Semantics of Reactor Systems
An Introduction

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Let the input and output data types be $T$ (They could be different, but for simplicity, we assume here they are the same.) Then at each tag $n$,

$$s_i(n) \in T \cup \{\bot, \varepsilon\}$$

where $\bot$ represents “unknown” and $\varepsilon$ represents “absent.” Let $\tilde{T} = T \cup \{\bot, \varepsilon\}$, the type extended.
Reactor Model at a Tag

At tag $n$, each actor provides a function

$$f(n) : \tilde{T} \rightarrow \tilde{T}$$

The function may be different at each tag. Typically, the actor is a state machine, where at each state, it defines an input-output function, and it switches state between tags.
Reactor with Multiple Inputs and Outputs

At each tag $n$, the actor provides a function

$$f(n) : \tilde{T}^2 \rightarrow \tilde{T}$$
At each tag \( n \), the actor provides a function

\[
f(n) : \tilde{T}^0 \rightarrow \tilde{T}
\]

where \( \tilde{T}^0 \) is a singleton set. Hence, at each tag, such an actor produces a fixed output (not dependent on any input).
Reactor with No Outputs

At each tag $n$, the actor provides a function $f(n): \tilde{T} \rightarrow \tilde{T}^0$ where $\tilde{T}^0$ is a singleton set. Hence, at each tag, the function provides no information.
Composition of Reactors
Fixed Point

At each tag $n$, the model defines a fixed point $s(n) \in \tilde{T}^3$ s.t.

$$(f(n))(s(n)) = s(n)$$

Key questions:
1. Does a fixed point exist?
2. Is the fixed point unique?
Every Composition: a Fixed Point Problem

Even if there is no feedback:
Posets

Let the set

\[ \tilde{T} = T \cup \{ \perp, \varepsilon \} \]

be a partially ordered set (poset) where \( \perp < \varepsilon \), \( \perp < t \) for all \( t \in T \) are the only order relations.

Let \( \tilde{T}^N \) be a poset by pointwise extension of the order relation. E.g., in \( \tilde{T}^2 \)

\[ (t_1, t_2) < (p_1, p_2) \iff t_1 < p_1 \text{ and } t_2 < p_2. \]
Monotonic Functions

A function $f(n) : \tilde{T}^N \rightarrow \tilde{T}^N$ is monotonic if for $t, p \in \tilde{T}^N$ s.t. $t < p$,

$$(f(n))(t) < (f(n))(p).$$
Form of the Kleene Fixed-Point Theorem

For any monotonic function $f: A \to A$ on a poset $(A, \leq)$ with finite depth, let

$$C = \{f^n(\perp) : n \in \mathbb{N}\}.$$  

$C$ is a chain. Then $\bigvee C = f(\bigvee C)$ is the least fixed point of $f$, where $\bigvee X$ is the least upper bound of the set $X$.

Intuition: The least fixed point is obtained by applying the function first to bottom, then to the result, then to that result, etc.
Composition of Reactors
Composition of Reactors in LF
Composition of Reactors in LF
Start with everything unknown.

\[ C = \left\{ \begin{bmatrix} \bot & \bot & \bot \end{bmatrix}, \cdots \right\} \]
Apply the function once.

\[ C = \left\{ \begin{bmatrix} \bot & \bot & \bot \\ \bot & \bot & \bot \\ \bot & \bot & \bot \end{bmatrix}, F \left( \begin{bmatrix} \bot & \bot & \bot \\ \bot & \bot & \bot \\ \bot & \bot & \bot \end{bmatrix} \right) = \begin{bmatrix} \bot & \bot & \bot \\ \bot & \bot & \bot \\ \bot & \bot & v_1 \end{bmatrix} \right\} \]
Apply the function again.

\[ C = \left\{ \begin{bmatrix} \bot & \bot & v_2 \end{bmatrix}, \begin{bmatrix} \bot & \bot & v_1 \end{bmatrix}, \ldots \right\} \]
Apply the function again.

\[ C = \left\{ \begin{bmatrix} \bot & \bot & v_2 & v_2 \\ \bot & \bot & v_1 \\ \bot & v_2 & v_1 \\ \bot & v_3 & v_1 \end{bmatrix} , \cdots \right\} \]
Fixed Point Reached!

\[ C = \left\{ \begin{bmatrix} \bot & \bot & \bot \\ \bot & \bot & V_2 \\ V_1 & V_1 & V_3 \end{bmatrix}, \begin{bmatrix} \bot & \bot & V_2 \\ \bot & \bot & V_2 \\ V_2 & V_2 & V_3 \end{bmatrix}, \begin{bmatrix} \bot & \bot & V_2 \\ \bot & \bot & V_2 \\ V_2 & V_3 & V_1 \end{bmatrix}, \begin{bmatrix} \bot & \bot & V_2 \\ \bot & \bot & V_2 \\ V_2 & V_3 & V_1 \end{bmatrix}, \cdots \right\} \]
Alternative Metric-Space Model

- signal: \( s : T \rightarrow V \), a partial function from tags to values.
- real time with an origin: \( T = \mathbb{R}_+ \), the non-negative reals.
- superdense time: \( T = \mathbb{R}_+ \times \mathbb{N} \), with lexicographic order.
- tag set: \( \mathcal{T}(s) \subseteq T \) is the set of tags where \( s \) is defined.
- graph: \( G(s) = \{ (t, v) \mid t \in \mathcal{T}(s), v = s(t) \} \).
- set of signals: \( (T \rightarrow V) \).
- order embedding: A one-to-one monotonic \( f : X \rightarrow Y \), for posets \( X, Y \)
- discrete set: \( D \) where there exists an order embedding \( f : D \rightarrow \mathbb{N} \)
- discrete signal: signal \( s \) where \( \mathcal{T}(s) \) is a discrete set.
- discrete actor: for discrete inputs, output is discrete.
Reactor Model on Signals

\[ F : (T \rightarrow V)^m \rightarrow (T \rightarrow V)^n, \quad m, n \in \mathbb{N}. \]

Definitions extend to tuples of signals. E.g.

- singleton set: \((T \rightarrow V)^0 = \{\lambda\}\), where \(\lambda\) is the empty signal
- tag set: \(\mathcal{T}(s_1, \cdots, s_m) = \mathcal{T}(s_1) \cup \cdots \cup \mathcal{T}(s_m)\).
- discrete tuple: \((s_1, \cdots, s_m)\) where \(\mathcal{T}(s_1, \cdots, s_m)\) is discrete.
Metric Space

A metric space \((X, d)\) is a set \(X\) with a metric distance function 
\[d: X \times X \to \mathbb{R}_+\] such that for all \(x, y, z \in X\),

1. \(d(x, y) = 0\) if and only if \(x = y\), identity of indiscernables
2. \(d(x, y) = d(y, x)\), symmetry
3. \(d(x, z) \leq d(x, y) + d(y, z)\), triangle inequality

Exercise: Show that these three properties imply that for all \(x, y \in X\), \(d(x, y) \geq 0\).
Ultrametric

If the metric distance function $d$ also satisfies

4. $d(x, z) \leq \max(d(x, y), d(y, z))$, ultrametric inequality

for all $x, y, z \in X$, then $(X, d)$ is an ultrametric space and $d$ an ultrametric distance function.

Exercise: Show that (4) implies (3).
Cantor Metric

Consider $X = (\mathbb{R}_+ \rightarrow V)$. Let $s_1, s_2 \in X$ be two signals that are identical up to (but possibly not including) time $\tau \in \mathbb{R}_+$. The Cantor metric is then defined to be

$$d(s_1, s_2) = \frac{1}{2\tau}.$$ 

This is also called the Baire distance.
Examples

What is the distance between $s_1, s_2 \in (\mathbb{R}_+ \rightarrow \mathbb{R})$, where

- $G(s_1) = \{(1, 0)\}$, $G(s_2) = \{(1, 0), (2, 0)\}$

What is the distance between $s_1, s_2 \in (\mathbb{R} \rightarrow \mathbb{R})$, where

- $G(s_1) = \{(k, 0) \mid k \in \mathbb{Z}\}$, $G(s_2) = \{(k, 1) \mid k \in \mathbb{Z}\}$
Examples

What is the distance between $s_1, s_2 \in (\mathbb{R}_+ \rightarrow \mathbb{R})$, where

- $\mathcal{G}(s_1) = \{(1, 0)\}$, $\mathcal{G}(s_2) = \{(1, 0), (2, 0)\}$ $\frac{1}{4}$

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Examples

What is the distance between $s_1, s_2 \in (\mathbb{R}_+ \rightarrow \mathbb{R})$, where

- $G(s_1) = \{(1, 0)\}$, $G(s_2) = \{(1, 0), (2, 0)\}$  \(\frac{1}{4}\)

What is the distance between $s_1, s_2 \in (\mathbb{R} \rightarrow \mathbb{R})$, where

- $G(s_1) = \{(k, 0) \mid k \in \mathbb{Z}\}$, $G(s_2) = \{(k, 1) \mid k \in \mathbb{Z}\}$  \(\infty\)

There is no such distance, so $(\mathbb{R} \rightarrow \mathbb{R})$ is not a metric space with the Cantor metric.
Cantor Metric

\[ d(s_1, s_2) = \frac{1}{2^\tau}. \]

What is the meaning of “identical up to (but possibly not including) time \( \tau \in \mathbb{R}_+ \)”? It can be defined as follows:

\[ \tau = \bigwedge \mathcal{T}(G(s_1) \cup G(s_2) \setminus G(s_1) \cap G(s_2)), \]

or the greatest lower bound of the tags where \( s_1 \) and \( s_2 \) differ. We define \( \bigwedge \emptyset = \infty \), so if \( s_1 = s_2 \), then \( d(s_1, s_2) = 0 \).
Cantor Metric

\[ d(s_1, s_2) = \frac{1}{2^\tau}. \]

where \( \tau \) is the greatest lower bound of the tags where \( s_1 \) and \( s_2 \) differ.

- If \( s_1 = s_2 \), then \( \tau = \bigwedge \emptyset = \infty \), so \( d(s_1, s_2) = 0 \).
- If \( s_1 \) and \( s_2 \) differ everywhere, then \( \tau = \bigwedge \mathbb{R}_+ = 0 \), so \( d(s_1, s_2) = 1 \).

Hence, for any signals in \( s_1, s_2 \in (\mathbb{R}_+ \rightarrow V) \),

\[ 0 \leq d(s_1, s_2) \leq 1. \]
The Cantor Metric is an Ultrametric

1. $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$, identity of indiscernables
2. $d(s_1, s_2) = d(s_2, s_1)$, symmetry
4. $d(s_1, s_3) \leq \max(d(s_1, s_2), d(s_2, s_3))$, ultrametric inequality

The first two are obvious. To see that property 4 holds, let $2^{-\tau}$ be the max on the right side. This means that $s_1$ is identical to $s_2$ up to time $\tau$ (at least), and $s_3$ is also identical to $s_2$ up to $\tau$ (at least). Hence, $s_1$ is identical to $s_3$ up to $\tau$.

Exercise: Prove formally using the definition of $\tau$. 
Cantor Metric for Discrete Signals

For discrete signals, the situation is simpler. The Cantor metric is

\[ d(s_1, s_2) = \frac{1}{2^\tau}. \]

where \( \tau \) is the time of the earliest event in \( s_1 \) not in \( s_2 \) or vice versa.

There is no need in this case for the GLB.
Causality

Consider a discrete actor $F: (\mathbb{R}_+ \rightarrow V) \rightarrow (\mathbb{R}_+ \rightarrow V)$. It is

- causal if $d(F(s_1), F(s_2)) \leq d(s_1, s_2)$,
- strictly causal if $d(F(s_1), F(s_2)) < d(s_1, s_2)$, and
- delta causal if there exists a real $\delta < 1$ s.t. $d(F(s_1), F(s_2)) \leq \delta d(s_1, s_2)$.

If for all discrete signals $s_1, s_2$, where $s_1 \neq s_2$. 
Causality

Consider a discrete actor $F: (\mathbb{R}^+ \rightarrow V) \rightarrow (\mathbb{R}^+ \rightarrow V)$. It is

- causal if $d(F(s_1), F(s_2)) \leq d(s_1, s_2)$,
- strictly causal if $d(F(s_1), F(s_2)) < d(s_1, s_2)$, and
- delta causal if there exists a real $\delta < 1$ s.t. $d(F(s_1), F(s_2)) \leq \delta d(s_1, s_2)$.

If for all discrete signals $s_1, s_2$, where $s_1 \neq s_2$.

Intuition: For two possible inputs, the outputs differ no earlier than the inputs.
Contracting Functions

A function $F: X \to X$ on a metric space $X$ is

- contracting if $d(F(s_1), F(s_2)) \leq d(s_1, s_2)$,
- strictly contracting if $d(F(s_1), F(s_2)) < d(s_1, s_2)$, and
- delta contracting if there exists a real $\delta < 1$ s.t. $d(F(s_1), F(s_2)) \leq \delta d(s_1, s_2)$.

If for all $s_1, s_2 \in X$, where $s_1 \neq s_2$. 
Generalize to Multiple Inputs and Outputs

For discrete tuples $\bar{s}_1, \bar{s}_2 \in (\mathbb{R}_+ \rightarrow V)^m$, the Cantor metric is

$$d(\bar{s}_1, \bar{s}_2) = \frac{1}{2^{\tau}}.$$ 

where $\tau$ is the time of the earliest event in any element of $\bar{s}_1$ that is not in the corresponding element of $\bar{s}_2$ or vice versa.
Generalize to Multiple Inputs and Outputs

For tuples $\bar{s}_1, \bar{s}_2 \in (\mathbb{R}_+ \rightarrow V)^m$, the Cantor metric is

$$d(\bar{s}_1, \bar{s}_2) = \frac{1}{2^\tau}.$$ 

where $\tau$ is the time of the earliest event in any element of $\bar{s}_1$ that is not in the corresponding element of $\bar{s}_2$ or vice versa.

What if $m = 0$? Singleton set, signals are identical, so $d(\bar{s}_1, \bar{s}_2) = 0$. 
Source and Sink Actors

Quiz:
Is this contracting, strictly contracting, or delta contracting?

How about this?
Source and Sink Actors

Quiz:
Is this contracting, strictly contracting, or delta contracting? delta contracting, with $\delta = 0$.

![Diagram of F, f_n]

How about this? delta contracting, with $\delta = 0$.

![Diagram of F, f_n]
**Metric Space Fixed Point Theorem**

Let \((X, d)\) be a metric space and \(F: X \rightarrow X\) be strictly contracting. Then \(F\) has at most one fixed point.

**Proof:** It is enough to show

\[ x \neq x' \implies F(x) \neq x \text{ or } F(x') \neq x' \]

Suppose to the contrary that \(x \neq x'\), but \(F(x) = x\) and \(F(x') = x'\). But this is not possible because it would imply

\[ d(x, x') = d(F(x), F(x')) < d(x, x'). \]
Limits

An element $x \in X$ is the limit of a sequence $\{x_k\}_{k \in \mathbb{N}}$, iff for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $d(x_k, x) < \epsilon$.

The sequence is then said to converge to $x$, denoted by $x_k \to x$. 
Examples

What do the following sequences in $s_0, s_1, \cdots \in (\mathbb{R}_+ \to \mathbb{R})$ converge to?

1. $G(s_k) = \{(k, 0)\}$
2. $G(s_k) = \{(n, 0) \mid n \leq k\}$
3. $G(s_k) = \{(1 - 1/2^n, 0) \mid n \leq k\}$
Examples

What do the following sequences in $s_0, s_1, \cdots \in (\mathbb{R}_+ \rightarrow \mathbb{R})$ converge to?

1. $G(s_k) = \{(k, 0)\}$ $s_k \rightarrow \lambda$
2. $G(s_k) = \{(n, 0) \mid n \leq k\}$ $s_k \rightarrow s$ where $G(s) = \{(n, 0) \mid n \in \mathbb{N}\}$
3. $G(s_k) = \{(1 - 1/2^n, 0) \mid n \leq k\}$ Nothing. Does not converge.

We might like to view the latter two as computations, constructing a sequence of approximations to a desired signal. But in the latter case, the sequence fails to converge to the desired signal. Maybe there is a better metric space?
Banach Fixed Point Theorem

Let \((X, d)\) be a complete metric space and \(F: X \rightarrow X\) be delta contracting. Then \(F\) has a unique fixed point, and the following sequence converges to that fixed point:

\[
\begin{align*}
    s_0 &= s \\
    s_{k+1} &= F(s_k), \quad k \in \mathbb{N}.
\end{align*}
\]
Cauchy Sequences

A sequence \( \{x_k\}_{k \in \mathbb{N}} \) is Cauchy iff for all \( \epsilon > 0 \), there exists \( n \in \mathbb{N} \) such that for all \( k, l \geq n \), \( d(x_k, x_l) < \epsilon \).

A metric space \((X, d)\) is complete iff every Cauchy sequence converges to some \( x \in X \).

Intuitively, a Cauchy sequence is a sequence of signals whose earliest time at which they differ goes to infinity, so such a sequence converges.
Cauchy Sequences

A sequence \( \{x_k\}_{k \in \mathbb{N}} \) is Cauchy iff for all \( \epsilon > 0 \), there exists \( n \in \mathbb{N} \) such that for all \( k, l \geq n \), \( d(x_k, x_l) < \epsilon \).

Consider a sequence in \( (\mathbb{R}^+ \to \mathbb{R}) \) given by

\[
G(x_k) = \{(k, 0)\}.
\]

Is this sequence Cauchy? Yes. \( d(x_m, x_n) = 1/2^{\min(m,n)} \to 0 \).

Does it converge? To what? Yes, to \( \lambda \).
Cauchy Sequences

A sequence $\{x_k\}_{k \in \mathbb{N}}$ is Cauchy iff for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k, l \geq n$, $d(x_k, x_l) < \epsilon$.

Consider a sequence in $(\mathbb{R}_+ \to \mathbb{R})$ given by

$$G(x_k) = \{(i, 0) \mid i \in \mathbb{N}, i \leq k\}.$$

Is this sequence Cauchy? Yes. $d(x_m, x_n) = 1/2^{\min(m,n)} \to 0$.

Does it converge? To what? Yes. To $\{(i, 0) \mid i \in \mathbb{N}\}$. 
Discrete Signals with Cantor Metric is a Complete Metric Space

Theorem

The set of discrete signals (or tuples) with the Cantor metric form a complete metric space.

To show that every Cauchy sequence \( \{s_k\}_{k \in \mathbb{N}} \) converges, note that for every \( t \in T \), there is a subsequence \( \{s_{n(t)}, s_{n(t)+1}, \ldots\} \), for some \( n(t) \in \mathbb{N} \), of signals that are identical up to and including \( t \) (choose \( \epsilon = 1/2^t \)). Let \( s \in (\mathbb{R}_+ \rightarrow V) \) be such that for every \( t \in \mathbb{R}_+ \), \( s(t) = s_{n(t)}(t) \). Then it is easy to show that \( \{s_k\}_{k \in \mathbb{N}} \rightarrow s \).
Summary

- Cantor metric provides an ultrametric space for $(\mathbb{R}_+ \rightarrow V)$
- Cantor metric equates causality with contracting functions (but only for discrete signals)
- Cantor metric provides a metric space for $(\mathbb{R} \rightarrow V)$ (but only for discrete signals) $d$
- Banach theorem provides constructive convergence (but only for delta causal systems).