Leader Selection for Performance and Control of Complex Networks

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A World of Networks

Networks in nature

Man-made networks
Example: Search and Rescue

- Network: Consists of aerial and ground robots; wireless network
- Each node: computes relative location, trajectory & coverage
- Communication and coordination tasks are performed in a distributed, autonomous manner

Features

- Node
  - Mobile
  - Battery resource limited
  - Forms wireless links; Limited communication range; lossy channel between nodes
- Network
  - Time varying topology due to mobility and node/link failure
- Formation and Control
  - Needs to be distributed and adaptively computable
  - Control and communication protocols need to be resource efficient
**Node Dynamics**

- Each node monitors and controls its own internal state (e.g., position, velocity, angular velocity)
- Receives state information as input from neighbors
- Computes and broadcasts internal state using predefined rule(s)
  - Common approach: weighted averaging of neighbor states
- Link noise affects the estimates and state update (e.g., errors in position, velocity estimates)

**Leader-Follower System**

- In a large network, impractical to provide control inputs to each node
- Instead, small subset of leader nodes act as control inputs to aid/influence the remaining follower nodes to a desired state
- Leader inputs propagate through network via local state updates
  - Propagation causes delay before followers reach desired states
Examples of Leader-Follower Systems

• Steering formations of unmanned vehicles
• Anchor-based localization in sensor networks
• Influence propagation in social networks
• Control of gene regulation and expression
• Synchronization of neuronal networks and biological oscillators

Main questions:
– What is/are the metrics for choosing leaders?
– Given a specific metric, which nodes are the best/effective leaders?
– How to efficiently compute the best leader set?

An Example: A Flock of Boids

• In 2003, a filmmaker had a Problem:
  – How to simulate a group of hundreds of horses moving in a realistic, coordinated way?
  – Expensive (labor & CPU) to design a motion path for each horse

• Solution: Give a motion path for a few of the horses, and make the rest follow

• Idea originally proposed in [Reynolds `87] on simulating a flock of birds (\textit{“boids”})

\textbf{VIDEO}
Today’s Talk: Leader Selection

• Metrics for choosing leaders:
  – Performance (robustness to link noise, convergence error)
  – Controllability
  – Joint performance and controllability

• Characterizing the optimal leader set:
  – Supermodular structure of leader selection metrics

• Efficiently computing the optimal leader set:
  – Polynomial-time algorithms with provable optimality gap

Outline

• Motivating Application
• Leader-Follower System Requirements
• Leader Selection in Complex Networks
  – Robustness to link noise
  – Minimizing convergence error
  – Performance and controllability
• Conclusions and Future Work
**System Model**

- Network of $n$ nodes, indexed $V = \{1, \ldots, n\}$, edge set $E$
- Each follower node has state $x_i(t)$ with dynamics
  \[ \dot{x}_i(t) = - \sum_{j \in N(i)} W_{ij}(x_i - x_j) + \epsilon_{ij}(t) \]
- $\epsilon_{ij}(t)$ zero-mean white process with variance $\nu_{ij}$, $\nu_{ij} = \nu_{ji}$
- Nodes in the leader set, denoted $S$, maintain a constant state $x^*$
- Weights $W_{ij}$ given by $W_{ij} = \nu_{ij}^{-1}$
- Dynamics have vector form
  \[ \dot{x}_f(t) = -Lx(t) + w(t), \] where $w$ is a zero-mean white process and $L$ is defined by
  \[
  L_{ij} = \begin{cases}
  W_{ij}, & (i, j) \in E \\
  \sum_{l \in N(i)} W_{il}, & i = j \\
  0, & i \in S \\
  0, & \text{else}
  \end{cases}
  \]

**Related Work – Link Noise**

- Analysis of mean-square error due to noise in a network with given leaders and dynamics
  - Noise in agent state updates [Patterson & Bamieh 2010, Young et al 2010]
  - Noise in communication links [Barooah & Hespanha 2006]
  - Quantization noise [Kar & Moura 2009]
  - Does not provide optimality guarantees
- **Our contribution**: Efficient approach with provable bounds on the optimality of the leader set
Quantifying Error Due to Link Noise

- The Laplacian matrix $L$ can be decomposed as
  \[ L = \begin{pmatrix} L_{ff} & L_{fl} \\ 0 & 0 \end{pmatrix}. \]
  - $L_{ff}$ and $L_{fl}$ represent the influence of followers and leaders
- Theorem (Barooah et al. `06): The mean-square error in the follower node states in steady-state is equal to
  \[ \lim_{t \to \infty} E\|x(t) - x^*1\|_2 = \text{tr}(L_{ff}^{-1}). \]
- Define the error due to link noise as the metric
  \[ R(S) = \text{tr}(L_{ff}^{-1}) \]
  - trace of the steady-state covariance matrix of follower nodes
- Define $R(S, u) = (L_{ff}^{-1})_{uu}$ for $u \in V \setminus S$ as the variance of each follower node

Problem Formulation

- Selecting up to $k$ leaders to minimize error due to link noise
  \[
  \begin{align*}
  \text{minimize} & \quad R(S) \\
  \text{s.t.} & \quad |S| \leq k
  \end{align*}
  \]
- Selecting the minimum-size leader set to achieve a bound $\alpha$ on error due to link noise
  \[
  \begin{align*}
  \text{minimize} & \quad |S| \\
  \text{s.t.} & \quad R(S) \leq \alpha
  \end{align*}
  \]
- Our Approach: Prove supermodularity of $R(S)$ as a function of $S$
  - Leads to efficient algorithms for minimizing supermodular functions up to a provable bound
Supermodularity

- Let $V$ be a finite set; a function $f : 2^V \to \mathbb{R}$ is supermodular if for any $S \subseteq T \subseteq V$ and $v \in V \setminus T$,
  \[ f(S) - f(S \cup \{v\}) \geq f(T) - f(T \cup \{v\}) \]

- A diminishing returns property for set functions — e.g., cost functions
- If $f$ is supermodular, then $-f$ is submodular
- Efficient approximation algorithms for minimizing supermodular functions exist

An Example

- Consider a collection of balls of different colors (e.g., Red, Green Blue)
- A set $S$ of balls is placed in a box
- Define $f(S) = \#$ of colors not found in the box

\[ f(S) \quad \text{Number of colors not found is reduced by one} \]

\[ f(T) \quad \text{No effect on number of colors represented} \]
Proving $R(S)$ is Supermodular

- **First step**: $R(S,u)$ is equal to the graph effective resistance between $u$ and $S$ (set $S$ to 0 volts and node $u$ to one volt; measure the resistance between the set $S$ and node $u$)

- **Second step**: The effective resistance is proportional to the commute time $\kappa(S,u)$ of a random walk from $u$ to $S$ — Generalization of [Chandra et al `89] (point to point)

- **Third step**: The commute time $\kappa(S,u)$ is supermodular as a function of $S$

Commute Time is Supermodular

- **Commute time**: Expected time for random walk starting at $u$ to reach any node in $S$ and return to $u$

- For any $S \subseteq T$, need to show

  $$\kappa(S,u) - \kappa(S \cup \{v\},u) \geq \kappa(T,u) - \kappa(T \cup \{v\},u)$$

- $\kappa(T,u) = 2$ steps
- $\kappa(T \cup \{v\},u) = 2$ steps
- $\kappa(S \cup \{v\},u) = 4$ steps
- $\kappa(S,u) = 6$ steps
Back to $R(S)$

- $R(S,u)$, which is proportional to $\kappa(S,u)$ is supermodular

- $R(S) = \sum_{u \in V \setminus S} R(S,u)$ is supermodular

- The supermodularity property leads to provable guarantees for a greedy algorithm

Choosing up to $k$ Leaders

- Choose set $S$ of $k$ leaders that minimizes total error

  $\begin{align*}
  \text{minimize} & \quad R(S) = \sum_{u \in V} R(S, u) \\
  \text{s.t.} & \quad |S| \leq k
  \end{align*}$

**Greedy Selection Procedure:**

- Initialize leader set $S = \emptyset$
- At each iteration, add the node $v$ to $S$ that maximizes $R(S) - R(S+v)$ (largest incremental decrease in error)
- Stop after $k$ iterations

**Theorem:**

$R^*$ is optimum, 

$$
R^* \leq R_{\text{max}} \leq \max_i \sum_u R(i,u)
$$
Choosing Leaders to Meet a Given Error Bound

- Choose minimum-size set $S$ to meet bound $\alpha$ on error

$$\minimize |S| \quad \text{s.t.} \quad \sum_{u \in V} R(S, u) = R(S) \leq \alpha$$

**Greedy Selection Procedure:**

- Initialize leader set $S = \emptyset$
- At each iteration, add the node $v$ to $S$ that maximizes $R(S) - R(S + v)$
- Stop when $R(S) \leq \alpha$

**Theorem:** $|S| \leq |S^*| \left(1 + \log \left( \frac{R_{\max}}{R(S^*)} \right) \right)$

$S^*$ is optimum set, $R_{\max} \triangleq \max_i \sum_{u \in V} R(i, u)$

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Choosing Leaders when Switching Between Topologies

- Network may switch between topologies $G_1, \ldots, G_r$
- First problem: Minimize average error

$$\frac{1}{r} \sum_{i=1}^r R(S|G_i)$$

- Nonnegative weighted sum of supermodular functions
- Second problem: Minimize worst-case error

$$\max_{i=1, \ldots, r} R(S|G_i)$$

- Not a supermodular function! Cannot use techniques above. Need a different metric.
A Metric for Minimizing Worst-case Error

- Consider the optimization problem

\[
\begin{align*}
\text{minimize} & \quad |S| \\
\text{s.t.} & \quad \max_{i=1,\ldots,r} R(S|G_i) \leq \alpha
\end{align*}
\]

- This is equivalent to

\[
\begin{align*}
\text{minimize} & \quad |S| \\
\text{s.t.} & \quad \frac{1}{r} \sum_{i=1}^{r} \max \{ R(S|G_i), \alpha \} \leq \alpha
\end{align*}
\]

Lemma: \( F_i(S) \equiv \max \{ R(S|G_i), \alpha \} \) is a supermodular function of \( S \)

- Hence, the equivalent optimization problem can be approximated using a greedy algorithm

Numerical Results – Static Case

- Simulated network of \( n=100 \) randomly positioned nodes
- Edge between two nodes if within communication range
- Supermodular optimization provides lowest bound
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Convergence Error

• **Goals of system:**
  – Ensure that follower nodes converge to a desired state
  – Reduce deviations from desired state prior to convergence
• **Question:** How to minimize these convergence errors via leader selection?

![Diagram showing convergence error with desired value x* and follower state over time](image-url)
System Model

• Consider follower node dynamics without noise:

\[ \dot{x}_i(t) = - \sum_{j \in N(i)} W_{ij} (x_i - x_j) \]

• Weights \( W_{ij} \) are arbitrary and nonnegative

• Each leader node \( j \in S \) maintains distinct constant state \( x_i^* \)

• Vector form \( \dot{x}(t) = -Lx(t) \)

Related Work – Convergence Error

• Convergence analysis for given leader set
  – Fixed and switching networks [Jadbabaie et al 2003]
  – Stochastic networks [Hatano & Mesbahi, 2005]
  – Spectral bounds on convergence rate [Rahmani et al 2009]

• Link weight selection to minimize convergence error [Boyd 2006]
  – Semidefinite programming approach
  – Does not consider impact of leader nodes

• Our contribution: Efficient approach for selecting leader nodes to minimize convergence error
Choosing Leaders under Convergence Error Metric

- Let $A = \{x_j^* : j \in S\}$, and let $\overline{A}$ denote the convex hull of $A$.
- The convergence error (or containment error) at time $t$ is defined by the distance to the convex hull:
  \[ f_t(S) = \left( \sum_{i \in V \setminus S} (d(x_i(t), \overline{A}))^p \right)^{1/p} = \left( \sum_{i \in V \setminus S} \min_{y \in A} \{|x_i(t), S) - y|^p\} \right)^{1/p} \]

- Problem of selecting up to $k$ leaders:
  \[
  \begin{align*}
  \text{minimize} & \quad f_t(S) \\
  \text{s.t.} & \quad |S| \leq k
  \end{align*}
  \]

- Problem of selecting minimum-size leader set:
  \[
  \begin{align*}
  \text{minimize} & \quad |S| \\
  \text{s.t.} & \quad f_t(S) \leq \alpha
  \end{align*}
  \]

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Our Approach

1. Derive upper bound $\tilde{f}_t(S)$ on convergence error at time $t$ that is independent of the initial state $x(0)$
2. Establish a connection between the upper bound and the probability that a random walk on the graph reaches the leader set in time $t$
3. Prove that the probability of reaching the leader set is supermodular as a function of $S$
4. Prove that the upper bound on the convergence error is supermodular as a function of $S$
Selecting up to k Leaders

- Supermodularity implies that a simple greedy algorithm gives a provable bound on the optimal leader set
- To select a set of up to k leaders to minimize $\hat{f}_t(S)$:
  - Initialize $S = \emptyset$
  - At each iteration, choose $v^*$ that maximizes $\hat{f}_t(S) - \hat{f}_t(S \cup \{v\})$
  - Set $S = S \cup \{v^*\}$, terminate when $|S| = k$

**Theorem:** If $S^*$ is the optimal set, then

$$\hat{f}_t(S) \leq \left(1 - \frac{1}{e}\right) \hat{f}_t(S^*) + \frac{1}{e} f_{\text{max}}$$

where $f_{\text{max}} \triangleq \max \{f_t(\{v\}) : v \in V\}$

(Follows from Nemhauser et al ‘78)

Selecting Leaders to Achieve Error Bound

- In order to achieve a bound $\alpha$ on $\hat{f}_t(S)$:
  - Initialize $S' = \emptyset$
  - At each iteration select $v^*$ that maximizes $\hat{f}_t(S) - \hat{f}_t(S \cup \{v\})$
  - Set $S = S \cup \{v^*\}$, terminate when $\hat{f}_t(S) \leq \alpha$

**Theorem:** Let $S^*$ be the minimum-size set with $\hat{f}_t(S) \leq \alpha$. We have:

$$\frac{|S'|}{|S^*|} \leq 1 + \ln \left(\frac{f_{\text{max}}}{\alpha}\right)$$
Simulation Results

- Simulated (Matlab) an undirected graph with n=100 nodes
- Two nodes share link if within communication range
- Supermodular optimization provides lowest convergence error and requires fewest leaders

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Controllability

- A system with dynamics
  \[ \dot{x}(t) = Ax(t) + Bu(t) \]
  is controllable if for any states \( a, b \) with \( x(0) = a \) and any \( T > 0 \), there exists \( \{u(t) : 0 \leq t \leq T\} \) such that \( x(T) = b \).

- Equivalently, it is possible to drive the state \( x \) from any initial state to any final state in finite time.

Related Work

- Controllability analysis for a given leader set
  - Necessary and sufficient graph spectrum conditions for controllability [Tanner 2004]
  - Necessary graph-based conditions for controllability [Rahmani et al 2009]
  - Controllability of dynamic networks [Liu et al 2008]

- Efficient algorithm for leader selection for controllability [Liu et al 2011]
  - No performance guarantees

- Our approach: We present an approach for leader selection based on joint performance and controllability
Our Approach: Joint Performance and Controllability

- **Goal of this topic**: Joint optimization of controllability and performance criteria
- **Approach**: Introduce a graph controllability index (GCI)
  - Characterizes the largest controllable subgraph of the network
  - Prove submodularity of the GCI
  - Formulate joint performance and controllability as a submodular optimization problem

Graph Controllability Index

- Define **GCI** as the largest controllable subgraph:
  \[ c(S) \triangleq \max \{|V'| : (V', E') \subseteq G \text{ is controllable from } S \} \]
- Controllability can then be traded off with a performance metric \( f(S) \) via the optimization problem
  \[
  \begin{align*}
  \text{maximize} & \quad \frac{1}{\lambda} c(S) - \lambda f(S) \\
  \text{s.t.} & \quad |S| \leq k
  \end{align*}
  \]
- Possible objective functions \( f(S) \):
  - Mean-square error due to link noise
  - Convergence error
- Computation of GCI is based on structural controllability of the graph
Structural Controllability

- Consider a system with state $x(t) \in \mathbb{R}^m$, input $u(t) \in \mathbb{R}^l$, and dynamics $\dot{x}(t) = Ax(t) + Bu(t)$
- Structural controllability [Lin `74] holds if, for almost every choice of the nonzero entries of $(A,B)$, system is controllable
- Define a graph $G$ with vertex set $\{v_1, \ldots, v_m, w_1, \ldots, w_l\}$ by adding edge $(v_j, v_i)$ if $A_{ij} \neq 0$ and edge $(w_j, v_i)$ if $B_{ij} \neq 0$
  - Here, $A = L_{ff}$ and $B = L_{fl}$
- **Theorem** (Lin `74): $(A,B)$ satisfies structural controllability iff:
  1. For each $v_i$, there exists $w_j$ such that a path exists from $w_j$ to $v_i$
  2. For each $T \subseteq \{v_1, \ldots, v_m\}$, $|T| \leq |N(T)|$, where $N(T)$ is set of neighbors of $T$

Controllability and Matching

- If graph is connected, then SC holds iff, for any $A \subseteq V \setminus S$, $|A| \leq |N(A)|$
- Consider the bipartite representation of $G$
- By Hall Marriage Theorem [Brualdi `10], SC is equivalent to existence of a perfect matching from $N(V \setminus S)$ into $V \setminus S$.
- We prove submodularity of the GCI using this connection to graph matching
Algorithms for Maximizing GCI

- A greedy approach maximizes GCI up to provable bound
- At each iteration, select the agent $v$ such that
  \[
  \frac{1}{n}(c(S + v) - c(S)) - \lambda(f(S + v) - f(S))
  \]
  is maximized
- Special case: $\lambda=0$
  - Reduces to optimization over controllability only
  - If $k$ is sufficiently large, then algorithm returns the minimum-size leader set needed for SC in polynomial time
  - Reduces to a graph matching, resulting in efficient leader selection with identical guarantees as existing methods

Simulation Results

- Erdos-Renyi random graph $G(n,p)$, with $n=100$ and $p=0.05$ simulated
- Simulations use total mean-square error as performance metric
- Submodular approach outperforms random and degree-based
- Achieves controllability when possible
- Degree-based selection provides controllability in roughly three-fourths of cases
Conclusions

- Presented a unifying, supermodular optimization framework for leader selection based on:
  - Robustness to link noise
  - Smooth convergence to desired state
  - Controllability and performance
- Developed efficient algorithms with provable guarantees in static and dynamic networks
- Based on connections between networked dynamical systems and the theory of random walks on graphs

Future Work

- Distributed algorithms for leader selection
- Controllability of dynamic networks
- Different application domains:
  - Biological networks
  - Social networks
  - Unmanned vehicular networks
- Leader selection for security as well as control
References


Questions & Discussion

- Thank you for your time and attention