# Leader Selection for Performance and Control of Complex Networks

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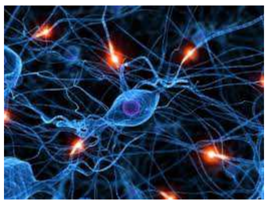
Joint work with Andrew Clark and Radha Poovendran

# A World of Networks



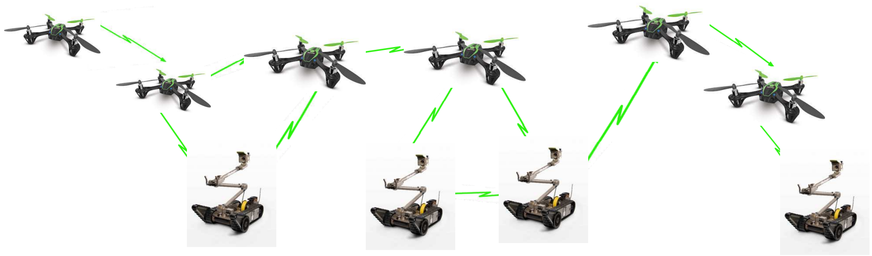


**Networks in nature**


**Man-made networks**

# Example: Search and Rescue

- Network: Consists of aerial and ground robots; wireless network
- Each node: computes relative location, trajectory & coverage
- Communication and coordination tasks are performed in a distributed, autonomous manner

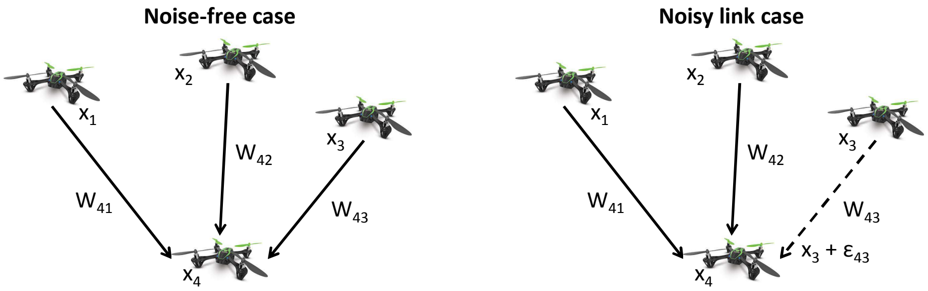




# Features

- Node
  - Mobile
  - Battery resource limited
  - Forms wireless links; Limited communication range; lossy channel between nodes
- Network
  - Time varying topology due to mobility and node/link failure
- Formation and Control
  - Needs to be distributed and adaptively computable
  - Control and communication protocols need to be resource efficient

# Node Dynamics

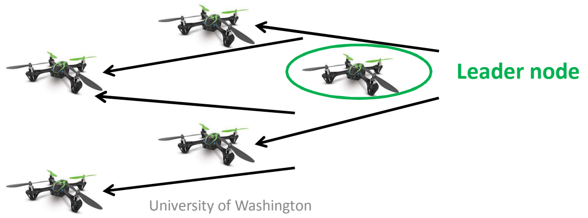



- Each node monitors and controls its own internal state (e.g., position, velocity, angular velocity)
- Receives state information as input from neighbors
- Computes and broadcasts internal state using predefined rule(s)
  - Common approach: weighted averaging of neighbor states
- Link noise affects the estimates and state update (e.g., errors in position, velocity estimates)



# Leader-Follower System

- In a large network, impractical to provide control inputs to each node
- Instead, small subset of leader nodes act as control inputs to aid/influence the remaining follower nodes to a desired state
- Leader inputs propagate through network via local state updates
  - Propagation causes delay before followers reach desired states



## Examples of Leader-Follower Systems

- Steering formations of unmanned vehicles
- Anchor-based localization in sensor networks
- Influence propagation in social networks
- Control of gene regulation and expression
- Synchronization of neuronal networks and biological oscillators
- Main questions:
  - What is/are the metrics for choosing leaders?
  - Given a specific metric, which nodes are the best/effective leaders?
  - How to efficiently compute the best leader set?



## An Example: A Flock of Boids

- In 2003, a filmmaker had a Problem:
  - How to simulate a group of hundreds of horses moving in a realistic, coordinated way?
  - Expensive (labor & CPU) to design a motion path for each horse
- Solution**:** Give a motion path for a few of the horses, and make the rest follow
- Idea originally proposed in [Reynolds `87] on simulating a flock of birds ("boids")


VIDEO

## Today's Talk: Leader Selection

- Metrics for choosing leaders:
  - Performance (robustness to link noise, convergence error)
  - Controllability
  - Joint performance and controllability
- Characterizing the optimal leader set:
  - Supermodular structure of leader selection metrics
- Efficiently computing the optimal leader set:
  - Polynomial-time algorithms with provable optimality gap



## Outline

- Motivating Application
- Leader-Follower System Requirements
- Leader Selection in Complex Networks
  - Robustness to link noise
  - Minimizing convergence error
  - Performance and controllability
- Conclusions and Future Work

# System Model

- Network of n nodes, indexed V = {1,…,n}, edge set E
- Each follower node has state $x_i(t)$ with dynamics

$$\dot{x}_i(t) = -\sum_{j \in N(i)} W_{ij}(x_i - x_j) + \epsilon_{ij}(t)$$

- $\varepsilon_{ij}(t)$ zero-mean white process with variance $\nu_{ij}$, $\nu_{ij} = \nu_{ji}$
- Nodes in the leader set, denoted S, maintain a constant state x*
- Weights $W_{ij}$ given by $W_{ij} = \nu_{ij}^{-1}$
- Dynamics have vector form $\dot{\mathbf{x}}_f(t) = -L\mathbf{x}(t) + \mathbf{w}(t)$ , where w is a zero-mean white process and L is defined by

$$L_{ij} = \begin{cases} W_{ij}, & (i,j) \in E \\ \sum_{l \in N(i)} W_{il}, & i = j \\ 0, & i \in S \\ 0, \text{else} & \end{cases}$$

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# Related Work – Link Noise

- Analysis of mean-square error due to noise in a network with given leaders and dynamics
  - Noise in agent state updates [Patterson & Bamieh 2010, Young et al 2010]
  - Noise in communication links [Barooah & Hespanha 2006]
  - Quantization noise [Kar & Moura 2009]
- Leader selection under link noise via convex relaxation [Lin et al 2011, Fardad et al 2011]
  - Does not provide optimality guarantees
- Our contribution: Efficient approach with provable bounds on the optimality of the leader set

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## Quantifying Error Due to Link Noise

- The Laplacian matrix L can be decomposed as

$$L = \left(\begin{array}{c|c} L_{ff} & L_{fl} \\ 0 & 0 \end{array}\right),$$

  - $L_{ff}$ and $L_{fl}$ represent the influence of followers and leaders
- Theorem (Barooah et al `06): The mean-square error in the follower node states in steady-state is equal to

$$\lim_{t\to\infty} \mathbf{E}||\mathbf{x}(t) - x^*\mathbf{1}||_2 = \mathbf{tr}(L_{ff}^{-1})$$

- Define the error due to link noise as the metric

$$R(S) = \mathbf{tr}(L_{ff}^{-1})$$

  - trace of the steady-state covariance matrix of follower nodes
- Define $R(S,u) = (L_{ff}^{-1})_{uu}$ for u∈V\S as the variance of each follower node



## Problem Formulation

- Selecting up to k leaders to minimize error due to link noise

$$\begin{array}{ll} \text{minimize} & R(S) \\ \text{s.t.} & |S| \leq k \end{array}$$

- Selecting the minimum-size leader set to achieve a bound α on error due to link noise

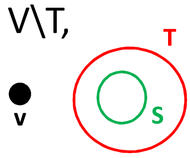
$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & R(S) \leq \alpha \end{array}$$

- **Our Approach:** Prove **supermodularity** of R(S) as a function of S
  - Leads to efficient algorithms for minimizing supermodular functions up to a provable bound

## Supermodularity

- Let V be a finite set; a function $f : 2^V \to \mathbb{R}$ is supermodular if for any S $\subseteq$ T $\subseteq$ V and v $\in$ V\T,

$$f(S) - f(S \cup \{v\}) \geq f(T) - f(T \cup \{v\})$$

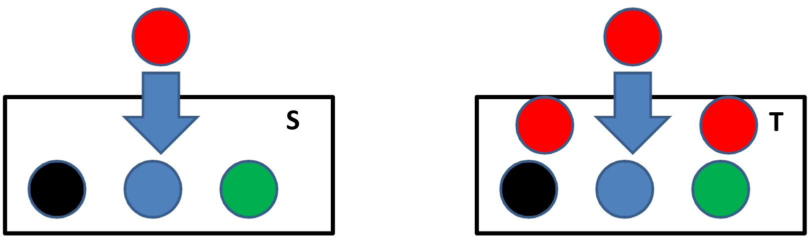



- A diminishing returns property for set functions
  - e.g., cost functions
- If f is supermodular, then -f is submodular
- Efficient approximation algorithms for minimizing supermodular functions exist



## An Example

- Consider a collection of balls of different colors (e.g., Red, Green Blue)
- A set S of balls is placed in a box
- Define f(S) = # of colors **not** found in the box




**Number of colors not found is reduced by one**

**No effect on number of colors represented**

# Proving R(S) is Supermodular

- **First step:** R(S,u) is equal to the graph effective resistance between u and S (set S to 0 volts and node u to one volt; measure the resistance between the set S and node u)
- **Second step:** The effective resistance is proportional to the commute time κ(S,u) of a random walk from u to S
  - Generalization of [Chandra et al `89] (point to point)
- **Third step:** The commute time κ(S,u) is supermodular as a function of S

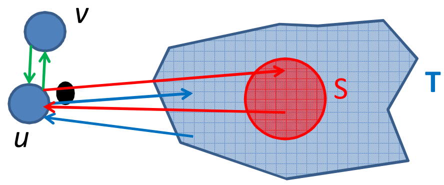


# Commute Time is Supermodular

- **Commute time:** Expected time for random walk starting at u to reach any node in S and return to u
- For any S⊆T, need to show

$$\kappa(S,u) - \kappa(S \cup \{v\}, u) \geq \kappa(T,u) - \kappa(T \cup \{v\}, u)$$

**κ(T,u) = 2 steps**
**κ(T∪{v},u) = 2 steps**
**κ(S∪{v},u) = 4 steps**
**κ(S,u) = 6 steps**



# Back to R(S)

- R(S,u), which is proportional to κ(S,u) is supermodular
- $R(S) = \sum_{u \in V \setminus S} R(S,u)$ is supermodular
- The supermodularity property leads to provable guarantees for a greedy algorithm



# Choosing up to k Leaders

- Choose set *S* of *k* leaders that minimizes total error

$$\begin{array}{ll} \text{minimize} & R(S) = \sum_{u \in V} R(S,u) \\ \text{s.t.} & |S| \leq k \end{array}$$

<u>Greedy Selection Procedure:</u>

- Initialize leader set S = ∅
- At each iteration, add the node *v* to *S* that maximizes
  *R(S) – R(S+v) (largest incremental decrease in error)*
- Stop after *k* iterations

**Theorem:** $\left(1 - \frac{1}{e}\right) R^* + \frac{1}{e} R_{max}$

*R** is optimum, $R_{max} \triangleq \max_i \sum_{u \in V} R(i,u)$

# Choosing Leaders to Meet a Given Error Bound

- Choose minimum-size set *S* to meet bound *α* on error

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & \sum_{u \in V} R(S,u) = R(S) \leq \alpha \end{array}$$

Greedy Selection Procedure:

- Initialize leader set S = ∅
- At each iteration, add the node *v* to *S* that maximizes *R(S) – R(S+v)*
- Stop when *R(S) ≤ α*

**Theorem**: $|S| \leq |S^{*}| \left(1 + \log\left\{\frac{R_{max}}{R(S^{*})}\right\}\right)$

*S** is optimum set, $R_{max} \triangleq \max_{i} \sum_{u \in V} R(i,u)$



# Choosing Leaders when Switching Between Topologies

- Network may switch between topologies $G_1,\ldots,G_r$
- First problem: Minimize average error

$$\frac{1}{r}\sum_{i=1}^{r} R(S|G_i)$$

  – Nonnegative weighted sum of supermodular functions
- Second problem: Minimize worst-case error

$$\max_{i=1,\ldots,r} R(S|G_i)$$

  – Not a supermodular function! Cannot use techniques above. Need a different metric.

# A Metric for Minimizing Worst-case Error

- Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & \max_{i=1,\ldots,r} R(S|G_i) \leq \alpha \end{array}$$

- This is equivalent to

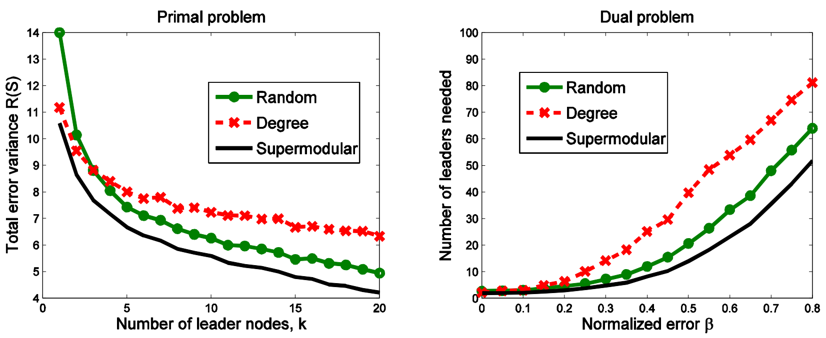
$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & \frac{1}{r}\sum_{i=1}^{r} \max\{R(S|G_i), \alpha\} \leq \alpha \end{array}$$

**Lemma:** $F_i(S) \triangleq \max\{R(S|G_i), \alpha\}$ is a supermodular function of S

- Hence, the equivalent optimization problem can be approximated using a greedy algorithm



# Numerical Results – Static Case




- Simulated network of n=100 randomly positioned nodes
- Edge between two nodes if within communication range
- Supermodular optimization provides lowest bound

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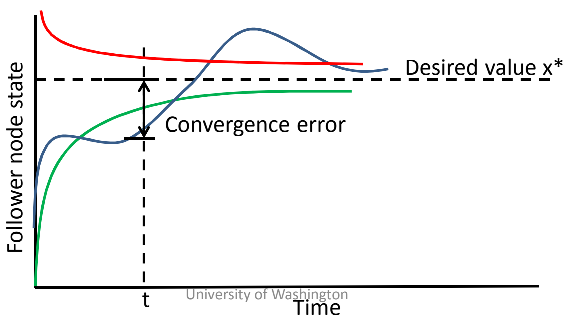
# Convergence Error

- **Goals of system:**
  - Ensure that follower nodes converge to a desired state
  - Reduce deviations from desired state prior to convergence
- **Question:** How to minimize these convergence errors via leader selection?



## System Model

- Consider follower node dynamics without noise:

$$\dot{x}_i(t) = -\sum_{j \in N(i)} W_{ij}(x_i - x_j)$$

- Weights $W_{ij}$ are arbitrary and nonnegative
- Each leader node $j \in S$ maintains distinct constant state $x_i^*$
- Vector form $\dot{\mathbf{x}}(t) = -L\mathbf{x}(t)$



## Related Work – Convergence Error

- Convergence analysis for given leader set
  - Fixed and switching networks [Jadbabaie et al 2003]
  - Directed and time-delayed networks [Olfati-Saber et al 2004]
  - Stochastic networks [Hatano & Mesbahi, 2005]
  - Spectral bounds on convergence rate [Rahmani et al 2009]
- Link weight selection to minimize convergence error [Boyd 2006]
  - Semidefinite programming approach
  - **Does not consider impact of leader** nodes
- Our contribution: Efficient approach for selecting leader nodes to minimize convergence error

# Choosing Leaders under Convergence Error Metric

- Let $A = \{x_j^* : j \in S\}$, and let $\overline{A}$ denote the convex hull of A
- The convergence error (or containment error) at time t is defined by the distance to the convex hull

$$f_t(S) \triangleq \left( \sum_{i \in V \setminus S} (d(x_i(t,S), \overline{A})^p) \right)^{1/p} = \left( \sum_{i \in V \setminus S} \min_{y \in \overline{A}} \{|x_i(t,S) - y|^p\} \right)^{1/p}$$

- Problem of selecting up to k leaders:

$$\begin{array}{ll} \text{minimize} & f_t(S) \\ \text{s.t.} & |S| \leq k \end{array}$$

- Problem of selecting minimum-size leader set:

$$\begin{array}{ll} \text{minimize} & |S| \\ \text{s.t.} & f_t(S) \leq \alpha \end{array}$$



# Our Approach

1. Derive upper bound $\hat{f}_t(S)$ on convergence error at time t that is independent of the initial state **x**(0)
2. Establish a connection between the upper bound and the probability that a random walk on the graph reaches the leader set in time t
3. Prove that the probability of reaching the leader set is supermodular as a function of S
4. Prove that the upper bound on the convergence error is supermodular as a function of S

## Selecting up to k Leaders

- Supermodularity implies that a simple greedy algorithm gives a provable bound on the optimal leader set
- To select a set of **up to k leaders** to minimize $\hat{f}_t(S)$:
  - Initialize S = ∅
  - At each iteration, choose v* that maximizes

    $$\hat{f}_t(S) - \hat{f}_t(S \cup \{v\})$$
  - Set S = S ∪ {v*}, terminate when |S| = k

**Theorem:** If S* is the optimal set, then

$$\hat{f}_t(S) \leq \left(1 - \frac{1}{e}\right) \hat{f}_t(S^*) + \frac{1}{e} f_{max}$$

where $f_{max} \triangleq \max\{\hat{f}_t(\{v\}) : v \in V\}$

(Follows from Nemhauser et al `78)



## Selecting Leaders to Achieve Error Bound

- In order to achieve a bound α on $\hat{f}_t(S)$ :
  - Initialize S' = ∅
  - At each iteration select v* that maximizes

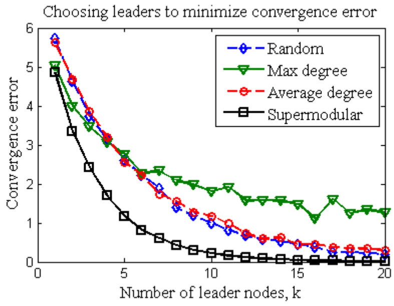
    $$\hat{f}_t(S) - \hat{f}_t(S \cup \{v\})$$
  - Set S = S ∪ {v*}, terminate when $\hat{f}_t(S) \leq \alpha$

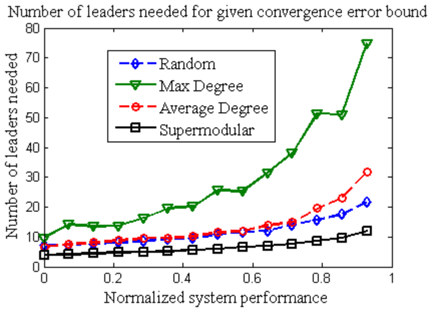
**Theorem:** Let S* be the minimum-size set with $\hat{f}_t(S) \leq \alpha$

We have:

$$\frac{|S'|}{|S^*|} \leq 1 + \ln\left(\frac{f_{max}}{\alpha}\right)$$

# Simulation Results




- Simulated (Matlab) an undirected graph with n=100 nodes
- Two nodes share link if within communication range
- Supermodular optimization provides lowest convergence error and requires fewest leaders



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## Controllability

- A system with dynamics

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

is controllable if for any states **a**, **b** with **x**(0) = **a** and any T > 0, there exists {u(t) : 0≤ t ≤ T} such that **x**(T) = **b.**

- Equivalently, it is possible to drive the state **x** from any initial state to any final state in finite time



## Related Work

- Controllability analysis for a given leader set
  – Necessary and sufficient graph spectrum conditions for controllability [Tanner 2004]
  – Necessary graph-based conditions for controllability [Rahmani et al 2009]
  – Controllability of dynamic networks [Liu et al 2008]
- Efficient algorithm for leader selection for controllability [Liu et al 2011]
  – No performance guarantees
- Our approach: We present an approach for leader selection based on joint performance and controllability

# Our Approach: Joint Performance and Controllability

- **Goal of this topic:** Joint optimization of controllability and performance criteria
- **Approach:** Introduce a graph controllability index (GCI)
  - Characterizes the largest controllable subgraph of the network
  - Prove submodularity of the GCI
  - Formulate joint performance and controllability as a submodular optimization problem



# Graph Controllability Index

- Define **GCI** as the largest controllable subgraph:

$$c(S) \triangleq \max\{|V'| : (V', E') \subseteq G \text{ is controllable from } S\}$$

- Controllability can then be traded off with a performance metric f(S) via the optimization problem

$$\begin{array}{ll} \text{maximize} & \frac{1}{n}c(S) - \lambda f(S) \\ \text{s.t.} & |S| \leq k \end{array}$$

- Possible objective functions f(S):
  - Mean-square error due to link noise
  - Convergence error
- Computation of GCI is based on structural controllability of the graph

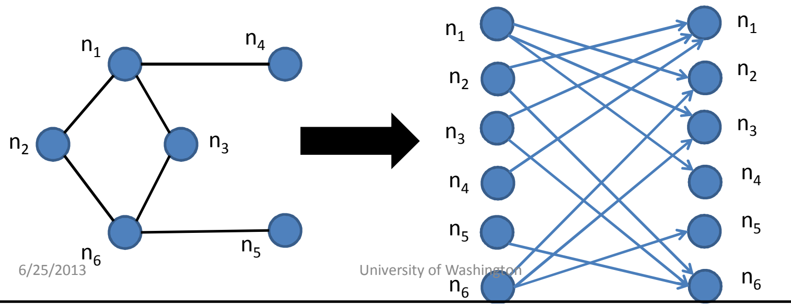
# Structural Controllability

- Consider a system with state $\mathbf{x}(t) \in \mathbf{R}^m$, input $\mathbf{u}(t) \in \mathbf{R}^l$, and dynamics $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$
- *Structural controllability [Lin `74]* holds if, for almost every choice of the nonzero entries of (A,B), system is controllable
- Define a graph G with vertex set $\{v_1,\ldots,v_m,w_1,\ldots,w_l\}$ by adding edge $(v_j, v_i)$ if $A_{ij} \neq 0$ and edge $(w_j, v_i)$ if $B_{ij} \neq 0$
  - Here, $A = L_{ff}$ and $B = L_{fl}$
- **Theorem** (Lin `74): (A,B) satisfies structural controllability iff:
  1. For each $v_i$, there exists $w_j$ such that a path exists from $w_j$ to $v_i$
  2. For each $T \subseteq \{v_1,\ldots, v_m\}$, $|T| \leq |N(T)|$, where N(T) is set of neighbors of T



# Controllability and Matching

- If graph is connected, then SC holds iff, for any $A \subseteq V\backslash S$, $|A| \leq |N(A)|$
- Consider the bipartite representation of G
- By Hall Marriage Theorem [Brualdi `10], SC is equivalent to existence of a **perfect matching** from N(V\S) into V\S.
- We prove submodularity of the GCI using this connection to graph matching



# Algorithms for Maximizing GCI

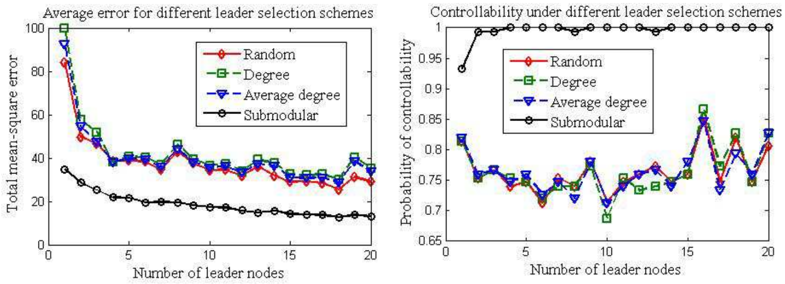
- A greedy approach maximizes GCI up to provable bound
- At each iteration, select the agent v such that

$$\frac{1}{n}(c(S+v)-c(S))-\lambda(f(S+v)-f(S))$$

  is maximized
- Special case: $\lambda$=0
  - Reduces to optimization over controllability only
  - If k is sufficiently large, then algorithm returns the minimum-size leader set needed for SC in polynomial time
  - Reduces to a graph matching, resulting in efficient leader selection with identical guarantees as existing methods



# Simulation Results




- Erdos-Renyi random graph G(n,p), with n=100 and p=0.05 simulated
- Simulations use total mean-square error as performance metric
- Submodular approach outperforms random and degree-based
- Achieves controllability when possible
- Degree-based selection provides controllability in roughly three-fourths of cases

## Conclusions

- Presented a unifying, supermodular optimization framework for leader selection based on:
  - Robustness to link noise
  - Smooth convergence to desired state
  - Controllability and performance
- Developed efficient algorithms with provable guarantees in static and dynamic networks
- Based on connections between networked dynamical systems and the theory of random walks on graphs



## Future Work

- Distributed algorithms for leader selection
- Controllability of dynamic networks
- Different application domains:
  - Biological networks
  - Social networks
  - Unmanned vehicular networks
- Leader selection for security as well as control

# References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Web link 1: http://www.ee.washington.edu/research/ncs/research.html
Web link 2: http://www.ee.washington.edu/research/nsl/faculty/radha/



# Questions & Discussion

- Thank you for your time and attention