical constructions we use are "standard" in the sense that they are familiar to practitioners of (applied) dynamical systems or optimization theory (or both). Since this paper represents (to the best of our knowledge) the first application of some techniques from nonsmooth analysis to the class of discontinuous dynamical systems introduced in section 2, we briefly review mathematical concepts and introduce notation that will be used to state and prove results throughout this paper and suggest textbook references where the interested reader could obtain a complete exposition.

3.1. Notation.

To simplify the statement of our definitions and results, we fix notation of some objects in $\mathbb{R}^n$: $+\mathbb{1} \in \mathbb{R}^n$ denotes the vector of all ones and $-\mathbb{1}$ its negative; e_j is the jth standard Euclidean basis vector; and $B_n = \{-1, +1\}^n \subset \mathbb{R}^n$ is the set of corners of the n-dimensional cube. We let $\operatorname{sign} : \mathbb{R}^n \to \{-1, +1\}^n$ be the vectorized signum function taking its values in the Euclidean cube's corners; i.e.,

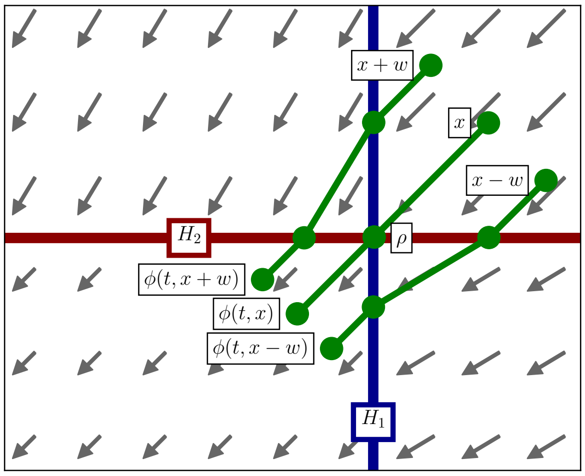
$$\forall x \in \mathbb{R}^n, j \in \{1, \ldots, d\} : e_j^\top \operatorname{sign}(x) = \begin{cases} -1, & x_j < 0, \\ +1, & x_j \geq 0. \end{cases} \tag{14}$$

To fix notation, in the following paragraphs we will briefly recapitulate standard constructions from topology, differential topology, and dynamical systems theory, and we refer the reader to [35] for details. If $U \subset X$ is a subset of a topological space, then $\operatorname{Int} U \subset X$ denotes its *interior* and ∂U denotes its *boundary*. Let $f : X \to Y$ be a map between topological spaces. If $U \subset X$, then $f|_U : U \to Y$ denotes the *restriction*. If $V \subset Y$, then $f^{-1}(V) = \{x \in X : f(x) \in V\}$ denotes the *preimage of* V *under* f.

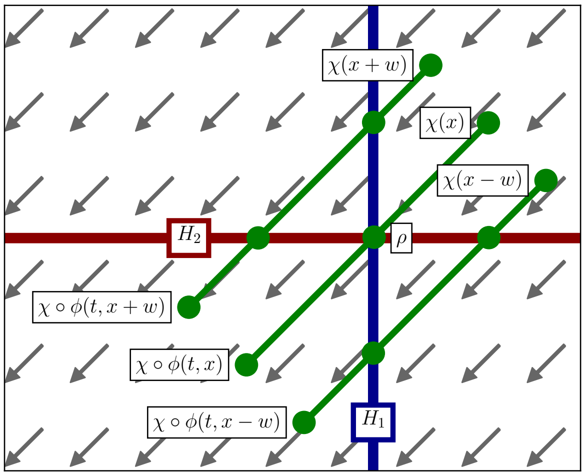
Given C^r manifolds D, N, we let $C^r(D, N)$ denote the set of C^r functions from D to N. $H \subset D$ is a C^r *codimension–*k *submanifold* of the d–dimensional manifold D if every $x \in H$ has a neighborhood $U \subset D$ over which there exists a C^r diffeomorphism $h : U \to \mathbb{R}^d$ such that

$$H \cap U = h^{-1}\left(\left\{y \in \mathbb{R}^d : y_{k+1} = \cdots = y_d = 0\right\}\right). \tag{15}$$

If $f \in C^r(D, N)$, then at every $x \in D$ there exists an induced linear map $Df(x) : T_xD \to T_{f(x)}N$ called the *pushforward* (in coordinates, $Df(x)$ is the Jacobian linearization of f at $x \in D$) where T_xD denotes the tangent space to the manifold D at the point $x \in D$. Globally, the pushforward is a C^{r-1} map $Df : TD \to TN$ where TD is the tangent bundle associated



(a) $\dot{x} = F(x)$



(b) $\dot{\widetilde{x}} = D\chi \circ F \circ \chi^{-1}(\widetilde{x})$

Figure 2. (a) *Piecewise–constant planar vector field* $F : \mathbb{R}^2 \to T\mathbb{R}^2$ *from* (4). (b) *Pushforward of* F *via the piecewise–linear ("flowbox") homeomorphism* $\chi : D \to D$ *from* (8).

with the manifold D; we recall that TD is naturally a $2d$–dimensional C^r manifold. When $N = \mathbb{R}$, we will invoke the standard identification $T_yN \simeq \mathbb{R}$ for all $y \in N$ and regard $Df(x)$ as a linear map from T_xD (i.e., an element of the *cotangent space* T_x^*D) into $\mathbb{R}$ for every $x \in D$; we recall that the *cotangent bundle* T^*D is naturally a $2d$–dimensional C^r manifold. If $U \subset D$ and $f : U \to N$ is a map, then a map $\widetilde{f} : D \to N$ is a C^r *extension of* f if $\widetilde{f}$ is C^r and $\widetilde{f}|_U = f$.

Following [35, Chapter 8], a (possibly discontinuous or nondifferentiable) map $F : D \to TD$ is a *(rough)*[4] *vector field* if $\pi \circ F = \mathrm{id}_D$, where $\pi : TD \to D$ is the natural projection and id_D is

[4]We will constrain the class of vector fields under consideration in section 4.1, but for expediency drop the *rough* modifier in what follows.

the identity map on D. A vector field may, under appropriate conditions, yield an associated *flow* $\phi : \mathcal{F} \to D$ defined over an open subset $\mathcal{F} \subset \mathbb{R} \times D$ called a *flow domain*; in this case for every $x \in D$ the set $\mathcal{F}^x = \mathcal{F} \cap (\mathbb{R} \times \{x\})$ is an open interval containing the origin, the restriction $\phi|_{\mathcal{F}^x} : \mathcal{F}^x \to D$ is absolutely continuous, and the derivative with respect to time is $D_t\phi(t, x) = F(\phi(t, x))$ for almost every $t \in \mathcal{F}^x$. A flow is *maximal* if it cannot be extended to a larger flow domain. An *integral curve* for F is an absolutely continuous function $\xi : I \to D$ over an open interval $I \subset \mathbb{R}$ such that $\dot{\xi}(t) = F(\xi(t))$ for almost all $t \in I$; it is *maximal* if it cannot be extended to an integral curve on a larger open interval.

3.2. Piecewise–differentiable functions and nonsmooth analysis.

The notion of piecewise–differentiability we employ was originally introduced by Robinson [42]; since the recent monograph from Scholtes [45] provides a more comprehensive exposition, we adopt the notational conventions therein. Let $r \in \mathbb{N} \cup \{\infty\}$ and $D \subset \mathbb{R}^d$ be open. A continuous function $f : D \to \mathbb{R}^n$ is called *piecewise–*C^r if for every $x \in D$ there exists an open set $U \subset D$ containing x and a finite collection $\{f_j : U \to \mathbb{R}^n\}_{j\in\mathcal{J}}$ of C^r–functions such that for all $x \in U$ we have $f(x) \in \{f_j(x)\}_{j\in\mathcal{J}}$. The functions $\{f_j\}_{j\in\mathcal{J}}$ are called *selection functions* for $f|_U$, and f is said to be a *continuous selection* of $\{f_j\}_{j\in\mathcal{J}}$. A selection function f_j is said to be *active* at $x \in U$ if $f(x) = f_j(x)$. We let $PC^r(D, \mathbb{R}^n)$ denote the set of piecewise–C^r functions from D to $\mathbb{R}^n$. Note that PC^r is closed under composition and pointwise maximum or minimum of a finite collection of functions. Any $f \in PC^r(D, \mathbb{R}^n)$ is locally Lipschitz continuous, and a Lipschitz constant for f is given by the supremum of the induced norms of the (Fréchet) derivatives of the set of selection functions for f. The definition of piecewise–C^r may at first appear unrelated to the intuition that a function ought to be piecewise–differentiable precisely if its "domain can be partitioned locally into a finite number of regions relative to which smoothness holds" [43, section 1]. However, as shown in [43, Theorem 2], piecewise–C^r functions are always piecewise–differentiable in this intuitive sense.

Piecewise–differentiable functions possess a first–order approximation $Df : TD \to T\mathbb{R}^n$ called the *Bouligand derivative* (or B–derivative) [45, Chapter 3]; this is the content of Lemma 4.1.3 in [45]. Significantly, and unlike the directional derivative, this B–derivative supports generalization of many techniques familiar from calculus, including the chain rule [45, Theorem 3.1.1] (and hence product and quotient rules [45, Corollary 3.1.1]), a fundamental theorem of calculus [45, Proposition 3.1.1], and an implicit function theorem [41, Corollary 20]. We let $Df(x; v)$ denote the B–derivative of f evaluated along the tangent vector $v \in T_xD$. The B–derivative is positively homogeneous; i.e., $\forall v \in T_xD, \lambda \geq 0 : Df(x; \lambda v) = \lambda Df(x; v)$.

4. Local and global flow.

In this section we rederive in our present nonsmooth setting the erstwhile familiar fundamental construction associated with a vector field: its flow. We begin in section 4.1 by introducing the class of vector fields under consideration, namely, event–selected C^r vector fields. Subsequently, in section 4.2 we construct a candidate flow function via composition of piecewise–differentiable functions. Finally, in section 4.3 we show that this candidate function is indeed the flow of the event–selected C^r vector field.

4.1. Event–selected vector fields discontinuities.

The flow of a discontinuous vector field $F : D \to TD$ over an open domain $D \subset \mathbb{R}^d$ can exhibit pathological behaviors ranging from nondeterminism to discontinuous dependence on initial conditions. We will investigate local properties of the flow when the discontinuities are confined to a finite collection of smooth submanifolds through which the flow passes transversally, as formalized in the following definitions.

Definition 1. *Given a vector field $F : D \to TD$ over an open domain $D \subset \mathbb{R}^d$ and a function $h \in C^r(U, \mathbb{R})$ defined on an open subset $U \subset D$, we say that h* is an event function for F on U *if there exists a positive constant $f > 0$ such that $Dh(x)F(x) \geq f$ for all $x \in U$. A codimension–1 embedded submanifold $\Sigma \subset U$ for which $h|_\Sigma$ is constant is referred to as a* local section for F.

Note that if h is an event function for F on a set containing $\rho \in D$, then necessarily $Dh(\rho) \neq 0$.

We will show in section 4.3 that vector fields that are differentiable everywhere except a finite collection of local sections give rise to a well–defined flow that is piecewise–differentiable. This class of *event–selected* vector fields is defined formally as follows.

Definition 2. *Given a vector field $F : D \to TD$ over an open domain $D \subset \mathbb{R}^d$, we say that F* is event–selected C^r at $\rho \in D$ *if there exists an open set $U \subset D$ containing ρ and a collection $\{h_j\}_{j=1}^n \subset C^r(U, \mathbb{R})$ such that*

1. *(event functions) h_j is an event function for F on U for all $j \in \{1, \ldots, n\}$;*
2. *(C^r extension) for all $b \in \{-1, +1\}^n = B_n$, with*

$$D_b = \{x \in U : b_j(h_j(x) - h_j(\rho)) \geq 0\},$$

$F|_{\operatorname{Int} D_b}$ admits a C^r extension $F_b : U \to TU$.

(Note that for any $b \in B_n$ such that $\operatorname{Int} D_b = \emptyset$ the latter condition is satisfied vacuously.) We let $EC^r(D)$ denote the set of vector fields that are event–selected C^r at every $x \in D$.

Before proceeding, we digress briefly to comment on the preceding definitions. The first condition in Definition 2, analogous to [15, equation (7.69)], is imposed to preclude *sliding modes* [51] and the *branching* [12] phenomenon illustrated in SM5; in particular, it ensures that trajectories progress monotonically[5] with respect to each event surface. In the second condition, all that is required for each $b \in B_n$ is that $F|_{\operatorname{Int} D_b}$ admits a C^r extension; note that the flow derived below is unaffected by the extension chosen. Note as well that the event surfaces are not required to be transverse; if two event surfaces locally coincided, this redundancy would manifest by yielding some regions $b \in B_n$ for which $\operatorname{Int} D_b = \emptyset$. For illustrations of event–selected C^r vector fields in the plane $D = \mathbb{R}^2$, refer to Figures 3 and 4.

4.2. Construction of the piecewise–differentiable flow. The following constructions will be used to state and prove results throughout the chapter. Suppose $F : D \to TD$ is event–selected C^r at $\rho \in D$. By definition there exists a neighborhood $\rho \in U \subset D$ and associated event functions $\{h_j\}_{j=1}^n \subset C^r(U, \mathbb{R})$ that divide U into regions $\{D_b\}_{b \in B_n}$ defined by $D_b = \{x \in U : (h_j(x) - h_j(\rho))b_j \geq 0\}$. The boundary of each D_b is contained in the collection of event surfaces $\{H_j\}_{j=1}^n$ defined for each $j \in \{1, \ldots, n\}$ by $H_j = \{x \in U : h_j(x) = h_j(\rho)\}$. For each $j \in \{1, \ldots, n\}$ and $b \in B_n$, we refer to the surface H_j as an *exit boundary in positive*

[5] Note that if $h \in C^r(U, \mathbb{R})$ is such that $Dh(x)F(x) \leq -f$ for all $x \in U$, then $-h$ is an *event function* as in Definition 1, so monotonic *positive* progress was stipulated without loss of generality.

time for D_b if $h_j(D_b) \subset (-\infty, 0]$; we refer to H_j as an *exit boundary in negative time* if $h_j(D_b) \subset [0, +\infty)$. In addition, the definition of event–selected C^r implies that there is a collection of C^r vector fields $\{F_b : U \to TU\}_{b\in B_n} \subset C^r(U, TU)$ such that $F|_{\operatorname{Int} D_b} = F_b|_{\operatorname{Int} D_b}$ for all $b \in B_n$.

4.2.1. Budgeted time–to–boundary. For each $b \in B_n$ with $\operatorname{Int} D_b \neq \emptyset$, let $\phi_b : \mathcal{F}_b \to U$ be a flow for F_b over a flow domain $\mathcal{F}_b \subset \mathbb{R} \times U$ containing $(0, \rho)$; recall that $\phi_b \in C^r(\mathcal{F}_b, U)$ since $F_b \in C^r(U, TU)$. Each $H \in \{H_j\}_{j=1}^n$ is a local section for F, and therefore a local section for F_b as well. This implies $F_b(\rho)$ is transverse to H (more precisely, $F_b(\rho) \notin T_\rho H$); thus the implicit function theorem [35, Theorem C.40] implies that there exists a C^r "time–to–impact" map $\tau_b^H : U_b^H \to \mathbb{R}$ defined on an open set $U_b^H \subset D$ containing ρ such that

$$\forall x \in U_b^H : (\tau_b^H(x), x) \in \mathcal{F}_b \text{ and } \phi_b(\tau_b^H(x), x) \in H. \tag{16}$$

The collection of maps $\{\tau_b^H\}_{b\in B_n}$ is jointly defined over the open set $U_b = \bigcap_{j=1}^n U_b^{H_j}$; note that U_b is nonempty since $\rho \in U_b$. Any $x \in U_b$ can be taken to any $H \in \{H_j\}_{j=1}^n$ by flowing with the vector field F_b for time $\tau_b^H(x) \in \mathbb{R}$. A useful fact we will recall in what follows is that if $y = \phi_b(\tau_b(x), x)$, then

$$D\tau_b^H(x) = \frac{-Dh(y) D_x\phi_b(t, x)}{Dh(y) F_b(y)}; \tag{17}$$

this follows from [25, section 11.2].

We now define functions $\tau_b^+, \tau_b^- : \mathbb{R} \times U_b \to \mathbb{R}$ that specify the time required to flow to the exit boundary of D_b in forward or backward time, respectively, without exceeding a given time budget:

$$\begin{aligned} &\forall (t, x) \in \mathbb{R} \times U_b : \tau_b^+(t, x) = \max\left\{0, \min\left(\{t\} \cup \left\{\tau_b^{H_j}(x) : b_j < 0\right\}_{j=1}^n\right)\right\}, \\ &\forall (t, x) \in \mathbb{R} \times U_b : \tau_b^-(t, x) = \min\left\{0, \max\left(\{t\} \cup \left\{\tau_b^{H_j}(x) : b_j > 0\right\}_{j=1}^n\right)\right\}. \end{aligned} \tag{18}$$

Since τ_b^+, τ_b^- are obtained via pointwise minimum and maximum of a finite collection of C^r functions, we conclude that $\tau_b^+, \tau_b^- \in PC^r(\mathbb{R} \times U_b, \mathbb{R})$. See Figure 3 for an illustration of the component functions of τ_b^+ in a planar vector field.

In what follows we will require the derivative of τ_b^+ with respect to t and x. In general this can be obtained via the chain rule [45, Theorem 3.1.1]. If we define $\nu_b^+ : U_b \to \mathbb{R} \cup \{+\infty\}$ using the convention $\min \emptyset = +\infty$ by

$$\forall x \in U_b : \nu_b^+(x) = \min\left\{\tau_b^{H_j}(x) : b_j < 0\right\}_{j=1}^n, \tag{19}$$

then we immediately conclude that for all $(t, x) \in \mathbb{R} \times U_b$ such that $\nu_b^+(x) \neq t \neq 0$, the forward–time budgeted time–to–boundary τ_b^+ is classically differentiable and

$$D\tau_b^+(t, x) = \begin{cases} \left[0,\ 0_d^\top\right], & t < 0, \\ \left[1,\ 0_d^\top\right], & 0 < t < \nu_b^+(x), \\ \left[0, D\tau_b^H(x)\right], & \nu_b^+(x) < t, \end{cases} \tag{20}$$

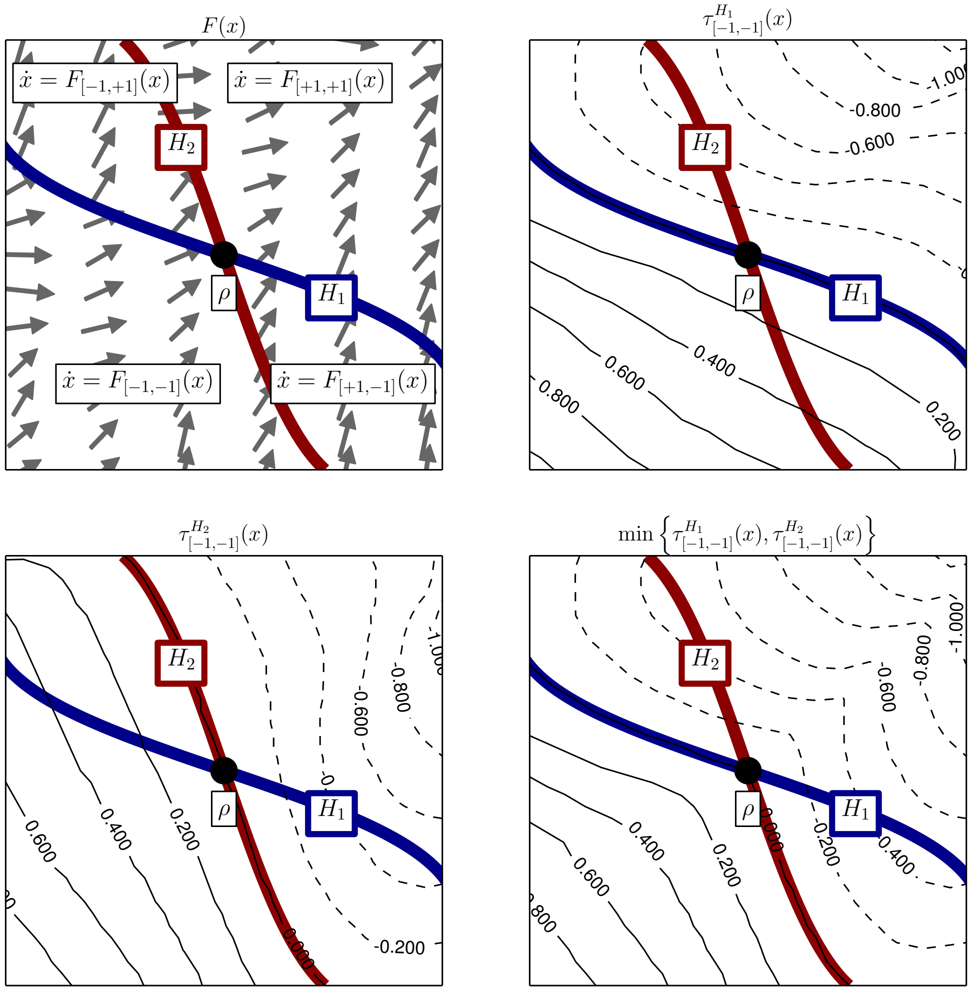


Figure 3. *Illustration of a vector field $F : D \to TD$ that is event–selected C^r at $\rho \in D = \mathbb{R}^2$. The functions $\{\tau^{H_j}_{[-1,-1]}\}_{j=1}^{2}$ specify the time required to flow via the vector field $F_{[-1,-1]}$ to the surface H_j. The pointwise minimum $\min\{\tau^{H_j}_{[-1,-1]}(x)\}_{j=1}^{2}$ is used in the definition of $\tau^{+}_{[-1,-1]}$ in (18).*

where in the third case $H \in \{H_j\}_{j=1}^{n}$ is such that $\tau_b^H(x) = \nu_b^+(x)$. To compute $D\tau_b^-(t,x)$, one may simply use the formula in (20) applied to the vector field $-F$; full details are provided in SM1.1.

4.2.2. Budgeted flow–to–boundary. By composing the flow ϕ_b with the budgeted time–to–boundary functions τ_b^+, τ_b^-, we now construct functions that flow points up to the exit boundary of D_b in forward or backward time over domains

$$\begin{aligned} \mathcal{V}_b^+ &= \left\{(t,x) \in \mathbb{R} \times U_b : (\tau_b^+(t,x), x) \in \mathcal{F}_b\right\}, \\ \mathcal{V}_b^- &= \left\{(t,x) \in \mathbb{R} \times U_b : (\tau_b^-(t,x), x) \in \mathcal{F}_b\right\}. \end{aligned} \tag{21}$$

(Note that $\mathcal{V}_b^+, \mathcal{V}_b^-$ are open since τ_b^+, τ_b^- are continuous and nonempty since $(0,\rho) \in \mathcal{V}_b^+, \mathcal{V}_b^-$.) For each $b \in B_n$ define the functions $\zeta_b^+ : \mathcal{V}^+ \to D, \zeta_b^- : \mathcal{V}^- \to D$ by

$$\begin{aligned} &\forall (t,x) \in \mathcal{V}_b^+ : \zeta_b^+(t,x) = \phi_b\left(\tau_b^+(t,x), x\right), \\ &\forall (t,x) \in \mathcal{V}_b^- : \zeta_b^-(t,x) = \phi_b\left(\tau_b^-(t,x), x\right). \end{aligned} \tag{22}$$

Clearly $\zeta_b^+ \in PC^r(\mathcal{V}_b^+, D)$ and $\zeta_b^- \in PC^r(\mathcal{V}_b^-, D)$ since they are obtained by composing PC^r functions [45, section 4.1]. Loosely speaking, the function ζ_b^+ coincides with ϕ_b for pairs (t,x) that do not cross the forward–time exit boundary of D_b. Yet unlike ϕ_b, it is the identity (stationary) flow over the remainder of its domain. More precisely, for $t < 0$ and for values of $t > \nu_b^+(x)$ the function $\tau_b^+(t,x)$ is constant (and hence the derivative with respect to time $D_t\zeta_b^+(t,x) = 0$), while for $t \in (0, \nu_b^+(x))$ we have $\zeta_b^+(t,x) = \phi_b(t,x)$ (and hence $D_t\zeta_b^+(t,x) = F_b(\phi_b(t,x))$).

Now fix $x \in D_b$, choose $a \in B_n \setminus b$, and for $t \in \mathbb{R}$ define

$$t_a^+(t) = \min\left\{\tau_a^{H_j}\left(\zeta_b^+(t,x)\right) : a_j < 0\right\}_{j=1}^n. \tag{23}$$

Applying the conclusions from the preceding paragraph, with $t' \in \mathbb{R}$ the composition

$$\zeta_a^+(t', \zeta_b^+(t,x)) \tag{24}$$

is classically differentiable with respect to both t' and t almost everywhere. Furthermore, we can deduce that the derivative of the composition with respect to t is $F_b(\phi_b(t,x))$ when $t \in \left(0, \nu_b^+(x)\right)$ and zero where it is otherwise defined; similarly, the derivative with respect to t' is $F_a\left(\phi_a(t', \zeta_b^+(t,x))\right)$ when $t' \in (0, t_a^+(t))$ and zero where it is otherwise defined. If we impose the relationship $t' = t - \tau_b^+(t,x)$, we have $t' = 0$ for any $t \in (0, \nu_b^+(x))$. The composition

$$\zeta_a^+(t - \tau_b^+(t,x), \zeta_b^+(t,x)) \tag{25}$$

follows the flow for F_b from x toward (but never passing) the exit boundary of D_b and then follows the flow of F_a from $\zeta_b^+(t,x)$ toward the exit boundary of D_a.

In what follows we will require the derivative of ζ_b^+ with respect to t and x. In general, this can be obtained via the chain rule [45, Theorem 3.1.1]. If we define $\nu_b^+ : U_b \to \mathbb{R}$ as in (19), then we immediately conclude that for all $(t,x) \in \mathbb{R} \times U_b$ such that $\nu_b^+(x) \neq t \neq 0$, the forward–time flow–to–boundary ζ_b^+ is classically differentiable and

$$D\zeta_b^+(t,x) = \begin{cases} [0_d,\ 0_{d\times d}], & t < 0, \\ [F_b(\phi_b(t,x)),\ D_x\phi_b(t,x)], & 0 < t < \nu_b^+(x), \\ [0_d, \Upsilon(t,x)], & \nu_b^+(x) < t, \end{cases} \tag{26}$$

where in the third case $\Upsilon(t,x) = F_b(\phi_b(\tau_b^+(t,x),x))D\tau_b^H(x) + D_x\phi_b(\tau_b^+(t,x),x)$ and $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^+(x)$. To compute $D\zeta_b^-(t,x)$, one may simply use the formula in (26) applied to the vector field $-F$; full details are provided in SM1.2.

4.2.3. Composite of budgeted time–to– and flow–to–boundary. Define $\varphi_b^+ : \mathcal{V}_b^+ \to \mathbb{R} \times D$, $\varphi_b^- : \mathcal{V}_b^- \to \mathbb{R} \times D$ by

$$\begin{aligned} &\forall (t,x) \in \mathcal{V}_b^+ : \varphi_b^+(t,x) = \left(t - \tau_b^+(t,x), \zeta_b^+(t,x)\right) = \left(t - \tau_b^+(t,x), \phi_b\left(\tau_b^+(t,x), x\right)\right), \\ &\forall (t,x) \in \mathcal{V}_b^- : \varphi_b^-(t,x) = \left(t - \tau_b^-(t,x), \zeta_b^-(t,x)\right) = \left(t - \tau_b^-(t,x), \phi_b\left(\tau_b^-(t,x), x\right)\right). \end{aligned} \tag{27}$$

Clearly $\varphi_b^+ \in PC^r(\mathcal{V}_b^+, \mathbb{R}\times D)$ and $\varphi_b^- \in PC^r(\mathcal{V}_b^-, \mathbb{R}\times D)$. Intuitively, the second component of the φ_b^+, φ_b^- functions flow according to F_b up to exit boundaries of D_b in forward or backward time, respectively, while the first component deducts the flow time $t - \tau_b^\pm(t,x)$ from the total time budget t. These functions satisfy an invariance property:

$$\begin{aligned} &\forall (t,x) \in \left(\mathcal{V}_b^+ \cap (-\infty, 0] \times U_b\right) : \varphi_b^+(t,x) = (t,x), \\ &\forall (t,x) \in \left(\mathcal{V}_b^- \cap [0, +\infty) \times U_b\right) : \varphi_b^-(t,x) = (t,x). \end{aligned} \tag{28}$$

We now combine (20) and (26) to obtain the derivative of φ_b^+ for all $(t,x) \in \mathbb{R} \times U_b$ such that $\nu_b^+(x) \neq t \neq 0$:

$$D\varphi_b^+(t,x) = \begin{cases} \begin{bmatrix} 1 & 0_d^\top \\ 0_d & I_d \end{bmatrix}, & t < 0, \\ \begin{bmatrix} 0 & 0_d^\top \\ F_b(\phi_b(t,x)) & D_x\phi_b(t,x) \end{bmatrix}, & 0 < t < \nu_b^+(x), \\ \begin{bmatrix} 1 & -D\tau_b^H(x) \\ 0_d & \Upsilon(t,x) \end{bmatrix}, & \nu_b^+(x) < t, \end{cases} \tag{29}$$

where in the third case $\Upsilon(t,x) = F_b(\phi_b(\tau_b^+(t,x),x))D\tau_b^H(x) + D_x\phi_b(\tau_b^+(t,x),x)$ and $H \in \{H_j\}_{j=1}^n$ is such that $\tau_b^H(x) = \nu_b^+(x)$. To compute $D\varphi_b^-(t,x)$, one may simply use the formula in (26) applied to the vector field $-F$; full details are provided in SM1.3.

4.2.4. Construction of flow via composition. Consider now the formal composition

$$\phi = \pi_2 \circ \left(\prod_{b=-\mathbb{1}}^{+\mathbb{1}} \varphi_b^+\right) \circ \left(\prod_{b=+\mathbb{1}}^{-\mathbb{1}} \varphi_b^-\right), \tag{30}$$

where $\pi_2 : \mathbb{R} \times D \to D$ is the canonical projection and $\prod_{b=-\mathbb{1}}^{+\mathbb{1}}$ denotes composition in lexicographic order (similarly $\prod_{b=+\mathbb{1}}^{-\mathbb{1}}$ denotes composition in reverse lexicographic order). The set $\phi^{-1}(D) \subset \mathbb{R} \times D$ is open (since ϕ is continuous) and nonempty (since combining (28) and (30) implies $\phi(0,\rho) = \rho$). Therefore there exist open neighborhoods $J \subset \mathbb{R}$ of 0 and $V \subset D$ of ρ such that $\mathcal{F} = J \times V \subset \phi^{-1}(D)$. Clearly $\phi \in PC^r(\mathcal{F}, D)$ since it is obtained by composing PC^r functions. Its derivative can be computed by applying the chain rule [45, Theorem 3.1.1]; alternatively, it can be obtained for almost all $(t,x) \in \mathcal{F}$ as a product of the appropriate matrices given in (29) (SM8). The derivative with respect to time has a particularly simple form almost everywhere, as we demonstrate in the following lemma.

Lemma 3 (time derivative of flow). *If the vector field $F : D \to TD$ is event–selected C^r at $\rho \in D$, then for almost all $(t,x) \in \mathcal{F}$ the flow $\phi \in PC^r(\mathcal{F}, D)$ defined by (30) is differentiable with respect to time and*

$$D_t\phi(t,x) = F(\phi(t,x)). \tag{31}$$

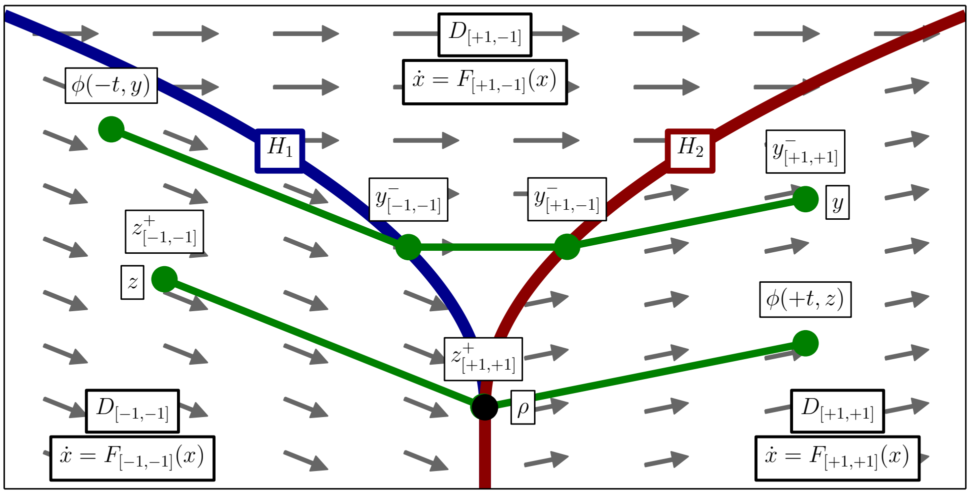


Figure 4. *Illustration of a vector field $F : D \to TD$ that is event–selected C^r near $\rho \in D = \mathbb{R}^2$. The vector field is discontinuous across the C^r codimension–1 submanifolds $H_1, H_2 \subset D$. For each $b \in B_n = \{[-1,-1], [+1,-1], [-1,+1], [+1,+1]\}$, if $\operatorname{Int} D_b \neq \emptyset$, then the vector field restricts as $F|_{\operatorname{Int} D_b} = F_b|_{\operatorname{Int} D_b}$ where $F_b : U_b \to TU_b$ is a smooth vector field over a neighborhood $\rho \in U_b \subset D$. An initial condition $z \in D_{[-1,-1]}$ flows in forward time to $\phi(+t, y) \in D_{[+1,+1]}$ through $z^+_{[+1,+1]} \in H_1 \cap H_2$. An initial condition $y \in D_{[+1,+1]}$ flows in backward time to $\phi(-t, y) \in D_{[-1,-1]}$ through $y^-_{[-1,+1]} \in H_1$ and $y^-_{[-1,-1]} \in H_2$.*

Proof. Choose $x \in D$ such that $(0, x) \in \mathcal{F}$. We will show that $\phi|_{\mathcal{F}^x}$ is classically differentiable for almost all times $t \in \mathcal{F}^x$. Let $t^- = \inf \mathcal{F}^x$, $t^+ = \sup \mathcal{F}^x$ so that $0 \in \mathcal{F}^x = (t^-, t^+)$. We construct a partition of $[0, t^+)$ as follows. For each $b \in B_n$, let $(t^+_b, x^+_b) = \left(\prod_{a<b} \varphi^+_a\right)(t^+, x)$ where the composition is over all $a \in B_n$ that occur before b lexicographically; refer to Figure 4 for an illustration of the sequence $\{y_b\}_{b \in B_n}$ generated by an initial condition $y \in D_{-\mathbb{1}}$. Note that $\left\{t^+ - t^+_b\right\}_{b \in B_n}$ is (lexicographically) nondecreasing and $t^+_{+\mathbb{1}} = \tau^+_{+\mathbb{1}}(t^+_{+\mathbb{1}}, x^+_{+\mathbb{1}})$. Defining the interval

$$J_b = [t^+ - t^+_b, t^+ - t^+_b + \tau^+_b(t^+_b, x^+_b)], \tag{32}$$

we have $[0, t^+) \subset \bigcup_{b \in B_n} J^+_b$ and $\operatorname{Int} J^+_a \cap \operatorname{Int} J^+_b = \emptyset$ for all $a \in B_n \setminus \{b\}$. Observe that

$$\forall t \in \operatorname{Int} J^+_b : \phi(t, x) = \pi_2 \circ \varphi^+_b(t - (t^+ - t^+_b), x^+_b) \in \operatorname{Int} D_b, \tag{33}$$

where the condition is vacuously satisfied if $\operatorname{Int} J^+_b = \emptyset$. Therefore, for all $t \in \operatorname{Int} J^+_b$, the piecewise–differentiable function ϕ is classically differentiable with respect to time at (t, x), and we have

$$\begin{aligned} D_t \phi(t, x) &= D\pi_2\, D_t \varphi^+_b(t - (t^+ - t^+_b), x^+_b) \\ &= F_b(\pi_2 \circ \varphi^+_b(t - (t^+ - t^+_b), x^+_b)) \\ &= F(\pi_2 \circ \varphi^+_b(t - (t^+ - t^+_b), x^+_b)) \\ &= F(\phi(t, x)). \end{aligned} \tag{34}$$

Applying an analogous argument in backward time, we conclude that $D_t \phi(t, x) = F(\phi(t, x))$ for almost all $t \in (t^-, t^+) = \mathcal{F}^x$. Since $(0, x) \in \mathcal{F}$ was arbitrary, the lemma follows. ∎

4.3. Piecewise–differentiable flow. We now show that the piecewise–differentiable function $\phi \in PC^r(\mathcal{F}, D)$ defined in (30) is in fact a flow for the discontinuous vector field F. See Figure 4 for an illustration of this flow.

Theorem 4 (local flow). *Suppose the vector field $F : D \to TD$ is event–selected C^r at $\rho \in D$. Then there exists a flow $\phi : \mathcal{F} \to D$ for F over a flow domain $\mathcal{F} \subset \mathbb{R} \times D$ containing $(0, \rho)$ such that $\phi \in PC^r(\mathcal{F}, D)$ and*

$$\forall (t, x) \in \mathcal{F} : \phi(t, x) = x + \int_0^t F(\phi(s, x))\, ds. \tag{35}$$

Proof. We claim that $\phi \in PC^r(\mathcal{F}, D)$ from (30) satisfies (35). Applying the fundamental theorem of calculus [45, Proposition 3.1.1] in conjunction with Lemma 3 and positive–homogeneity of the derivative (31), we find

$$\begin{aligned} \phi(t, x) &= \phi(0, x) + \int_0^1 D\phi(tu, x; t, 0)du \\ &= x + \int_0^t D\phi(s, x; t, 0)\frac{1}{t}ds \\ &= x + \int_0^t D_t\phi(s, x)ds \\ &= x + \int_0^t F(\phi(s, x))ds. \end{aligned} \tag{36}$$

■

If the vector field $F : D \to TD$ is event–selected C^r at every point in the domain D, we may stitch together the local flows obtained from Theorem 4 (local flow) to obtain a global flow.

Corollary 5 (global flow). *If $F \in EC^r(D)$, then there exists a unique maximal flow $\phi \in PC^r(\mathcal{F}, D)$ for F. This flow has the following properties:*

(a) *For each $x \in D$, the curve $\phi^x : \mathcal{F}^x \to D$ is the unique maximal integral curve of F starting at x.*

(b) *If $s \in \mathcal{F}^x$, then $\mathcal{F}^{\phi(s,x)} = \mathcal{F}^x - s = \{t - s : t \in \mathcal{F}^x\}$.*

(c) *For each $t \in \mathbb{R}$, the set $D_t = \{x \in D : (t, x) \in \mathcal{F}\}$ is open in D and $\phi_t : D_t \to D_{-t}$ is a piecewise–C^r homeomorphism with inverse ϕ_{-t}.*

Proof. This follows from a straightforward modification of the analogous Theorem 9.12 in [35] (simply replace all occurrences of the word "smooth" with "PC^r"). We recapitulate the argument in SM3 in the supplemental materials. ■

Remark 1. *Existence, uniqueness, and Lipschitz continuity of the flow $\phi : \mathcal{F} \to D$ has been established previously for a more general class of discontinuous vector fields $F : D \to TD$ than those considered in Definition 2; see, for instance,* [18, Chapter 2, section 8, corollary to Theorem 3]. *Our contribution is the observation that, by restricting to event–selected C^r vector fields, the flow is in fact piecewise–C^r, i.e., $\phi \in PC^r(\mathcal{F}, D)$. Although significant labor was required to construct the candidate flow function in* (30) *via composition of PC^r functions,* *the task in Theorem 4 (local flow) of verifying that this function is the flow of the vector field F—i.e., ensuring ϕ from (30) satisfies (35)—required only a straightforward application of PC^r calculus. Similarly, PC^r calculus enables the proof of Corollary 5 (global flow) to proceed wholly analogously to the proof of the corresponding classical result.*

We conclude this section with a technical observation that will prove useful in what follows: if a vector field is event–selected C^r at every point along an integral curve, the following lemma shows that it is actually C^r at all but a finite number of points along the curve.

Lemma 6 (EC^r implies C^r almost everywhere). *Suppose the vector field $F : D \to TD$ is event–selected C^r at every point along an integral curve $\xi : I \to D$ for F over a compact interval $I \subset \mathbb{R}$. Then there exists a finite subset $\delta \subset \xi(I)$ such that F is C^r on $\xi(I) \setminus \delta$.*

Proof. Let $\delta \subset \xi(I)$ be the set of points where F fails to be C^r. If $|\delta| = \infty$, then since $\xi(I)$ is compact there exists an accumulation point $\alpha \in \xi(I)$. Since F is event–selected C^r at α, there exists $\varepsilon > 0$ such that F is C^r at every point in the set $(B_\varepsilon(\alpha) \cap \xi(I)) \setminus \{\alpha\}$, but this violates the existence of an accumulation point $\alpha \in \delta$. Therefore, $|\delta| < \infty$. ∎

5. Time–to–impact, (Poincaré) impact map, and flowbox.

We now leverage the fact that event–selected C^r vector fields yield piecewise–differentiable flows to obtain useful constructions familiar from classical (smooth) dynamical systems theory. Using an inverse function theorem [41, Corollary 20], we construct piecewise–differentiable time–to–impact maps for local sections in section 5.1. We then apply this construction to infer the existence of piecewise–differentiable (Poincaré) impact maps associated with periodic orbits in section 5.2 and piecewise–differentiable flowboxes in section 5.3.

5.1. Piecewise–differentiable time–to–impact.

We begin in this section by constructing piecewise–differentiable time–to–impact maps.

Theorem 7 (time–to–impact). *Suppose the vector field $F : D \to TD$ is event–selected C^r at $\rho \in D$. If $\sigma \in C^r(U, \mathbb{R})$ is an event function for F on an open neighborhood $U \subset D$ of ρ, then there exists an open neighborhood $V \subset D$ of ρ and piecewise–differentiable function $\mu \in PC^r(V, \mathbb{R})$ such that*

$$\forall x \in V : \sigma \circ \phi(\mu(x), x) = \sigma(\rho), \tag{37}$$

where $\phi \in PC^r(\mathcal{F}, D)$ is a flow for F and $(0, \rho) \in \mathcal{F}$.

Proof. Theorem 4 (local flow) ensures the existence of a flow $\phi \in PC^r(\mathcal{F}, D)$ such that $\mathcal{F} \subset \mathbb{R} \times D$ contains $(0, \rho)$. Let $\alpha = \sigma \circ \phi$, and note that there exist open neighborhoods $T \subset \mathbb{R}$ of 0 and $W \subset D$ of ρ such that $\alpha \in PC^r(T \times W, \mathbb{R})$.

We aim to apply an implicit function theorem to show that $\alpha(s, x) = \sigma(\rho)$ has a unique piecewise–differentiable solution $s = \mu(x)$ near $(0, \rho)$. To do so, we need to establish the function α is *completely coherently oriented* with respect to its first argument.

Specializing Definition 16 in [41], a sufficient condition for α to be completely coherently oriented with respect to its first argument at $(0, \rho)$ is that the (scalar) derivatives $D\alpha_j(0, \rho; 1, 0)$ of all essentially active selection functions $\{\alpha_j : j \in I^e(\alpha, (0, \rho))\}$ have the same sign. Lemma 3 implies the time derivatives of all essentially active selection functions for ϕ at $(0, \rho)$ are contained in the collection $\{F_b(\rho) : b \in B_n, D_b \neq \emptyset\}$, where $\{F_b : b \in B_n\}$ are the C^r vector fields that define F near ρ. Since σ is an event function for F, there exists $f > 0$ such that

$$\forall b \in B_n : D\sigma(\rho)F_b(\rho) \geq f > 0. \tag{38}$$

This implies that α is completely coherently oriented with respect to time at $(0, \rho)$. Therefore, we may apply Corollary 20 in [41] to obtain an open neighborhood $0 \in V \subset \mathbb{R}$ and a piecewise–differentiable function $\mu \in PC^r(V, \mathbb{R})$ such that (37) holds. ∎

Corollary 8 (time–to–impact). *Suppose the vector field $F : D \to TD$ is event–selected C^r at every point along an integral curve $\xi : [0, t] \to D$ for F. If $\sigma \in C^r(U, \mathbb{R})$ is an event function for F on an open set $U \subset D$ containing $\xi(t)$, then there exists an open neighborhood $\xi(0) \in V \subset D$ and piecewise–differentiable function $\mu \in PC^r(V, \mathbb{R})$ that satisfies (37).*

Proof. Corollary 5 ensures the existence of a flow $\phi \in PC^r(\mathcal{F}, D)$ such that $\mathcal{F} \subset \mathbb{R} \times D$ contains $[0, t] \times \{\xi(0)\}$. Let $\widetilde{\mu} \in PC^r(\widetilde{V}, \mathbb{R})$ be the impact time function for σ obtained by applying Corollary 7 at $\xi(t) = \phi(t, \xi(0))$. Then with $V = \{x \in D : \phi(t, x) \in \widetilde{V}\}$, noting that V is nonempty since $\xi(0) \in V$ and open since ϕ is continuous, the function $\mu : V \to \mathbb{R}$ defined by $\mu(x) = t + \widetilde{\mu} \circ \phi(t, x)$ is piecewise–C^r and satisfies (37). ∎

Remark 2. *We are not the first to observe that vector field discontinuities influence the time required to flow to a submanifold Σ of state space D; previous authors have computed the first–order effect these discontinuities exert on the flow time via* saltation matrices [3, 30] *and* discontinuity mappings [15, 19]. *Our contribution is the observation that, by restricting to event–selected C^r vector fields, the function $\mu : V \to \mathbb{R}$ that computes the time–to–impact from an open set $V \subset D$ to a surface $\Sigma \subset D$ is in fact piecewise–C^r, i.e., $\mu \in PC^r(V, \mathbb{R})$. In addition to generalizing the time–to–impact construction to apply in the presence of an arbitrary number of surfaces of discontinuity that are not required to be transverse (the one surface case is considered in [3, 19], while [15, 30] consider the case of two transverse surfaces), this immediately establishes the existence of higher–order approximations of μ. As with the proof of Theorem 4 (local flow), PC^r calculus enabled us to prove Theorem 7 (time–to–impact) using the same technique as the corresponding classical result, namely, an implicit function theorem.*

5.2. Piecewise–differentiable (Poincaré) impact map.

We now apply Theorem 7 (time–to–impact) in the important case where the integral curve is a *periodic orbit* to construct a piecewise–differentiable (Poincaré) impact map.

Definition 9. *An integral curve $\gamma : \mathbb{R} \to D$ is a* periodic orbit *for the vector field $F : D \to TD$ if there exists $t > 0$ such that $\gamma(t) = \gamma(0)$ and $D_t\gamma(s) \neq 0$ for all $s \in [0, t]$. The minimal $t > 0$ for which $\gamma(t) = \gamma(0)$ is referred to as the* period *of γ, and we say that γ is a t–periodic orbit for F. We let $\Gamma = \gamma(\mathbb{R})$ denote the image of γ.*

Suppose the vector field $F : D \to TD$ is event–selected C^r at every point along a t–periodic orbit γ for F. Then given a local section $\Sigma \subset D$ for F that intersects $\Gamma = \gamma(\mathbb{R})$ at $\{\rho\} = \Gamma \cap \Sigma$, Corollary 8 implies there exists a piecewise–differentiable impact time function $\mu \in PC^r(V, \mathbb{R})$ defined over an open neighborhood $V \subset D$ of ρ such that $\mu(\rho) = t$. With $V \cap \Sigma$, we let

$\psi : V \to \Sigma$ be the piecewise–differentiable impact map defined by

$$\forall x \in V : \psi(x) = \phi(\mu(x), x). \tag{39}$$

Theorem 10 (Poincaré map). *Suppose the vector field $F : D \to TD$ is event–selected C^r at every point along a periodic orbit γ for F. Then given a local section $\Sigma \subset D$ for F that intersects $\Gamma = \gamma(\mathbb{R})$ at $\{\rho\} = \Gamma \cap \Sigma$, there exists an open neighborhood $V \subset D$ of ρ such that the impact map (39) restricts to a piecewise–differentiable (Poincaré) map $P \in PC^r(S, \Sigma)$ on $S = V \cap \Sigma$.*

Proof. Without loss of generality assume $\gamma(0) \in \Sigma$. Let T be the period of γ, apply Theorem 7 (time–to–impact) to $\gamma|_{[0,T]}$ to obtain an open set $V \subset D$ containing $\gamma(0)$ and a piecewise–C^r impact time map $\mu \in PC^r(V, \mathbb{R})$, and define $\psi : V \to \Sigma$ as in (39). Then with $S = V \cap \Sigma$, the restriction $P = \psi|_S$ is a piecewise–C^r Poincaré map for γ. ■

Since the Poincaré map $P : S \to \Sigma$ yielded by Theorem 10 (Poincaré map) is piecewise–differentiable, it admits a first–order approximation (its Bouligand derivative) $DP : TS \to T\Sigma$ that can be used to assess the local exponential stability of the fixed point $P(\rho) = \rho$. This topic will be investigated in more detail in section 7.2.

5.3. Piecewise–differentiable flowbox. Theorem 7 (time–to–impact) enables us to easily derive a canonical form for the flow near an event–selected vector field discontinuity.

Theorem 11 (flowbox). *Suppose the vector field $F : D \to TD$ is event–selected C^r at $\rho \in D$, and let $\phi : \mathcal{F} \to D$ be the flow obtained from Theorem 4 (local flow). Then there exists a piecewise–differentiable homeomorphism $\chi \in PC^r(V, W)$ between neighborhoods $V \subset D$ of ρ and $W \subset \mathbb{R}^d$ of 0 such that*

$$\forall x \in V, t \in \mathcal{F}^x : \chi \circ \phi(t, x) = \chi(x) + te_1, \tag{40}$$

where $e_1 \in \mathbb{R}^d$ is the first standard Euclidean basis vector.

Proof. Let $\sigma \in C^r(U, \mathbb{R})$ be an event function for F on a neighborhood $\rho \in U \subset D$ that is linear.[6] Theorem 7 (time–to–impact) implies that there exists a piecewise–differentiable time–to–impact map $\mu \in PC^r(V, \mathbb{R})$ on a neighborhood $V \subset D$ of ρ such that

$$\forall x \in V : \sigma \circ \phi(\mu(x), x) = \sigma(\rho), \tag{41}$$

i.e., $\phi(\mu(x), x)$ lies in the codimension–1 subspace $\Sigma = \sigma^{-1}(\sigma(\rho))$. Define $\chi : V \to \mathbb{R} \times \Sigma$ by

$$\forall x \in V : \chi(x) = (-\mu(x), \phi(\mu(x), x)). \tag{42}$$

Clearly $\chi \in PC^r(V, \mathbb{R} \times \Sigma)$, and hence χ is continuous. Furthermore, it is clear that χ is injective since (i) $\pi_\Sigma \chi(x) = \pi_\Sigma \chi(y)$ implies that x and y lie along the same integral curve, and (ii) distinct points along an integral curve pass through Σ at distinct times. It follows from Brouwer's open mapping theorem [10, 23] that the image $W = \chi(V)$ is an open subset

[6] Existence of a linear event function is always guaranteed. For instance, take the linear approximation at ρ of any nonlinear event function for F.

of $\mathbb{R}^d$. This implies that χ is a homeomorphism between V and W. With $\iota : \mathbb{R} \times \Sigma \to \mathbb{R} \times D$ denoting the canonical inclusion, the inverse of $\chi \in PC^r(V, W)$ is $\phi \circ \iota|_W \in PC^r(W, V)$; thus χ is a PC^r homeomorphism. Finally, using the semigroup property of the flow ϕ and the fact that $\mu \circ \phi(t, x) = \mu(x) - t$ for all $x \in V, t \in \mathcal{F}^x$,

$$\begin{aligned} \forall x \in V,\ t \in \mathcal{F}^x : \chi \circ \phi(t, x) &= (-\mu \circ \phi(t, x), \phi(\mu \circ \phi(t, x), \phi(t, x))) \\ &= (t - \mu(x), \phi(\mu(x) - t, \phi(t, x))) \\ &= (t - \mu(x), \phi(\mu(x), x)) \\ &= \chi(x) + te_1. \end{aligned} \tag{43}$$

■

Thus the flow is conjugate via a piecewise–differentiable homeomorphism to a *flowbox* [25, section 11.2], [35, Theorem 9.22].

6. Perturbed flow.

In this section we study how the flow associated with an event–selected C^r vector field varies under perturbations to both the smooth vector field components (in section 6.1) and the event functions (in section 6.2).

6.1. Perturbation of vector fields.

Suppose $F : D \to TD$ is event–selected C^r at $\rho \in D$ with respect to the components of $h \in C^r(D, \mathbb{R}^n)$. Then by Definition 2 there exists $U \subset D$ containing ρ such that for each $b \in B_n$ either $\operatorname{Int} D_b = \emptyset$ or $D_b \subset U$ and $F|_{\operatorname{Int} D_b}$ admits a C^r extension $F_b : U \to TU$. We note that F is determined on U up to a set of measure zero from h and the function $\widehat{F} \in C^r\left(\coprod_{b \in B_n} U, \coprod_{b \in B_n} TU\right)$ defined by $\widehat{F}|_{\{b\} \times U} = F_b|_U$. Note that we regard $C^r\left(\coprod_{b \in B_n} U, \coprod_{b \in B_n} TU\right)$ as a vector space under pointwise addition of tangent vectors and the norm

$$\left\| \widehat{F} \right\|_{C^r} = \sum_{b \in B_n} \left\| \widehat{F}|_{\{b\} \times U} \right\|_{C^r}. \tag{44}$$

Thus in what follows we consider perturbations to event–selected C^r vector fields in the space $C^r\left(\coprod_{b \in B_n} U, \coprod_{b \in B_n} TU\right)$.

Theorem 12 (vector field perturbation). *Let $F \in C^r\left(\coprod_{b \in B_n} D, \coprod_{b \in B_n} TD\right)$, $h \in C^r(D, \mathbb{R}^n)$ determine an event–selected C^r vector field at $\rho \in D$, $r \geq 1$. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\widetilde{F} \in B_\delta^{C^r}(F)$*

(a) *pairing h with $\widetilde{F}$ determines an event–selected C^r vector field at ρ;*

(b) *the perturbed flow $\widetilde{\phi} : \widetilde{\mathcal{F}} \to D$ obtained by applying Theorem 4 (local flow) to $\widetilde{F}$ satisfies $\widetilde{\phi} \in B_\varepsilon^{C^0}(\phi)$ on $\widetilde{\mathcal{F}} \cap \mathcal{F}$ and $(0, \rho) \in \widetilde{\mathcal{F}} \cap \mathcal{F}$;*

(c) *there exists a piecewise–differentiable homeomorphism $\eta \in PC^r(U, \widetilde{U})$ defined between neighborhoods $U, \widetilde{U} \subset D$ of ρ such that $\eta|_{B_\delta(\rho)} \in B_\varepsilon^{C^0}(\mathrm{id}_{B_\delta(\rho)})$ and*

$$\eta \circ \phi(t, x) = \widetilde{\phi}(t, \eta(x)) \tag{45}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ such that $x \in U$, $t \in \mathcal{F}^x \cap \widetilde{\mathcal{F}}^{\eta(x)}$, and $\phi(t, x) \in U$.

Proof. Since F is event–selected C^r with respect to h at ρ, there exists $f > 0$ such that for all x sufficiently close to ρ every component of $Dh(x)F(x)$ is bounded below by f. Then so long as $0 < \delta < f$, every component of $Dh(x)\widetilde{F}(x)$ is bounded below by $f - \delta > 0$, establishing claim (a).

We claim that (b) follows from [18, Theorem 1 in section 8 of Chapter 2], which we reproduce as Theorem SM3 (differential inclusion perturbation) in SM4 in the supplemental materials. Indeed, given any $G \in C^r(\coprod_{b\in B_n} D, \coprod_{b\in B_n} TD)$ for which (G, h) determines an event–selected C^r vector field, define a set–valued map $\overline{G} : D \to 2^{TD}$ as follows:

$$\forall x \in D : \overline{G}(x) = \operatorname{conv}\left\{G|_{\{b\}\times D}(x) : b \in B_n,\ x \in D_b\right\}. \tag{46}$$

At any $x \in D$, it is clear that $\overline{G}(x)$ is nonempty, bounded, closed, and convex. Furthermore, it is clear that $\overline{G}$ is upper semicontinuous at x in the sense defined in SM4 in the supplemental materials. Therefore, the map $\overline{G}$ satisfies Assumption 1 (differential inclusion basic conditions) over the domain of the flow for G. It is straightforward to verify that solutions to the differential inclusion $\dot{x} \in \overline{G}(x)$ coincide with those of the differential equation $\dot{x} = G(x)$ since the derivatives of the (absolutely continuous) solution functions agree almost everywhere. Claim (b) then follows by applying Theorem SM3 (differential inclusion perturbation) to $\overline{F}$ determined from F by (46) and $\overline{\widetilde{F}}$ determined from $\widetilde{F} \in B_\delta^{C^r}(F)$ by (46).

For claim (c), apply Theorem 11 (flowbox) to ϕ and $\widetilde{\phi}$ to obtain $\chi \in PC^r(V, W)$ and $\widetilde{\chi} \in PC^r(\widetilde{V}, \widetilde{W})$ such that

$$\forall x \in V \cap \widetilde{V}, t \in \mathcal{F}^x \cap \widetilde{\mathcal{F}}^x : \chi \circ \phi(t, x) = \chi(x) + t\, e_1,\ \widetilde{\chi} \circ \widetilde{\phi}(t, x) = \widetilde{\chi}(x) + t\, e_1. \tag{47}$$

Then with $U = \chi^{-1}(\widetilde{W})$, $\widetilde{U} = \widetilde{\chi}^{-1} \circ \chi(U)$ (both sets are nonempty since $\rho \in U \cap \widetilde{U}$ and open since χ and $\widetilde{\chi}$ are homeomorphisms), the piecewise–differentiable homeomorphism $\eta = \widetilde{\chi}^{-1} \circ \chi|_U \in PC^r(U, \widetilde{U})$ provides conjugacy between ϕ and $\widetilde{\phi}$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ such that $x \in U$, $t \in \mathcal{F}^x \cap \widetilde{\mathcal{F}}^{\eta(x)}$, and $\phi(t, x) \in U$:

$$\begin{aligned} \eta \circ \phi(t, x) &= \widetilde{\chi}^{-1} \circ \chi \circ \phi(t, x) \\ &= \widetilde{\chi}^{-1}\left(\chi(x) + t\, e_1\right) \\ &= \widetilde{\chi}^{-1}\left(\widetilde{\chi} \circ \widetilde{\chi}^{-1} \circ \chi(x) + t\, e_1\right) \\ &= \widetilde{\chi}^{-1}\left(\widetilde{\chi} \circ \eta(x) + t\, e_1\right) \\ &= \widetilde{\phi}(t, \eta(x)). \end{aligned} \tag{48}$$

We now wish to choose $\delta > 0$ sufficiently small to ensure $\eta|_{B_\delta(\rho)} \in B_\varepsilon^{C^0}(\mathrm{id}_{B_\delta(\rho)})$. Recalling from (42) that

$$\forall x \in V : \chi(x) = (-\mu(x), \phi(\mu(x), x)), \tag{49}$$

where $\mu \in PC^r(V, \mathbb{R})$ is the time–to–impact map for the event surface used to define χ, we have

$$\begin{aligned} \|\chi(x) - \widetilde{\chi}(x)\| &\le |\mu(x) - \widetilde{\mu}(x)| + \left\|\phi(\mu(x), x) - \widetilde{\phi}(\widetilde{\mu}(x), x)\right\| \\ &\le |\mu(x) - \widetilde{\mu}(x)| + \|\phi(\mu(x), x) - \phi(\widetilde{\mu}(x), x)\| \\ &\quad + \left\|\phi(\widetilde{\mu}(x), x) - \widetilde{\phi}(\widetilde{\mu}(x), x)\right\| \\ &\le (1 + L_\phi)\, |\mu(x) - \widetilde{\mu}(x)| + \varepsilon_\phi, \end{aligned} \tag{50}$$

where $L_\phi > 0$ is a Lipschitz constant for ϕ on $\overline{B_\delta(0,\rho)}$, claim (b) ensures $\widetilde{\phi} \in B^{C^0}_{\varepsilon_\phi}(\phi)$ for any desired $\varepsilon_\phi > 0$, and we have restricted to $x \in V \cap \widetilde{V} \cap B_\delta(\rho)$ for which $(\widetilde{\mu}(x), x) \in \mathcal{F}$ and $(\mu(x), x) \in \widetilde{\mathcal{F}}$. Applying [41, Lemma 9, Theorem 11] to μ, we conclude that $\delta > 0$ can be chosen sufficiently small to ensure $\widetilde{\mu} \in B^{C^0}_{\varepsilon_\mu}(\mu)$ for any desired $\varepsilon_\mu > 0$. Therefore, $(1 + L_\phi)\varepsilon_\mu + \varepsilon_\phi$ can be made arbitrarily small in (50); hence we may apply [41, Theorem 11] to choose $\delta > 0$ sufficiently small to ensure $\widetilde{\chi}^{-1} \in B^{C^0}_{\varepsilon}(\chi^{-1})$ for any desired $\varepsilon > 0$. Thus $\delta > 0$ may be chosen sufficiently small to ensure $B_\delta(\rho) \subset U$ and

$$\|\eta(x) - x\| = \left\|\widetilde{\chi}^{-1} \circ \chi(x) - \chi^{-1} \circ \chi(x)\right\| \le \varepsilon, \tag{51}$$

whence $\eta|_{B_\delta(\rho)} \in B^{C^0}_{\varepsilon}(\mathrm{id}_{B_\delta(\rho)})$. This completes the proof of claim (c). ■

Remark 3. *Persistence and C^0–closeness of the flow $\widetilde{\phi} : \widetilde{\mathcal{F}} \to D$ associated with a perturbation $\widetilde{F}$ of a discontinuous vector field $F : D \to TD$ has been established previously for a more general class of vector fields than those considered in Definition 2; see, for instance,* [18, Chapter 2, section 8, Theorem 1]. *Our contribution is the observation that, by restricting to event–selected C^r vector fields, the perturbed flow is in fact piecewise–C^r, i.e., $\widetilde{\phi} \in PC^r(\widetilde{\mathcal{F}}, D)$, and is conjugate to the unperturbed flow via a PC^r homeomorphism. The existence of a PC^r "flow-box" homeomorphism, provided by Theorem 11 (flowbox), was pivotal in the derivation of this fact.*

6.2. Perturbation of event functions.

It is a well–known fact that the solution of n equations in n unknowns generically varies continuously with variations in the equations. This observation provides a basis for studying structural stability of the flow associated with event–selected C^r vector fields when there are exactly $n = d = \dim D$ event functions, since for a collection of event functions $\{h_j\}_{j=1}^d \subset C^r(D, \mathbb{R})$ whose composite $h \in C^r(D, \mathbb{R}^d)$ satisfies $\det Dh(\rho) \ne 0$, the existence of a unique intersection point $\widetilde{\rho}$ and the set of possible transition sequences undertaken by nearby trajectories are unaffected by a sufficiently small perturbation $\widetilde{h}$ of h. We now combine this observation with the previous theorem. Subsequently, we will present an embedding technique that enables immediate generalization to cases where $Dh(\rho)$ is not invertible (whether because $n < d$, $n > d$, or $n = d$ and $\det Dh(\rho) = 0$).

Theorem 13 (event function perturbation). *Let $F \in C^r\left(\coprod_{b\in B_n} D, \coprod_{b\in B_n} TD\right)$, $h \in C^r(D, \mathbb{R}^d)$ determine an event–selected C^r vector field at $\rho \in D$ and suppose $Dh(\rho)$ is invertible, $r \ge 1$. Then for all $\varepsilon > 0$ sufficiently small there exists $\delta > 0$ such that for all $\widetilde{F} \in B^{C^r}_\delta(F)$, $\widetilde{h} \in B^{C^r}_\delta(h)$:*

(a) *there exists a unique $\widetilde{\rho} \in B_\delta(\rho)$ such that $\widetilde{h}(\widetilde{\rho}) = 0$ and $\widetilde{h}(x) \ne 0$ for all $x \in B_\delta(\rho) \setminus \{\widetilde{\rho}\}$;*
(b) *pairing $\widetilde{h}$ with $\widetilde{F}$ determines an event–selected C^r vector field at $\widetilde{\rho}$;*
(c) *the perturbed flow yielded by Theorem 4 (local flow), $\widetilde{\phi} : \widetilde{\mathcal{F}} \to D$, satisfies $\widetilde{\phi} \in B^{C^0}_\varepsilon(\phi)$ on $\widetilde{\mathcal{F}} \cap \mathcal{F} \ne \emptyset$;*
(d) *there exists a piecewise–differentiable homeomorphism $\eta \in PC^r(U, \widetilde{U})$ defined between neighborhoods $U, \widetilde{U} \subset D$ containing $\{\rho, \widetilde{\rho}\}$ such that $\eta|_{B_\delta(\rho)} \in B^{C^0}_\varepsilon(\mathrm{id}_{B_\delta(\rho)})$ and*

$$\eta \circ \phi(t, x) = \widetilde{\phi}(t, \eta(x)) \tag{52}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ such that $x \in U$, $t \in \mathcal{F}^x \cap \widetilde{\mathcal{F}}^{\eta(x)}$, and $\phi(t, x) \in U$.

Proof of Theorem 13 (event function perturbation). Smooth dependence of the intersection point follows from the implicit function theorem [1, Theorem 2.5.7] since C^r functions over compact domains comprise a Banach space [24, Chapter 2.1]. Specifically, if $h \in C^r(D, \mathbb{R}^n)$ satisfies $h(\rho) = 0$ for some $\rho \in D$ and $Dh(\rho)$ is invertible,[7] then there exist $\alpha, \beta > 0$ and $\widetilde{\rho} \in C^r(B_\alpha(h), B_\beta(\rho))$ such that for all $\widetilde{h} \in B_\alpha(h)$ the point $\widetilde{\rho}(\widetilde{h})$ is the unique zero of $\widetilde{h}$ on $B_\beta(\rho)$; i.e., $\widetilde{h}(\widetilde{\rho}(\widetilde{h})) = 0$ and for all $x \in B_\beta(\rho) \setminus \{\widetilde{\rho}(\widetilde{h})\}$ we have $\widetilde{h}(x) \neq 0$. This establishes (a); (b) follows from continuity.

For any $\delta' > 0$, we can choose $\delta > 0$ sufficiently small to ensure that $\widetilde{F} \in B_\delta^{C^r}(F)$, $\widetilde{h} \in B_\delta^{C^r}(h)$ implies $D\widetilde{h}^{-1} \circ \widetilde{F} \in B_{\delta'}^{C^r}\left(Dh^{-1} \circ F\right)$; let $\widetilde{F}' = D\widetilde{h}^{-1} \circ \widetilde{F}$, $F' = Dh^{-1} \circ F$. With $\widetilde{\phi}' : \widetilde{\mathcal{F}}' \to \mathbb{R}^d$, $\phi' : \mathcal{F}' \to \mathbb{R}^d$ denoting the flows for $\widetilde{F}'$, F', Theorem 12 (vector field perturbation) implies that $\delta' > 0$ can be chosen sufficiently small to ensure $\widetilde{\phi}' \in B_{\varepsilon'}^{C^0}(\phi')$ for any $\varepsilon' > 0$. Since $\widetilde{h}$ provides conjugacy between $\widetilde{\phi}$ and $\widetilde{\phi}'$, and similarly h provides conjugacy between ϕ and ϕ', we conclude that $\delta > 0$ can be chosen sufficiently small to ensure $\widetilde{\phi} \in B_\varepsilon^{C^0}(\phi)$ on $\widetilde{\mathcal{F}} \cap \mathcal{F}$. This establishes (c).

Let $\eta' \in PC^r(U', \widetilde{U}')$ be the conjugacy from Theorem 12 (vector field perturbation) relating ϕ' to $\widetilde{\phi}'$. Then $\eta = \widetilde{h} \circ \eta' \circ h^{-1}$ provides conjugacy between ϕ and $\widetilde{\phi}$ since

$$\begin{aligned} \eta \circ \phi(t, x) &= \widetilde{h} \circ \eta' \circ h^{-1} \circ \phi(t, x) \\ &= \widetilde{h} \circ \eta' \circ \phi'(t, h^{-1}(x)) \\ &= \widetilde{h} \circ \widetilde{\phi}'(t, \eta' \circ h^{-1}(x)) \\ &= \widetilde{\phi}(t, \widetilde{h} \circ \eta' \circ h^{-1}(x)) \\ &= \widetilde{\phi}(t, \eta(x)) \end{aligned} \tag{53}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ such that $x \in h(U)$, $t \in \mathcal{F}^x \cap \widetilde{\mathcal{F}}^{\eta(x)}$, and $\phi(t, x) \in U$. Furthermore, given $\varepsilon > 0$ we may choose $\delta > 0$ sufficiently small to ensure $\widetilde{h}^{-1} \in B_\delta^{C^0}(h^{-1})$ and $\eta' \in B_\delta^{C^0}(\mathrm{id})$, whence

$$\begin{aligned} \|\eta(x) - x\| &= \left\|\widetilde{h} \circ \eta' \circ h^{-1}(x) - x\right\| \\ &\le \left\|\widetilde{h} \circ \eta' \circ h^{-1}(x) - \widetilde{h} \circ h^{-1}(x)\right\| + \left\|\widetilde{h} \circ h^{-1}(x) - x\right\| \\ &\le L_{\widetilde{h}} \left\|\eta'(y) - y\right\| + \delta \\ &\le (1 + L_{\widetilde{h}})\delta \end{aligned} \tag{54}$$

for all $x \in B_\delta(0)$. Thus $\delta < \varepsilon/(1 + L_{\widetilde{h}})$ ensures $\eta|_{B_\delta(\rho)} \in B_\varepsilon^{C^0}(\mathrm{id}_{B_\delta(\rho)})$. This completes the proof of claim (d). ∎

Remark 4. *Now consider the case where $F : D \to TD$ is event–selected C^r at $\rho \in D$ with respect to the composite event function $h \in C^r(D, \mathbb{R}^n)$ but $Dh(\rho) \in \mathbb{R}^{n\times d}$ is not invertible (because either $n < d$, $n > d$, or $n = d$ and* $\det Dh(\rho) = 0$*). We will embed this d–dimensional system into a $(d+n)$–dimensional system to obtain an event–selected C^r vector field with respect to an invertible composite event function; this will enable application of the*

[7] Note that necessarily $n = \dim D$.

preceding theorem to the degenerate system determined by F *and* h*. For each* $b \in B_n$*, let* $S_b = \{x \in \mathbb{R}^{1\times d} : xF_b(\rho) > 0\}$ *be the open half–space of row vectors that have a positive inner product with* $F_b(\rho)$*. The set* $S = \cap_{b\in B_n} S_b$ *is open (since each* S_b *is open) and nonempty (since in particular* $Dh_1(\rho) \in S$*). Let* $A \in \mathbb{R}^{d\times d}$ *be an invertible matrix whose rows are selected from* S*; such a matrix always exists since* S *is open and nonempty. Now let* $\overline{D} = D \times \mathbb{R}^n$*, and define* $\overline{F} : \overline{D} \to T\overline{D}$ *and* $\overline{h} \in C^r(\overline{D}, \mathbb{R}^{d+n})$ *as follows:*

$$\forall (x, y) \in D \times \mathbb{R}^n : \overline{F}(x, y) = \begin{bmatrix} F(x) \\ 0 \end{bmatrix}, \ \overline{h}(x, y) = \begin{bmatrix} Ax \\ h(x) + y \end{bmatrix}. \tag{55}$$

Clearly $\overline{F}$ *is event–selected* C^r *at* $\overline{\rho} = (\rho, 0)$*, and* $D\overline{h}(\overline{\rho})$ *is invertible since*

$$D\overline{h}(\overline{\rho}) = \begin{bmatrix} A & 0 \\ Dh(x) & I_n \end{bmatrix} \in \mathbb{R}^{(d+n)\times(d+n)} \tag{56}$$

has linearly independent columns. Therefore, Theorem 13 (event function perturbation) may be applied to study the effect of perturbations on the flow $\overline{\phi} : \overline{\mathcal{F}} \to \overline{D}$ *for* $\overline{F}$*; the conclusions of the theorem can be specialized to the original flow* $\phi : \mathcal{F} \to D$ *for* F *as follows. With* $\mathcal{V} = \{(t, x) \in \mathcal{F} : (t, x, 0) \in \overline{\mathcal{F}}\}$ *let* $\iota : \mathcal{V} \to \widetilde{\mathcal{F}}$ *denote the embedding defined by* $\iota(t, x) = (t, x, 0)$ *for all* $(t, x) \in \mathcal{V}$*, and let* $\pi : \overline{D} \to D$ *denote the projection defined by* $\pi(x, y) = x$ *for all* $(x, y) \in \overline{D}$*. With these definitions we have*

$$\phi|_{\mathcal{V}} = \left(\pi \circ \overline{\phi} \circ \iota\right)|_{\mathcal{V}}. \tag{57}$$

7. Computation.

In this section, we apply the theoretical results from sections 4, 5, and 6 to derive procedures to compute the B–derivative of the flow and assess stability of a periodic orbit for an event–selected C^r vector field F. We begin in section 7.1 by developing a concrete procedure to compute the B–derivative of the piecewise–differentiable flow yielded by F. Subsequently, in section 7.2 we provide sufficient conditions ensuring exponential stability of a periodic orbit that passes through the intersection of multiple surfaces of discontinuity for F.

7.1. Variational equations and saltation matrices.

In this section we compute the B–derivative of the piecewise–differentiable flow by solving a jump–linear time–varying ordinary differential equation (ODE) along a trajectory. At trajectory points where the vector field is C^r, we recall in section 7.1.1 that the derivative is obtained by solving a time–varying ODE (the so–called *variational equation*) with no "jumps." At points where the vector field is discontinuous along one (or two transverse) event surface(s), in section 7.1.2 we note (as others have before us) that the ODE must be updated discontinuously (via a so–called *saltation matrix*). In the remainder of the section, we exploit properties of the piecewise–differentiable flow to derive a generalization of this procedure applicable in the presence of an arbitrary number of surfaces of discontinuity that are not required to be transverse.

7.1.1. C^r vector field.

Let $D \subset \mathbb{R}^d$ be an open domain and $F \in C^r(D, TD)$ a smooth vector field on D. It is a classical result [25, Theorem 1, section 15.2] that the derivative of the flow $\phi : \mathcal{F} \to D$ associated with F with respect to state can be obtained by solving a linear time–varying differential equation, the so–called *variational equation*, along a trajectory; i.e., if $(t, x) \in \mathcal{F}$ and $X : [0, t] \to \mathbb{R}^{d \times d}$ satisfies

$$\forall u \in [0, t] : \dot{X}(u) = D_x F(\phi(u, x))X(u), \ X(0) = I, \tag{58}$$

then the derivative of the flow with respect to time and state is given by

$$D_t\phi(t, x) = F(\phi(t, x)), \ D_x\phi(t, x) = X(t). \tag{59}$$

Here and in what follows we assume without loss of generality that $t > 0$; the $t < 0$ case can be addressed by applying the same reasoning to the vector field $-F$.

7.1.2. Event–selected C^r vector field. If the vector field is instead event–selected C^r, $F \in EC^r(D)$, adjustments must be made to (58) wherever a trajectory crosses a surface of discontinuity. Let $\phi : \mathcal{F} \to D$ denote the global flow of F yielded by Corollary 5 (global flow), and let $(t, x) \in \mathcal{F}$. As shown in [3, equation 1.4] (and subsequently [27, equations 57–60]), if for some $s \in (0, t)$ the vector field F is event–selected C^r at $\rho = \phi(s, x)$ with respect to a single surface of discontinuity, H, then the variational equation (58) must be updated discontinuously via multiplication by a so–called *saltation matrix*,

$$X(s^+) = \left[I + \frac{(F_{+\mathbb{1}}(\rho) - F_{-\mathbb{1}}(\rho))\, Dh(\rho)}{Dh(\rho)F_{-\mathbb{1}}(\rho)} \right] X(s^-), \tag{60}$$

where $X(s^+) = \lim_{u \to s^+} X(u)$, $X(s^-) = \lim_{u \to s^-} X(u)$, and $H \subset h^{-1}(0)$ near ρ.

As claimed in [30, equation 2.4] (and subsequently [15, Theorem 7.5], [16, equation 46], and [9, equation 27]), if for some $s \in (0, t)$ the vector field F is event–selected C^r at $\rho = \phi(s, x)$ with respect to multiple surfaces of discontinuity, then the variational equation (58) must be updated discontinuously via multiplication by one saltation matrix for each surface. Unlike the preceding cases, the flow will generally not possess a classical derivative with respect to state after time s. Previous authors compute the first–order effect of the flow using crossing times of perturbed trajectories. Due to the combinatorial complexity of this approach, these authors only derive the first–order approximation for two intersecting surfaces; though they claim that the approach readily extends to arbitrary numbers of intersecting surfaces, they leave the details to the reader.

The development in section 4 enables us to apply the PC^r calculus to rigorously derive the derivative of the flow along trajectories passing through an arbitrary collection $\{H_j\}_{j=1}^n$ of surfaces across which F is discontinuous using techniques familiar from calculus, namely, the chain rule. Without loss of generality[8] we assume F is C^r at every point in $\phi([0, t] \setminus \{s\}, x)$, and we let $\rho = \phi(s, x)$ as before.

[8]Lemma 6 ensures that there are a finite number of discontinuities along any integral curve of a vector field $F \in EC^r(D)$ defined over a compact time interval. Therefore, to evaluate the B–derivative of the flow after any number of discontinuities, one may iteratively apply the procedure described in what follows to a finite number of trajectory segments and combine the result using the chain rule [45, Theorem 3.1.1].

7.1.3. Sampled vector field associated with event–selected C^r vector field. We begin by noting that the B–derivative calculation in (29) depends only on first–order approximations of the flow and event functions $\{h_j\}_{j=1}^n$. For all $b \in B_n$ let

$$\widetilde{D}_b = \{x \in D : b_j\, Dh_j(\rho)(x - \rho) \geq 0\} \tag{61}$$

and consider the flow $\widetilde{\phi} : \widetilde{\mathcal{F}} \to D$ of the piecewise–constant vector field $\widetilde{F} : D \to TD$ defined by

$$\forall b \in B_n,\ x \in \widetilde{D}_b : \widetilde{F}(x) = F_b(\rho). \tag{62}$$

Applying (29) together with the chain rule [45, Theorem 3.1.1], we conclude that

$$\forall (v, w) \in T_{(0,\rho)}\mathcal{F} : D\phi(0, \rho; v, w) = D\widetilde{\phi}(0, \rho; v, w). \tag{63}$$

In other words, by sampling the event–selected C^r vector field F across its tangent planes we obtain a piecewise–constant event–selected C^∞ vector field $\widetilde{F}$ whose flow $\widetilde{\phi}$ agrees with the flow ϕ for F to first order. In this sense, we regard the piecewise–constant "sampled" vector field $\widetilde{F}$ as the analogue of the linearization of a smooth vector field in our nonsmooth setting. Note that, since the flow of the sampled system is obtained in (30) by composing a sequence of piecewise–affine functions, it is piecewise–affine:

$$\forall (v, w) \in T_{(0,\rho)}\mathcal{F} : \widetilde{\phi}(v, \rho + w) = \widetilde{\phi}(0, \rho) + D\widetilde{\phi}(0, \rho; v, w). \tag{64}$$

These observations enable us in the remainder of this section to derive several properties of the B–derivative that will prove useful in the applications presented in section 8.

7.1.4. Saltation matrix for multiple transition surfaces. Suppose $(v, w) \in T_{(t,x)}\mathcal{F} = \mathbb{R} \times \mathbb{R}^d$ is such that[9] for all $c > 0$ sufficiently small the trajectory initialized at $x + cw$ (i) passes through a unique sequence of m region interiors on its way to $\phi(t + cv, x + cw) \in D_{+\mathbb{1}}$; and (ii) does not pass through the intersection of nontangent surfaces. Let $\omega : \{1, \ldots, m\} \to B_n$ specify the sequence of region interiors, excluding $D_{+\mathbb{1}}$, and let $\eta : \{1, \ldots, m\} \to \{1, \ldots, n\}$ specify the corresponding sequence[10] of surfaces crossed. The B–derivative of the flow evaluated in the (v, w) direction is

$$D\phi(t, x; v, w) = D\phi(t - s, \rho) \left[\prod_{j=1}^{m} D\varphi^{+}_{\omega(j)}(0, \rho) \right] \begin{bmatrix} 0 \\ D\phi(s, x) \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}, \tag{65}$$

where $D\phi(t-s, \rho)$, $D\phi(s, x)$ are obtained as in (59) by solving the classical variational equation since F is smoothly extendable to a neighborhood of those segments of the trajectory and

[9]Since the flow $\widetilde{\phi}$ for the "sampled" vector field (62) is piecewise–affine, the set of tangent vectors that fail to satisfy the two specified conditions has measure zero. Since (once the base point has been fixed) the B–derivative is a continuous function of tangent vectors, it is determined by its values on the dense subset of tangent vectors that satisfy the two conditions.

[10]If H_j is tangent to H_i at ρ, then either H_j or H_i may be indexed by η; the choice will have no effect on the subsequent calculation.

for each $j \in \{1, \ldots, m\}$ the derivative $D\varphi^+_{\omega(j)}(0, \rho)$ is given by the matrix in the third case in (29) with the simplifications $\tau^+_{\omega(j)}(0, \rho) = 0$, $\phi_{\omega(j)}(0, \rho) = \rho$. Substituting $f = F_{\omega(j)}(\rho)$, $g^\top = Dh_{\eta(j)}(\rho)$ for clarity yields

$$D\varphi^+_{\omega(j)}(0, \rho) = \begin{bmatrix} 1 & \frac{1}{g^\top f} g^\top \\ 0 & I - \frac{1}{g^\top f} f\, g^\top \end{bmatrix} = I + \frac{1}{g^\top f} \begin{bmatrix} 1 \\ -f \end{bmatrix} \begin{bmatrix} 0 & g^\top \end{bmatrix} \tag{66}$$

since (17) simplifies to $D\tau^{H_{\eta(j)}}_{\omega(j)}(\rho) = -\frac{1}{g^\top f} g^\top$. Thus, the B–derivative in (65) is obtained by composing rank–1 updates of the identity with solutions to classical variational equations. In what follows we will make use of the saltation matrix $\Xi_\omega \in \mathbb{R}^{(d+1)\times(d+1)}$ given by

$$\Xi_\omega = \prod_{j=1}^{m} D\varphi^+_{\omega(j)}(0, \rho). \tag{67}$$

7.1.5. Flow between tangent transition surfaces. If the surfaces are tangent at the point $\rho = \phi(s, x) \in \bigcap_{j=1}^n H_j \neq \emptyset$ of intersection with the trajectory, a perturbed trajectory is not affected to first order by flow through the interior of a region between surfaces that are tangent; this follows from the equality in (63) relating the B–derivative of the original system to that of its "sampled" version. Indeed, consider the vector field illustrated in Figure 4 where the surfaces H_1 and H_2 are tangent at ρ. Evaluating the derivative $D\phi(t, z; 0, (0, \delta))$ for any $\delta > 0$ requires composition of $D\varphi_{-\mathbb{1}}$, $D\varphi_{[+1,-1]}$, and $D\varphi_{+\mathbb{1}}$,

$$D\phi(t, z; 0, (0, \delta)) = D\phi(t - s, \rho) D\varphi^+_{[+1,-1]}(0, \rho) D\varphi^+_{+\mathbb{1}}(0, \rho) D\phi(s, z) \begin{bmatrix} 0 \\ \begin{bmatrix} 0 \\ \delta \end{bmatrix} \end{bmatrix}, \tag{68}$$

since the perturbed trajectory $\phi(t, z + (0, \delta))$ passes through the interior of $D_{[+1,-1]}$. Combining (17), (65), and (66), after some algebra we obtain

$$D\varphi^+_{[+1,-1]}(0, \rho) D\varphi^+_{-\mathbb{1}}(0, \rho) = D\varphi^+_{-\mathbb{1}}(0, \rho). \tag{69}$$

In other words, $D\phi(t, z; 0, (0, \delta))$ is unaffected by flow through $D_{[+1,-1]}$. Intuitively, the time spent flowing through any region between surfaces that meet at a tangency at $\rho \in D$ depends quadratically on the distance from ρ; therefore it does not affect the first–order approximation of the flow through ρ. If $r > 1$ B–derivatives of the flow are desired, then it would be necessary to take these higher–order effects into account when evaluating the desired higher–order derivative.

7.1.6. Variational equation for event–selected C^r vector field. By synthesizing the preceding observations, we now provide a generalization of the variational equation in (58) applicable to the piecewise–differentiable flow yielded by an event–selected C^r vector field. We wish to evaluate $D\phi(t, x; v, w)$ where F is event–selected C^r at $\rho = \phi(s, x)$ for some $s \in (0, t)$ and F is C^r at every point in $\phi([0, t] \setminus \{s\}, x)$, and where $(v, w) \in T_{(t,x)}\mathcal{F}$. By (65), the desired derivative can be obtained by solving a jump–linear time–varying differential equation. With $\omega : \{1, \ldots, m\} \to B_n$ denoting the *word* associated with the tangent vector (v, w) from (65) and letting $\Xi_\omega \in \mathbb{R}^{(d+1)\times(d+1)}$ be the saltation matrix from (67), if $(\lambda, \xi) : [0, t] \to \mathbb{R} \times \mathbb{R}^d$ satisfies

$$\begin{gathered}\forall u \in [0,t] \setminus \{s\} : \begin{bmatrix} \dot\lambda(u) \\ \dot\xi(u) \end{bmatrix} = \begin{bmatrix} 0 \\ D_x F(\phi(u,x))\xi(u) \end{bmatrix}, \\ \begin{bmatrix} \lambda(0) \\ \xi(0) \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}, \begin{bmatrix} \lambda(s) \\ \xi(s) \end{bmatrix} = \Xi_\omega \begin{bmatrix} \lambda(s^-) \\ \xi(s^-) \end{bmatrix},\end{gathered} \tag{70}$$

then the B–derivative of the flow is given by

$$D\phi(t,x;v,w) = F(\phi(t,x))\lambda(t) + \xi(t). \tag{71}$$

More generally, (65) indicates that the selection functions for the piecewise–differentiable flow ϕ are indexed by the set of *words*, i.e., functions from $\{1,\dots,m\}$ into B_n that specify the sequence of regions a perturbed trajectory could pass through when flowing from $D_{-\mathbb{1}}$ to $D_{+\mathbb{1}}$:

$$\Omega = \{\omega : \{1,\dots,m\} \to B_n \mid m \le n,\ \omega \text{ is injective and increases from } -\mathbb{1} \text{ to } +\mathbb{1}\}; \tag{72}$$

here the phrase *ω increases from $-\mathbb{1}$ to $+\mathbb{1}$* means that $\omega(1) = -\mathbb{1}$, $\omega(m) = +\mathbb{1}$, and for each $j \in \{1,\dots,m-1\}$ there exists $I_j \subset \{1,\dots,n\}$ such that $\omega(j+1) - \omega(j) = 2\sum_{i\in I_j} e_i$. To evaluate the (Fréchet) derivative for the selection function ϕ_ω indexed by $\omega \in \Omega$, we solve a matrix–valued jump–linear time–varying differential equation to obtain $(\Lambda_\omega^\top, X_\omega) : [0,t] \to \mathbb{R}^{(d+1)\times d}$ via

$$\begin{gathered}\forall u \in [0,t] \setminus \{s\} : \begin{bmatrix} \dot\Lambda_\omega^\top(u) \\ \dot X_\omega(u) \end{bmatrix} = \begin{bmatrix} 0 \\ D_x F(\phi(u,x))X_\omega(u) \end{bmatrix}, \\ \begin{bmatrix} \Lambda_\omega^\top(0) \\ X_\omega(0) \end{bmatrix} = \begin{bmatrix} 0 \\ I_d \end{bmatrix}, \begin{bmatrix} \Lambda_\omega^\top(s) \\ X_\omega(s) \end{bmatrix} = \Xi_\omega \begin{bmatrix} \Lambda_\omega^\top(s^-) \\ X_\omega(s^-) \end{bmatrix}.\end{gathered} \tag{73}$$

Then the B–derivative of the selection function ϕ_ω with respect to state is given by

$$D_x\phi_\omega(t,x) = F(\phi(t,x))\Lambda_\omega^\top(t) + X_\omega(t). \tag{74}$$

As we demonstrate in the following section, evaluating (74) for all words $\omega \in \Omega$ provides a straightforward computational procedure[11] to check contractivity of a Poincaré map associated with a periodic orbit.

7.2. Stability of a periodic orbit.

We assume we are given an event–selected C^r vector field $F \in EC^r(D)$ over an open domain $D \subset \mathbb{R}^d$ containing a periodic orbit $\gamma : \mathbb{R} \to D$. Theorem 4 (local flow) and Corollary 5 (global flow) together yield a maximal flow $\phi \in PC^r(\mathcal{F}, D)$ for F. Theorem 10 (Poincaré map) yields a Poincaré map $P \in PC^r(S, \Sigma)$ over any local section $\Sigma \subset D$ that intersects $\Gamma = \gamma(\mathbb{R})$ at $\{\rho\} = \Gamma \cap \Sigma$. The Bouligand derivative $DP : TS \to T\Sigma$ of this piecewise–differentiable Poincaré map can be used to assess local exponential stability of the fixed point $P(\rho) = \rho$, as the following corollary shows; this generalizes Proposition 3 in [30] to stability of fixed points for arbitrary PC^r functions.

[11]Though straightforward, this procedure can be laborious since the number of elements in Ω grows factorially with the number n of surfaces of discontinuity.

Proposition 14 (contractivity test for stability of a periodic orbit). *Suppose $P \in PC^r(S, \Sigma)$ where $S \subset \Sigma$ has a fixed point $P(\rho) = \rho$ and DP is a contraction over tangent vectors near ρ; i.e., there exists $c \in (0, 1)$, $\delta > 0$, and $\|\cdot\| : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$ such that*

$$\forall x \in B_\delta(\rho) \subset S \cap \Sigma,\ v \in T_x\Sigma : \|DP(x; v)\| \le c\,\|v\|\,. \tag{75}$$

Then γ is an exponentially stable periodic orbit.

Proof. By the fundamental theorem of calculus [45, Proposition 3.1.1],

$$\begin{aligned}\forall x, y \in \overline{B_\delta(\rho)} : \|P(x) - P(y)\| &\le \int_0^1 \|DP(y + s(x-y); x-y)\|\, ds \\ &\le c\,\|x - y\|\,.\end{aligned} \tag{76}$$

We conclude that P is a contraction over the compact ball $\overline{B_\delta(\rho)}$, whence by the Banach contraction mapping principle [8], [35, Lemma C.35] its unique fixed point $P(\rho) = \rho$ is exponentially stable. ∎

In the remainder of this section we consider the case where P is a Poincaré map associated with a periodic orbit in an event–selected C^r vector field and demonstrate how the B–derivative of P can be obtained from the B–derivative of the flow ϕ. This provides a straightforward computational procedure to determine whether the contraction hypothesis in the above Proposition is satisfied using the variational equation developed in section 7.1.

To simplify the exposition, and since Lemma 6 (EC^r implies C^r almost everywhere) ensures that F is C^r at all but finitely many points in Γ, in the remainder of this section we let $\rho \in \Gamma$ be such that F is C^r at ρ. Let $\mu \in C^r(V, \mathbb{R})$ be the time–to–impact map for Σ on a neighborhood $V \subset D$ containing ρ; note that V can be chosen sufficiently small to ensure μ is continuously (as opposed to piecewise) differentiable since F is C^r at ρ. Let $\psi \in C^r(V, \Sigma)$ be the impact map given by $\psi(x) = \phi(\mu(x), x)$ for all $x \in V$; again note that ψ is continuously differentiable. By continuity of the flow there exists a neighborhood $U \subset S \subset \Sigma$ of ρ sufficiently small to ensure $\{\phi(t, x) : x \in U\} \subset V$, whence we have the equality

$$\forall x \in U : P(x) = \psi \circ \phi(t, x). \tag{77}$$

Applying the chain rule [45, Theorem 3.1.1] we find that

$$\forall w \in T_\rho\Sigma : DP(\rho; w) = D\psi(\rho) D\phi(t, \rho; 0, w), \tag{78}$$

where $D\psi(\rho) \in \mathbb{R}^{(d-1)\times d}$ is the (Fréchet) derivative of ψ. Following the conventions from section 7.1, let $\{\phi_\omega\}_{\omega\in\Omega}$ denote the set of selection functions for the flow ϕ. Now satisfying the contractivity condition (75) from Proposition 14 (contractivity test for periodic orbit stability), namely that DP is a contraction over tangent vectors near ρ, is clearly equivalent to finding $c \in (0, 1)$ and $\|\cdot\| : \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\forall \omega \in \Omega,\ w \in T_\rho\Sigma : \|D\psi(\rho) D_x\phi_\omega(t, \rho)(0, w)\| \le c\,\|w\|\,. \tag{79}$$

We emphasize that a single norm must be found relative to which the inequality in (79) is satisfied for all $\omega \in \Omega$; it would not suffice, for instance, to merely ensure that all the eigenvalues of $D\psi(\rho) D_x\phi_\omega(t, \rho)$ reside in the open unit ball.

The condition in (79) is equivalent to requiring that the *induced norm* of the linear operator $D\psi(\rho)D_x\phi_\omega(t,\rho)$ satisfy a bound on the induced norm,

$$\forall\omega \in \Omega : \|D\psi(\rho)D_x\phi_\omega(t,\rho)\|_i \le c. \tag{80}$$

These observations are summarized formally in the following proposition.

Proposition 15 (induced norm test for periodic orbit stability). *Let D be an open domain, suppose $\gamma : \mathbb{R} \to D$ is a t–periodic orbit for $F \in EC^r(D)$, let $\phi \in PC^r(\mathcal{F}, D)$ denote the maximal flow for F, and let $\{\phi_\omega\}_{\omega\in\Omega}$ denote a set of selection functions for ϕ. Let $\Sigma \subset D$ be a local section for F such that F is C^r at $\{\rho\} = \Gamma \cap \Sigma$ where $\Gamma = \gamma(\mathbb{R})$, and let $\psi \in C^r(V, \mathbb{R})$ be the impact map for Σ over a neighborhood $V \subset D$ containing ρ such that $F|_V$ is C^r. If there exist $c \in (0, 1)$ and $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}^d$ such that* (80) *holds, then γ is an exponentially stable periodic orbit.*

Remark 5. *As noted above,* (80) *is equivalent to stipulating that DP is a contraction over tangent vectors near ρ, which is the contractivity condition from Proposition* 14 *(contractivity test for periodic orbit stability). In* [30]*, Ivanov considered the stability of a fixed point of a piecewise–defined map. It is clear from his exposition that* [30, Proposition 3] *is intended to apply to the Poincaré map P associated with a periodic orbit that passes through multiple surfaces of discontinuity, but it was not shown in* [30] *that the Poincaré map in question had the piecewise–differentiable structure stipulated in* [30, Proposition 3]*. We demonstrate that P has the piecewise–defined form assumed in* [30, (3.1)] *and rigorously derive a stability condition in Proposition* 15 *(induced norm test for periodic orbit stability) that is equivalent to that in* [30, Proposition 3].

Remark 6. *In Proposition* 15 *(induced norm test for periodic orbit stability), the problem of finding the norm that ensures* (80) *holds is equivalent to that of finding a common Lyapunov function for a switched linear system, which remains an open problem in the theory of switched systems. We refer the interested reader to* [37, Section II–A] *for a survey of state–of–the–art approaches to this problem.*

8. Applications. We now illustrate the applicability of these results by appealing to a very simple family of event–selected C^r fields that abstractly captures the essential nature of the discontinuities arising in the physical settings mentioned in section 1. For instance, integrate–and–fire neuron models consist of a population of n subsystems that undergo a discontinuous change in membrane voltage and synaptic capacitance triggered by crossing a voltage threshold [9,29,32]. Since the discontinuities in state are confined to independent "reset" translations in membrane voltages [32, equation (2)], these transitions can be modeled locally as a first–order discontinuity in an event–selected C^r vector field. In the context of electrical power networks, when constituent elements—lines, cables, and transformers—encounter excessive voltages or currents, they trip fail–safe mechanisms that discontinuously change connectivity between elements [26, section II-A.2]. As discussed in section 2, legged animals and robots with four, six, and more limbs exhibit gaits with near–simultaneous touchdown of two or more legs [4,20,28]; since each touchdown introduces a discontinuity in velocities and/or forces, these transitions can give rise to first– and/or second–order discontinuities in an event–selected C^r vector field.

Motivated by these applications in neuroscience, electrical engineering, and biological and robotic locomotion, we now apply the results derived in the previous sections to analyze the effect of flowing near the intersection of multiple surfaces of discontinuity generated by a very simple but illustrative family of step functions. As noted in section 7.1, to describe this effect in general one must solve a collection of variational equations that grows factorially with the number of surfaces of discontinuity. Thus for clarity in sections 8.1 and 8.2 we focus on a simple family of examples arising from the presence of a generalized signum function. We demonstrate that populations of phase oscillators in both first– and second–order versions of this setting can be synchronized via piecewise–constant feedback.

8.1. Synchronization of first–order phase oscillators. In this section we study synchronization in a system consisting of d *first–order phase oscillators*, i.e., a control system of the form

$$\dot{q} = \nu \mathbb{1} + u(q), \tag{81}$$

where $q \in Q = (S^1)^d$, $\nu \in \mathbb{R}$ is a constant, and $u : Q \to TQ$ is a state–dependent feedback law. The state space is the d–dimensional torus $Q = (S^1)^d = \mathbb{R}^d/\mathbb{Z}^d$; we let $\pi : \mathbb{R}^d \to Q$ denote the canonical quotient projection, considered as a covering map [35, Appendix A]. In this section, we propose a piecewise–constant form for u and prove that it renders the synchronized orbit

$$\Gamma = \{q \in Q \mid \forall i, j \in \{1, \dots, d\} : q_i = q_j\} \tag{82}$$

locally exponentially stable for (81).

8.1.1. B–derivative of flow in Euclidean covering space via saltation matrices. First, we work in the Euclidean covering space, considering the vector field $F : \mathbb{R}^d \to T\mathbb{R}^d$ defined by

$$\forall x \in \mathbb{R}^d : F(x) = \nu \mathbb{1} - \delta \operatorname{sign}(x), \tag{83}$$

where $0 < \delta < \nu$ is a given constant and $\operatorname{sign} : \mathbb{R}^d \to B_d$ is the vectorized signum function defined as in (14).[12] Clearly F is event–selected C^∞ on $\mathbb{R}^d$ since the event surfaces coincide with the d standard coordinate planes; for clarity we let $0_d \in \mathbb{R}^d$ denote the intersection point (i.e., the origin). Let $\phi : \mathcal{F} \to \mathbb{R}^d$ be the global flow for F yielded by Corollary 5.

We aim to compute the B–derivative of the flow with respect to state along the trajectory passing through 0_d. For clarity we outline the computation here and relegate a detailed derivation to SM2.1. For any word $\omega \in \Omega$ we can obtain the derivative of the selection function ϕ_ω with respect to state from (74),

$$D_x \phi_\omega(0, 0_d) = F(\phi(0, x))\Lambda_\omega^\top(0) + X_\omega(0) = \Xi_\omega \begin{bmatrix} 0_d \\ I_d \end{bmatrix}, \tag{84}$$

[12] We note that there are three common definitions for the scalar signum function, depending on what value one chooses to assign to $0 \in \mathbb{R}$, and hence 3^d candidate definitions for a vectorized version. Since integral curves for F spend zero time at the signum's zero crossing, there is no loss of generality in our choice.

since $\Lambda_\omega^\top(0) = 0_d$ and $X_\omega(0) = I_d$. The saltation matrix Ξ_ω, given in general by (67), simplifies in this example to (SM13), whence we conclude as in (SM17) that

$$\forall \omega \in \Omega : D_x\phi_\omega(0, 0_d) = \frac{\nu - \delta}{\nu + \delta} I_d. \tag{85}$$

This shows that ϕ is in fact C^1 with respect to state at $(0, 0_d) \in \mathcal{F}$, and hence

$$\forall w \in T_{0_d}\mathbb{R}^d : D\phi(0, 0_d; 0, w) = \frac{\nu - \delta}{\nu + \delta} w; \tag{86}$$

i.e., the first–order effect of the nonsmooth flow associated with this piecewise–constant vector field is linear contraction at rate $\frac{\nu-\delta}{\nu+\delta}$ independent of the direction $w \in T_{0_d}\mathbb{R}^d$.

8.1.2. B–derivative of flow in Euclidean covering space via flowbox. Before continuing with the task at hand—namely, applying feedback of the form in (83) to demonstrate synchronization of the first–order phase oscillators in (81)—we digress momentarily to provide an alternate derivation of the result in (86) that yields additional intuition. This alternate derivation may be seen as a d–dimensional generalization of the construction introduced in section 2.3 to study the first–order locomotion model of section 2.1.

Let $\chi_0, \chi_0^{-1} : \mathbb{R} \to \mathbb{R}$ be the piecewise–linear homeomorphisms defined by

$$\forall s \in \mathbb{R} : \chi_0(s) = \begin{cases} s, & s < 0, \\ \frac{\nu+\delta}{\nu-\delta} s, & s \geq 0, \end{cases} \quad \forall \widetilde{s} \in \mathbb{R} : \chi_0^{-1}(\widetilde{s}) = \begin{cases} \widetilde{s}, & \widetilde{s} < 0, \\ \frac{\nu-\delta}{\nu+\delta} \widetilde{s}, & \widetilde{s} \geq 0, \end{cases} \tag{87}$$

and let $\chi, \chi^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ be the piecewise–linear homeomorphisms defined by

$$\forall x \in \mathbb{R}^d : \chi(x) = \left(\chi_0(x_1), \ldots, \chi_0(x_d)\right), \ \forall \widetilde{x} \in \mathbb{R}^d : \chi^{-1}(\widetilde{x}) = \left(\chi_0^{-1}(\widetilde{x}_1), \ldots, \chi_0^{-1}(\widetilde{x}_d)\right). \tag{88}$$

Note that $\chi_0 \circ \chi_0^{-1} = \mathrm{id}_{\mathbb{R}}$ and hence $\chi \circ \chi^{-1} = \mathrm{id}_{\mathbb{R}^d}$. Since furthermore $\chi \in PC^r(\mathbb{R}^d, \mathbb{R}^d)$, there is no ambiguity in the definition of the "pushforward" $\widetilde{F} = D\chi \circ F \circ \chi^{-1} : \mathbb{R}^d \to T\mathbb{R}^d$. In fact, the vector field $\widetilde{F}$ is constant,

$$\forall \widetilde{x} \in \mathbb{R}^d : \widetilde{F}(\widetilde{x}) = (\nu + \delta)\mathbb{1}, \tag{89}$$

and hence its flow $\widetilde{\phi} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ has the simple form

$$\forall (t, \widetilde{x}) \in \mathbb{R} \times \mathbb{R}^d : \widetilde{\phi}(t, \widetilde{x}) = \widetilde{x} + t(\nu + \delta)\mathbb{1}. \tag{90}$$

Since the homeomorphism χ provides conjugacy between the flows, we have

$$\forall (t, x) \in \mathcal{F} : \chi \circ \phi(t, x) = \widetilde{\phi}(t, \chi(x)) = \chi(x) + t(\nu + \delta)\mathbb{1}. \tag{91}$$

If $t \in \mathbb{R}$ and $x, w \in \mathbb{R}^d$ are such that $x, x+w, x-w \in D_{-\mathbb{1}}$ and $\phi(t, x), \phi(t, x+w), \phi(t, x-w) \in D_{+\mathbb{1}}$, the conjugacy in (91) can be used to evaluate the B–derivative of the flow $D\phi$, since

$$\begin{aligned} \phi(t, x + sw) &= \chi^{-1}\left(\chi(x + sw) + t(\nu + \delta)\mathbb{1}\right) \\ &= \frac{\nu - \delta}{\nu + \delta}\left((x + sw) + t(\nu + \delta)\mathbb{1}\right) \\ &= \frac{\nu - \delta}{\nu + \delta}(x + sw) + t(\nu - \delta)\mathbb{1}, \end{aligned} \tag{92}$$

and hence

$$\lim_{s\to 0^+} \frac{1}{s}\left(\phi(t, x + sw) - \phi(t, x)\right) = \frac{\nu - \delta}{\nu + \delta} w, \tag{93}$$

whence (86) follows directly. We emphasize that this outcome—the piecewise–differentiable flow is C^1 with respect to state—will not arise in general, but note that other examples in this vein can be obtained by applying other piecewise–linear homeomorphisms to a constant vector field (i.e., a flowbox) so long as the constant vector field is transverse to surfaces of nonsmoothness for the homeomorphism (needed to ensure the vector field is event–selected C^r).

We conclude by noting that this approach to computing $D\phi$ required a closed–form expression for the "flowbox" homeomorphism χ and its inverse χ^{-1}, which is equivalent to possessing a closed–form expression for the flow ϕ. Since such expressions are rarely available in applications of interest, in general we expect to rely on the technique developed in section 7.1 to compute the B–derivative of the flow.

8.1.3. Synchronization via piecewise–constant feedback.

Now, returning to the state space of interest, let $U_\Delta \subset Q$ be the following open set parameterized by $\Delta > 0$:

$$U_\Delta = \left\{ q \in TQ \mid \exists x \in \pi^{-1}(q) : \|x\|_1 \le \frac{\Delta}{d} \right\}; \tag{94}$$

for $\Delta > 0$ sufficiently small, U_Δ is "evenly covered" in the sense that $\pi|_{\pi^{-1}}(U_\Delta)$ is a homeomorphism [35, Appendix A]. Consider the effect of applying feedback of the form

$$\forall q \in Q : u(q) = \begin{cases} -\delta \operatorname{sign} \circ \pi^{-1}(q), & q \in U_\Delta, \\ 0, & q \in Q \setminus U_\Delta, \end{cases} \tag{95}$$

to (81). It is straightforward to show (as we do in SM2.1) that the synchronized orbit Γ defined in (82) is a periodic orbit for (81) under this feedback; we note that the closed–loop dynamics determine an event–selected C^∞ vector field on a neighborhood of Γ.

Now we choose a local section $\Sigma \subset Q \setminus U_\Delta$ for the closed–loop dynamics that is perpendicular to Γ and let $P \in PC^\infty(S, \Sigma)$ denote a Poincaré map for Γ over a neighborhood $S \subset \Sigma$ containing $\{\rho\} = \Gamma \cap \Sigma$. To compute $DP(\rho)$ we employ (78), which involves solving the jump–linear time–varying differential equation (73) with the saltation matrix update given by (86). Note that away from discontinuities introduced by the feedback (95), the vector field in (81) does not depend on the state. This implies that $D_x F \equiv 0$; hence the continuous–time portion of the variational dynamics (73) does not alter the derivative computation.

Focusing our attention on the discrete–time (saltation matrix) portion of the variational dynamics (73), the closed–loop dynamics are discontinuous at three points in Γ: $\{-\Delta\mathbb{1}, 0_d, +\Delta\mathbb{1}\}$. At 0_d, the saltation matrix is given by (86). At $\pm\Delta\mathbb{1}$, the update is determined by a single event surface that we chose to be perpendicular to Γ; although these updates affect $D\phi$, they have no effect on DP since they lie in the kernel of $D\psi$ in (78). We conclude that P is C^1 and

$$DP(\rho) = \frac{\nu - \delta}{\nu + \delta} I_{d-1}. \tag{96}$$

Therefore, the induced norm contraction hypothesis of Proposition 15 (induced norm test for periodic orbit stability) is satisfied with the standard Euclidean norm and $c = \frac{\nu-\delta}{\nu+\delta}$. We conclude that Γ is exponentially stable, whence the state feedback in (95) synchronizes the first–order phase oscillators in (81) at an exponential rate.

8.2. Synchronization of second–order phase oscillators. In this section we study synchronization in a system consisting of d *second–order phase oscillators*, i.e., a control system of the form

$$\ddot{q} = \alpha\mathbb{1} - \beta\dot{q} + u(q, \dot{q}), \tag{97}$$

where $q \in Q = \mathbb{R}^d/\mathbb{Z}^d$, $\alpha, \beta \in \mathbb{R}$ are constants, and $u : TQ \to T^*Q$ is a state–dependent feedback law. The state space is the tangent bundle TQ of the d–dimensional torus $Q = \mathbb{R}^d/\mathbb{Z}^d$; we let $\pi : \mathbb{R}^{2d} \to TQ$ denote the canonical quotient projection.

If $u \equiv \mu\mathbb{1}$ where $\mu \in \mathbb{R}$ is a constant, then (97) reduces to d decoupled cascades of a pair of scalar affine time–invariant systems; thus it is clear that $\ddot{q} \to 0$, and hence $\dot{q} \to \frac{\alpha+\mu}{\beta}\mathbb{1}$ as $t \to \infty$; this convergence is exponential with rate β. In this section, we propose a piecewise–constant form for the feedback u and prove that for all β sufficiently large there exists an exponentially stable periodic orbit that passes near $(0, \frac{\alpha}{\beta}\mathbb{1}) \in TQ$.

8.2.1. B–derivative of flow in Euclidean covering space. First, consider the vector field $F : \mathbb{R}^{2d} \to T\mathbb{R}^{2d}$ defined by

$$\forall (x, \dot{x}) \in \mathbb{R}^{2d} : F(x, \dot{x}) = \begin{bmatrix} \dot{x} \\ \alpha\mathbb{1} - \beta\dot{x} - \delta\,\mathrm{sign}(x) \end{bmatrix}, \tag{98}$$

where $0 < \delta < \alpha$ is a given constant. Clearly F is event–selected C^∞ on the open set

$$D = \left\{ (x, \dot{x}) \in \mathbb{R}^{2d} \mid \forall j \in \{1, \ldots, d\} : \dot{x}_j \neq 0 \right\} \subset \mathbb{R}^{2d} \tag{99}$$

since the event surfaces coincide with the first d standard coordinate planes in $\mathbb{R}^{2d}$; since F fails to be event–selected C^r at points with zero velocity, we exclude them from our analysis. Let $\phi : \mathcal{F} \to D$ denote the global flow for F yielded by Corollary 5.

We begin by computing the B–derivative of the flow with respect to state along the trajectory passing through a point $(0, \nu\mathbb{1}) \in D$ where $\nu > 0$. For clarity we outline the computation here and relegate a detailed derivation to SM2.2. For any word $\omega \in \Omega$ we can obtain the derivative of the selection function ϕ_ω with respect to state from (74),

$$D_x\phi_\omega(0, (0, \nu\mathbb{1})) = F(\phi(0, x))\Lambda_\omega^\top(0) + X_\omega(0) = \Xi_\omega \begin{bmatrix} 0_{2d} \\ I_{2d} \end{bmatrix}, \tag{100}$$

since $\Lambda_\omega^\top(0) = 0_{2d}$ and $X_\omega(0) = I_{2d}$. The saltation matrix Ξ_ω, given in general by (67), simplifies in this example to (SM26), whence we conclude as in (SM30) that

$$D_x\phi_\omega(0, (0, \nu\mathbb{1})) = \begin{bmatrix} I_d & 0 \\ -\frac{2\delta}{\nu} I_d & I_d \end{bmatrix}. \tag{101}$$

This shows that ϕ is in fact C^1 with respect to state at $(0,(0,\nu\mathbb{1}))\in\mathcal{F}$, and hence

$$\forall(p,\dot p)\in T_{(0,\nu\mathbb{1})}D : D\phi(0,(0,\nu\mathbb{1});0,(p,\dot p)) = \begin{bmatrix} I_d & 0 \\ -\frac{2\delta}{\nu}I_d & I_d \end{bmatrix}\begin{bmatrix} p \\ \dot p \end{bmatrix} =: \Xi \begin{bmatrix} p \\ \dot p \end{bmatrix}, \tag{102}$$

i.e., the first–order effect of the nonsmooth flow associated with this piecewise–constant vector field is a change in velocity $\dot p \mapsto \dot p - \frac{2\delta}{\nu}p$ that is proportional to the error in position p. Solving the variational equation as in section 7.1, a straightforward calculation (given for completeness in SM2.2) yields

$$\begin{bmatrix} p(s) \\ \dot p(s) \end{bmatrix} = \begin{bmatrix} I_d & \frac{1}{\beta}\left(1-e^{-\beta s}\right)I_d \\ 0 & e^{-\beta s}I_d \end{bmatrix}\begin{bmatrix} p(0^+) \\ \dot p(0^+) \end{bmatrix} =: X(s)\begin{bmatrix} p(0^+) \\ \dot p(0^+) \end{bmatrix}, \tag{103}$$

where $(p(0^+),\dot p(0^+))$ is determined from $(p(0),\dot p(0))\in T_{(0,\nu\mathbb{1})}D$ by applying (102). Combining (102) with (103), we conclude that the B–derivative with respect to state at time s is given by

$$\begin{aligned} \begin{bmatrix} p(s) \\ \dot p(s) \end{bmatrix} &= D\phi\left(s, \begin{bmatrix} x(0) \\ \dot x(0) \end{bmatrix}; 0, \begin{bmatrix} p(0) \\ \dot p(0) \end{bmatrix}\right) \\ &= X(s)\,\Xi \begin{bmatrix} p(0) \\ \dot p(0) \end{bmatrix} \\ &= \begin{bmatrix} I_d & \frac{1}{\beta}\left(1-e^{-\beta s}\right)I_d \\ 0 & e^{-\beta s}I_d \end{bmatrix}\begin{bmatrix} I_d & 0 \\ -\frac{2\delta}{\nu}I_d & I_d \end{bmatrix}\begin{bmatrix} p(0) \\ \dot p(0) \end{bmatrix} \\ &= \begin{bmatrix} I_d - \frac{2\delta}{\beta\nu}\left(1-e^{-\beta s}\right)I_d & \frac{1}{\beta}\left(1-e^{-\beta s}\right) \\ -\frac{2\delta}{\nu}e^{-\beta s}I_d & e^{-\beta s}I_d \end{bmatrix}\begin{bmatrix} p(0) \\ \dot p(0) \end{bmatrix}. \end{aligned} \tag{104}$$

Taking the limit as $s\to\infty$,

$$\begin{bmatrix} p(\infty) \\ \dot p(\infty) \end{bmatrix} = \lim_{s\to\infty}\begin{bmatrix} p(s) \\ \dot p(s) \end{bmatrix} = \begin{bmatrix} \left(1-\frac{2\delta}{\beta\nu}\right)p(0) \\ 0 \end{bmatrix}. \tag{105}$$

In plain language, (105) indicates that, to first order, the nonsmooth flow associated with the vector field (98) asymptotically (i) drives the initial velocity error $\dot p(0)$ to zero and (ii) multiplies the initial position error $p(0)$ by a factor of $c=(1-\frac{2\delta}{\beta\nu})$. If we ensure $\nu\in(\frac{\alpha}{\beta},\frac{\alpha+\delta}{\beta})$, then $c=(1-\frac{2\delta}{\beta\nu})\in(1-\frac{2\delta}{\alpha},1-\frac{2\delta}{\alpha+\delta})\subset(-1,+1)$, achieving contraction in positions. Finally, we note that the convergence in (105) is exponential with rate β.

8.2.2. Synchronization via piecewise–constant feedback. We now apply a construction analogous to that of section 8.1 to define a piecewise–constant feedback law that results in an exponentially stable periodic orbit that passes near $(0,\frac{\alpha}{\beta}\mathbb{1})\in TQ$. To that end, consider the following form for the control neighborhood $U_\Delta\subset TQ$ parameterized by $\Delta>0$:

$$\begin{aligned} U_\Delta = \Big\{(q,\dot q)\in TQ \mid &\left(\exists(x,\dot x)\in\pi^{-1}(q,\dot q) : \|x\|_1 \le \frac{\Delta}{d}\right) \\ &\wedge (\forall j\in\{1,\dots,d\} : \dot q_j > 0)\Big\}; \end{aligned} \tag{106}$$

for $\Delta > 0$ sufficiently small, U_Δ is "evenly covered" in the sense that $\pi|_{\pi^{-1}}(U_\Delta)$ is a homeomorphism [35, Appendix A]. Furthermore, "synchronized" points of the form $(\pm\Delta\mathbb{1}, \nu\mathbb{1})$ where $\nu > 0$ are in the boundary ∂U_Δ. We study the effect of applying feedback of the form

$$\forall(q,\dot{q}) \in TQ : u(q,\dot{q}) = \begin{cases} -\delta \operatorname{sign} \circ \pi^{-1}(q,\dot{q}), & (q,\dot{q}) \in U_\Delta, \\ 0, & (q,\dot{q}) \in TQ \setminus U_\Delta, \end{cases} \tag{107}$$

to (97). It is straightforward to show (as we do in SM2.2) that for all $\beta > 0$ sufficiently large there exists $\nu_\beta \in (\frac{\alpha}{\beta}, \frac{\alpha+\delta}{\beta})$ such that the trajectory initialized at $(0, \nu_\beta \mathbb{1})$ is periodic for the dynamics in (97) subject to the piecewise–constant forcing (107). We let $\Gamma_\beta \subset TQ$ denote the image of the periodic orbit, and let ν_β^- (resp., $\nu_\beta^+ > 0$) denote the speed of the orbit when the position is equal to $-\Delta\mathbb{1}$ (resp., $+\Delta\mathbb{1}$) so that $(-\Delta\mathbb{1}, \nu_\beta^-\mathbb{1}) \in \Gamma_\beta$ (resp., $(+\Delta\mathbb{1}, \nu_\beta^+\mathbb{1}) \in \Gamma_\beta$). Note that, by increasing β, ν_β^- can be made arbitrarily close to $\frac{\alpha}{\beta}$ and ν_β can be made arbitrarily close to $\frac{\alpha+\delta}{\beta}$, whence $\nu_\beta \in (\frac{\alpha}{\beta}, \frac{\alpha+\delta}{\beta})$. Further, note that the closed–loop dynamics determine an event–selected C^∞ vector field on a neighborhood of Γ_β.

Now we choose a local section $\Sigma_\beta \subset TQ \setminus U_\Delta$ for the closed–loop dynamics whose normal vector is parallel to $(\mathbb{1}, 0)$ at the point $\rho_\beta = (-\Delta\mathbb{1}, \nu_\beta^-\mathbb{1}) \in \Gamma_\beta \cap \Sigma_\beta$. Note that by construction $\Sigma_\beta \cap \partial U_\Delta$ is an open set containing ρ_β. Let $P_\beta \in PC^\infty(S_\beta, \Sigma_\beta)$ denote a Poincaré map for Γ_β over a neighborhood $S_\beta \subset \Sigma_\beta$ containing $\{\rho_\beta\}$. To compute $DP_\beta(\rho_\beta)$ we employ (78), which involves solving the jump–linear time–varying differential equation (73) with the saltation matrix update given by (102). Away from discontinuities introduced by the feedback (107), the state dependence of the vector field in (97) is confined to viscous drag on velocities. This implies that the continuous–time portion of the variational dynamics (73) is given by (103); i.e., the first–order effect of the flow contracts velocity error at an exponential rate and amplifies position error by an amount proportional to $1/\beta$.

Focusing our attention now on the discrete–time (saltation matrix) portion of the variational dynamics (73), the closed–loop dynamics are discontinuous at three points in Γ: $(-\Delta\mathbb{1}, \nu_\beta^-\mathbb{1})$, $(0, \nu\mathbb{1})$, and $(+\Delta\mathbb{1}, \nu_\beta^+\mathbb{1})$. At $(0, \nu\mathbb{1})$, the saltation matrix is given by (102). At $(\pm\Delta\mathbb{1}, \nu_\beta^\pm)$, the saltation matrix is determined by a single event surface whose normal vector is parallel to $(\mathbb{1}, 0)$. Although these updates affect $D\phi$, they have no effect on DP_β since they lie in the kernel of $D\psi$ in (78). We conclude that P_β is C^1 and

$$DP_\beta(\rho_\beta) = \begin{bmatrix} \left(1 - \frac{2\delta}{\beta\nu_\beta}\right) I_{d-1} & 0 \\ 0 & 0 \end{bmatrix} + E_\beta, \tag{108}$$

where the induced norm of the error term $\|E_\beta\|_i$ decreases exponentially with increasing β. Therefore, for all $\beta > 0$ sufficiently large the induced norm contraction hypothesis of Proposition 15 (induced norm test for periodic orbit stability) is satisfied with the standard Euclidean norm and $c \approx (1 - \frac{2\delta}{\beta\nu_\beta}) \in (1 - \frac{2\delta}{\alpha}, 1 - \frac{2\delta}{\alpha+\delta}) \subset (-1, +1)$. We conclude that Γ_β is exponentially stable for all $\beta > 0$ sufficiently large, whence the state feedback in (107) synchronizes the second–order phase oscillators in (97) at an exponential rate.

9. Discussion.

In this paper, we studied local properties of the flow generated by vector fields with "event–selected" discontinuities, that is, vector fields that are (i) smooth except along a finite collection of smooth submanifolds and (ii) "transverse" to these submanifolds in the sense that integral curves intersect them at isolated points in time. Basic properties of discontinuous vector fields have been studied in a more general setting, for instance yielding sufficient conditions ensuring existence of a continuous flow (see [18, Chapter 2] generally and [18, Theorem 3 in section 8] specifically). Our chief contribution is the introduction of techniques from nonsmooth analysis [45] to show that the flow associated with a vector field with event–selected discontinuities admits a strong first–order approximation, the (so–called [42]) *Bouligand* derivative. We employed this B–derivative to generalize fundamental constructions familiar from classical (smooth) dynamical systems theory, including impact maps, flowboxes, and variational equations, and to study the effect of perturbations, both infinitesimal and noninfinitesimal. In the classical setting, these constructions are obtained using the classical (alternately called Fréchet [45, section 3.1] or Jacobian [22, section 1.3]) derivative of the smooth flow; our construction of the nonsmooth object proceeded analogously to that of its smooth counterpart after replacing the classical derivative of the flow with our B–derivative. In future work, we expect to obtain generalizations of other techniques from the theories of smooth dynamical and control systems that depend primarily on the existence of first– or higher–order approximations of the flow, for instance stability analysis via control Lyapunov functions [6, 49] or infinitesimal contractivity [38, 50]; conditions for controllability based on the inverse function theorem [36, Theorem 8], [14, section II.C]; or necessary and sufficient conditions for optimality in nonlinear programs involving dynamical systems [40, Chapter 4].

More broadly, we believe our results support the study of a class of discontinuous vector fields that arise in neuroscience [32], electrical engineering [26], and biological and robotic locomotion [28]. In each of these disparate domains, behaviors of interest occur near the intersection of surfaces of discontinuity; hence the techniques we developed in this paper may be brought to bear. Thus we conclude with remarks about the formal applicability and practical relevance of our results in these applications.

9.1. Neuroscience.

Integrate–and–fire neuron models consist of a population of n subsystems that undergo a discontinuous change in membrane voltage triggered by crossing a voltage threshold [32],

$$\begin{gathered} \dot{v} = -\gamma v + u, \\ v(t^+) = 0 \text{ if } v(t^-) = \overline{v}, \end{gathered} \tag{109}$$

where $v \in \mathbb{R}$ is the membrane voltage; $\gamma \in \mathbb{R}$ is a dissipation constant; $u \in \mathbb{R}$ is an exogenous input; and $\overline{v} \in \mathbb{R}$ is the firing threshold. When driven by a periodic exogenous input, integrate–and–fire neuron populations can exhibit *phase locking* [32] or *local synchronization* [29] behavior, resulting in simultaneous or near–simultaneous firing. Of interest in applications is the computation of so–called[13] "Lyapunov exponents" using variational equations. We showed in section 7.1 how the variational equation must be supplemented by discontinuous updates via saltation matrices near such simultaneous–firing events. As noted in [9, section 4.1], neglecting this nonsmooth effect can result in erroneous conclusions.

[13] In practice, one computes singular values of finite–time sensitivity matrices, rather than the formal asymptotic Lyapunov exponent, as defined, for instance, in [44, section 3.4.1].

9.2. Electrical engineering. Electrical power systems undergo discontinuous changes in network topology triggered by excessive voltages or currents, leading to differential–algebraic models of the form [26, (2)–(4)]

$$\begin{aligned} \dot{x} &= f(x, y, z; p), \ 0 = g(x, y, z; p), \\ z(t^+) &= h(x(t), y(t), z(t^-); p) \text{ if } y_j(t^-) = 0, \end{aligned} \tag{110}$$

where $x \in \mathbb{R}^d$ contains dynamic states; $y \in \mathbb{R}^n$ contains algebraic states; $z \in \mathbb{R}^m$ contains discrete states; $p \in \mathbb{R}^\ell$ contains parameters; $f : \mathbb{R}^{d+n+m+\ell} \to \mathbb{R}^d$ is a smooth vector field; $g : \mathbb{R}^{d+n+m+\ell} \to \mathbb{R}^k$ is a smooth constraint function; and $h : \mathbb{R}^{d+n+m+\ell} \to \mathbb{R}^m$ is a smooth reset function. The update $z(t^+) = h(x(t), y(t), z(t^-); p)$ is applied when one of the algebraic states y_j crosses a prespecified threshold (e.g., a bus voltage limit), causing a discontinuity in the vector field governing the time evolution of x. In electrical power networks, discrete switches triggered by overexcitation limits can occur at arbitrary times with respect to one another. When the switches occur at distinct time instants, the trajectory sensitivity matrix (i.e., the F–derivative of the flow) computed as in [27] can provide quantitative insights for design and control. However, as noted in [27, section VIII], these calculations lose accuracy when event times become coincident; this is due to the fact that the flow is not classically differentiable along trajectories that undergo simultaneous discrete transitions. The procedure we developed in section 7.1 can be employed to compute a collection of trajectory sensitivity matrices (i.e., the B–derivative of the flow) that generalize the approach advocated in [27] to be applicable in power networks that undergo an arbitrary (but finite) number of simultaneous discrete transitions.

9.3. Biological and robotic locomotion. Legged locomotion of animals and robots involves intermittent interaction of limbs with terrain; their dynamics are given by [31, section II]

$$\begin{aligned} M(q)\ddot{q} &= f(q, \dot{q}) + \lambda(q, \dot{q}) Da(q), \\ \dot{q}(t^+) &= R(q(t))\dot{q}(t^-) \text{ if } a_j(t^-) = 0, \end{aligned} \tag{111}$$

where $q \in Q$ is a vector of generalized coordinates for the body and limbs; M is the inertia tensor; f is a vector of internal, applied, and Coriolis forces; $a : Q \to \mathbb{R}^n$ specifies n unilateral constraints of the form

$$\forall j \in \{1, \ldots, n\} : a_j(q) \geq 0; \tag{112}$$

and λ is a vector of reaction forces that ensure (112) are satisfied by (111) for all time. The update $\dot{q}(t^+) = R(q(t))\dot{q}(t^-)$ is triggered when one of the unilateral constraints a_j would be violated by a penetrating velocity; it generally causes a discontinuity in both the velocity and the forces acting on the system. Legged animals and robots with four, six, and more limbs exhibit gaits with near–simultaneous touchdown of two or more legs [4, 20, 28]. Steady–state gaits are commonly modeled as periodic orbits in the body reference frame [33, 34, 46]. In practice, gait stability is assessed using the linearization of the first–return or Poincaré map, since if this map is smooth and the eigenvalues of the linearization (the so–called *Floquet multipliers* [22, section 1.5]) lie within the unit disk, then the gait is exponentially stable [3, 21].

We showed in section 5.2 that the Poincaré map associated with a periodic orbit passing through the intersection of multiple surfaces of discontinuity is generally nonsmooth. This implies that it is not possible to assess the stability of such orbits using the F–derivative; this F–derivative does not exist. In section 7.2, we showed how the B–derivative of the Poincaré map can be employed instead to assess the stability of such orbits.

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