Semantics of Reactor Systems

An Introduction

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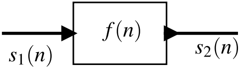
Data Types



Let the input and output data types be T (They could be different, but for simplicity, we assume here they are the same.) Then at each tag n,

$$s_i(n) \in T \cup \{\perp, \varepsilon\}$$

where $\perp$ represents “unknown” and ε represents “absent.”
Let $\tilde{T} = T \cup \{\perp, \varepsilon\}$, the type extended.

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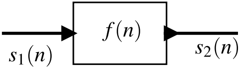
Reactor Model at a Tag

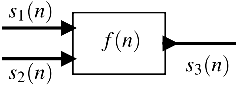


At tag *n*, each actor provides a function

$$f(n) : \tilde{T} \to \tilde{T}$$

The function may be different at each tag. Typically, the actor is a state machine, where at each state, it defines an input-output function, and it switches state between tags.

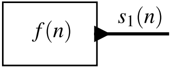
Reactor with Multiple Inputs and Outputs



At each tag *n*, the actor provides a function

$$f(n): \tilde{T}^2 \to \tilde{T}$$

Reactor with No Inputs

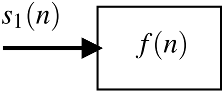


At each tag *n*, the actor provides a function

$$f(n) : \tilde{T}^0 \to \tilde{T}$$

where $\tilde{T}^0$ is a singleton set. Hence, at each tag, such an actor produces a fixed output (not dependent on any input).

Reactor with No Outputs

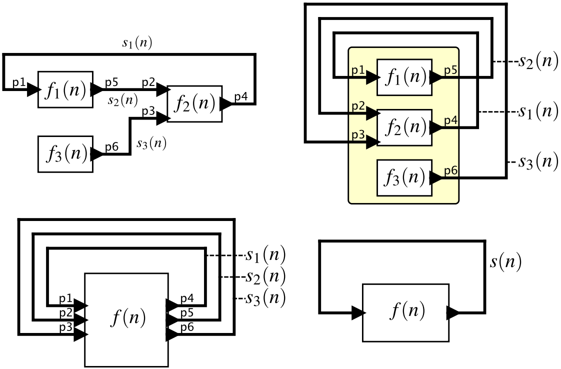


At each tag *n*, the actor provides a function

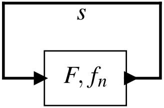
$$f(n) : \tilde{T} \to \tilde{T}^0$$

where $\tilde{T}^0$ is a singleton set. Hence, at each tag, the function provides no information.

Composition of Reactors



Fixed Point



At each tag n, the model defines a fixed point $s(n) \in \tilde{T}^3$ s.t.

$$(f(n))(s(n)) = s(n)$$

Key questions:

1. Does a fixed point exist?
2. Is the fixed point unique?

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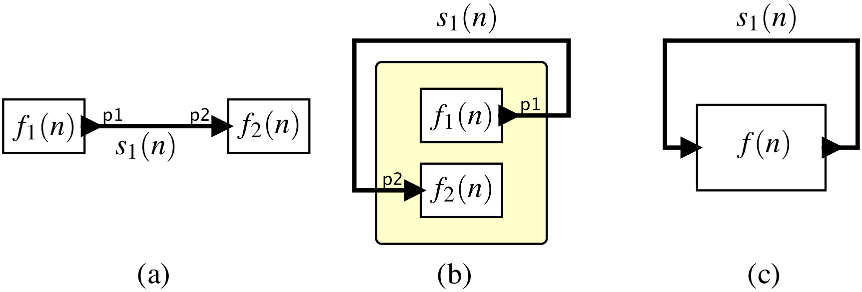
Every Composition: a Fixed Point Problem

Even if there is no feedback:



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Posets

Let the set

$$\tilde{T} = T \cup \{\perp, \varepsilon\}$$

be a partially ordered set (poset) where $\perp < \varepsilon$, $\perp < t$ for all $t \in T$ are the only order relations.

Let $\tilde{T}^N$ be a poset by pointwise extension of the order relation. E.g., in $\tilde{T}^2$

$$(t_1, t_2) < (p_1, p_2) \iff t_1 < p_1 \text{ and } t_2 < p_2.$$

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Monotonic Functions

A function $f(n) : \tilde{T}^N \to \tilde{T}^N$ is **monotonic** if for $t, p \in \tilde{T}^N$ s.t. $t < p$,

$$(f(n))(t) < (f(n))(p).$$

Form of the Kleene Fixed-Point Theorem

For any monotonic function $f\colon A \to A$ on a poset $(A, \leq)$ with finite depth, let

$$C = \{f^n(\bot) \; : \; n \in \mathbb{N}\} \, .$$

C is a chain. Then $\bigvee C = f(\bigvee C)$ is the least fixed point of f, where $\bigvee X$ is the least upper bound of the set X.

Intuition: The least fixed point is obtained by applying the function first to bottom, then to the result, then to that result, etc.

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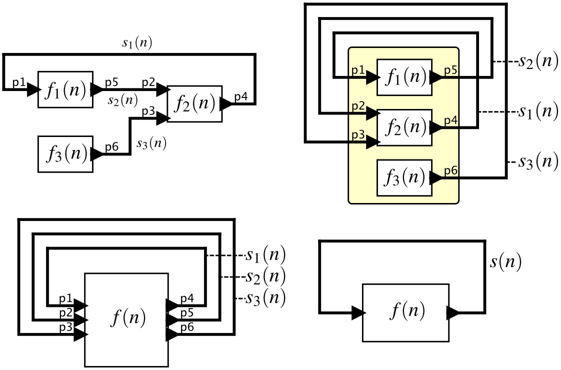
Composition of Reactors



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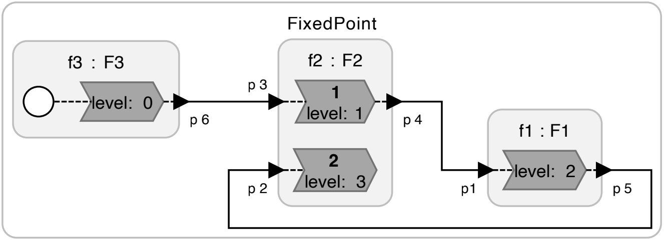
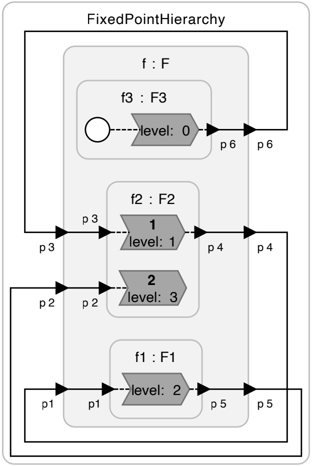
Composition of Reactors in LF



Composition of Reactors in LF



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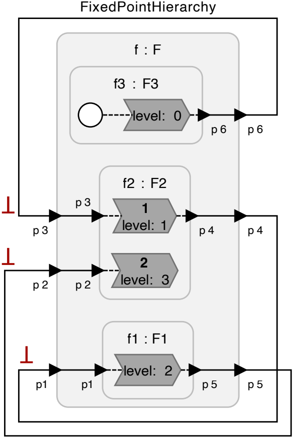
Start with everything unknown.

$$C = \left\{ \begin{bmatrix} \perp \\ \perp \\ \perp \end{bmatrix}, \cdots \right\}$$

At tag (0,0):



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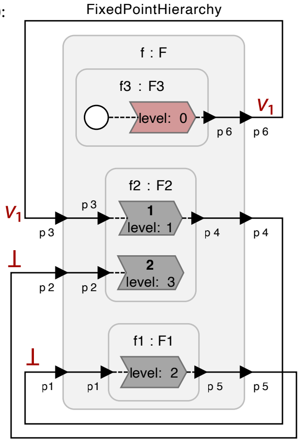
Apply the function once.

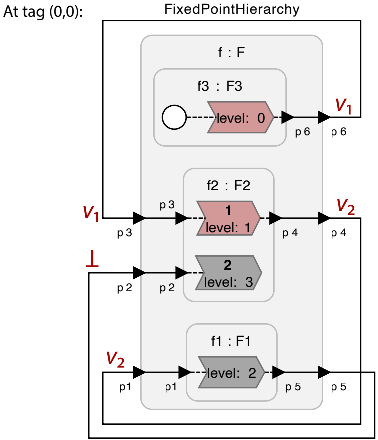
$$C = \left\{ \begin{bmatrix} \perp \\ \perp \\ \perp \end{bmatrix}, F\left(\begin{bmatrix} \perp \\ \perp \\ \perp \end{bmatrix} \right) = \begin{bmatrix} \perp \\ \perp \\ v_1 \end{bmatrix}, \cdots \right\}$$

At tag (0,0):



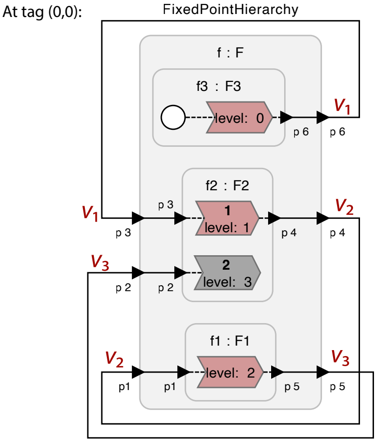
Apply the function again.

$$C = \left\{ \begin{bmatrix} \perp \\ \perp \\ \perp \end{bmatrix}, \begin{bmatrix} \perp \\ \perp \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ \perp \\ v_1 \end{bmatrix}, \cdots \right\}$$



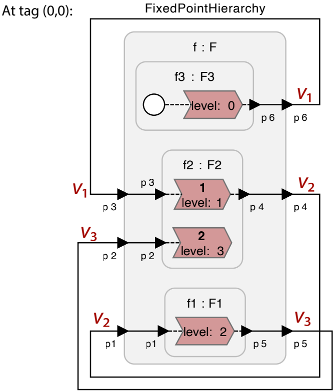
Apply the function again.

$$C = \left\{ \begin{bmatrix} \perp \\ \perp \\ \perp \end{bmatrix}, \begin{bmatrix} \perp \\ \perp \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ \perp \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ v_3 \\ v_1 \end{bmatrix}, \cdots \right\}$$



Fixed Point Reached!

$$C = \left\{ \begin{bmatrix} \perp \\ \perp \\ \perp \end{bmatrix}, \begin{bmatrix} \perp \\ \perp \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ \perp \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ v_3 \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ v_3 \\ v_1 \end{bmatrix}, \cdots \right\}$$



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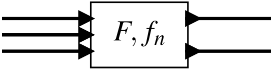
Alternative Metric-Space Model

- signal: $s\colon T \rightharpoonup V$, a partial function from tags to values.
- real time with an origin: $T = \mathbb{R}_+$, the non-negative reals.
- superdense time: $T = \mathbb{R}_+ \times \mathbb{N}$, with lexicographic order.
- tag set: $\mathcal{T}(s) \subseteq T$ is the set of tags where s is defined.
- graph: $\mathcal{G}(s) = \{(t, v) \mid t \in \mathcal{T}(s), v = s(t)\}$.
- set of signals: $(T \rightharpoonup V)$.
- order embedding: A one-to-one monotonic $f\colon X \to Y$, for posets X, Y
- discrete set: D where there exists an order embedding $f\colon D \to \mathbb{N}$
- discrete signal: signal s where $\mathcal{T}(s)$ is a discrete set.
- discrete actor: for discrete inputs, output is discrete.

Reactor Model on Signals



$$F\colon\ (T \rightharpoonup V)^m \to (T \rightharpoonup V)^n, \quad m, n \in \mathbb{N}.$$

Definitions extend to tuples of signals. E.g.

- singleton set: $(T \rightharpoonup V)^0 = \{\lambda\}$, where λ is the empty signal
- tag set: $\mathcal{T}(s_1, \cdots, s_m) = \mathcal{T}(s_1) \cup \cdots \cup \mathcal{T}(s_m)$.
- discrete tuple: $(s_1, \cdots, s_m)$ where $\mathcal{T}(s_1, \cdots, s_m)$ is discrete.

Metric Space

A metric space (X, d) is a set X with a metric distance function $d\colon X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$,

1. $d(x, y) = 0$ if and only if $x = y$, identity of indiscernables
2. $d(x, y) = d(y, x)$, symmetry
3. $d(x, z) \leq d(x, y) + d(y, z)$, triangle inequality

Exercise: Show that these three properties imply that for all $x, y \in X$, $d(x, y) \geq 0$.

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Ultrametric

If the metric distance function d also satisfies

4. $d(x,z) \leq \max(d(x,y), d(y,z))$, ultrametric inequality

for all $x, y, z \in X$, then (X, d) is an ultrametric space and d an ultrametric distance function.

Exercise: Show that (4) implies (3).

Cantor Metric

Consider $X = (\mathbb{R}_+ \rightharpoonup V)$. Let $s_1, s_2 \in X$ be two signals that are identical up to (but possibly not including) time $\tau \in \mathbb{R}_+$. The Cantor metric is then defined to be

$$d(s_1, s_2) = \frac{1}{2^\tau}.$$

This is also called the Baire distance.

Examples

What is the distance between $s_1, s_2 \in (\mathbb{R}_+ \rightharpoonup \mathbb{R})$, where

- $\mathcal{G}(s_1) = \{(1,0)\}, \quad \mathcal{G}(s_2) = \{(1,0),(2,0)\}$

What is the distance between $s_1, s_2 \in (\mathbb{R} \rightharpoonup \mathbb{R})$, where

- $\mathcal{G}(s_1) = \{(k,0) \mid k \in \mathbb{Z}\}, \quad \mathcal{G}(s_2) = \{(k,1) \mid k \in \mathbb{Z}\}$

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Examples

What is the distance between $s_1, s_2 \in (\mathbb{R}_+ \rightharpoonup \mathbb{R})$, where

- $\mathcal{G}(s_1) = \{(1, 0)\}, \quad \mathcal{G}(s_2) = \{(1, 0), (2, 0)\}$ 1/4

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- $\mathcal{G}(s_1) = \{(k, 0) \mid k \in \mathbb{Z}\}, \quad \mathcal{G}(s_2) = \{(k, 1) \mid k \in \mathbb{Z}\}$ ∞

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Examples

What is the distance between $s_1, s_2 \in (\mathbb{R}_+ \rightharpoonup \mathbb{R})$, where

- $\mathcal{G}(s_1) = \{(1, 0)\}, \quad \mathcal{G}(s_2) = \{(1, 0), (2, 0)\}$ 1/4

What is the distance between $s_1, s_2 \in (\mathbb{R} \rightharpoonup \mathbb{R})$, where

- $\mathcal{G}(s_1) = \{(k, 0) \mid k \in \mathbb{Z}\}, \quad \mathcal{G}(s_2) = \{(k, 1) \mid k \in \mathbb{Z}\}$ ∞

There is no such distance, so $(\mathbb{R} \rightharpoonup \mathbb{R})$ is not a metric space with the Cantor metric.

Cantor Metric

$$d(s_1, s_2) = \frac{1}{2^\tau}.$$

What is the meaning of “identical up to (but possibly not including) time $\tau \in \mathbb{R}_+$”? It can be defined as follows:

$$\tau = \bigwedge \mathcal{T}(\mathcal{G}(s_1) \cup \mathcal{G}(s_2) \setminus \mathcal{G}(s_1) \cap \mathcal{G}(s_2)),$$

or the greatest lower bound of the tags where s_1 and s_2 differ.
We define $\bigwedge \emptyset = \infty$, so if $s_1 = s_2$, then $d(s_1, s_2) = 0$.

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Cantor Metric

$$d(s_1, s_2) = \frac{1}{2^\tau}.$$

where τ is the greatest lower bound of the tags where s_1 and s_2 differ.

- If $s_1 = s_2$, then $\tau = \bigwedge \emptyset = \infty$, so $d(s_1, s_2) = 0$.
- If s_1 and s_2 differ everywhere, then $\tau = \bigwedge \mathbb{R}_+ = 0$, so $d(s_1, s_2) = 1$.

Hence, for any signals in $s_1, s_2 \in (\mathbb{R}_+ \rightharpoonup V)$,

$$0 \le d(s_1, s_2) \le 1.$$

The Cantor Metric is an Ultrametric

1. $d(s_1, s_2) = 0$ if and only if $s_1 = s_2$, identity of indiscernables
2. $d(s_1, s_2) = d(s_2, s_1)$, symmetry
4. $d(s_1, s_3) \leq \max(d(s_1, s_2), d(s_2, s_3))$, ultrametric inequality

The first two are obvious. To see that property 4 holds, let $2^{-\tau}$ be the max on the right side. This means that s_1 is identical to s_2 up to time τ (at least), and s_3 is also identical to s_2 up to τ (at least). Hence, s_1 is identical to s_3 up to τ.

Exercise: Prove formally using the definition of τ.

Cantor Metric for Discrete Signals

For discrete signals, the situation is simpler. The Cantor metric is

$$d(s_1, s_2) = \frac{1}{2^\tau}.$$

where τ is the time of the earliest event in s_1 not in s_2 or vice versa.

There is no need in this case for the GLB.

Causality

Consider a discrete actor $F\colon (\mathbb{R}_+ \rightharpoonup V) \to (\mathbb{R}_+ \rightharpoonup V)$. It is

- causal if $d(F(s_1), F(s_2)) \leq d(s_1, s_2)$,
- strictly causal if $d(F(s_1), F(s_2)) < d(s_1, s_2)$, and
- delta causal if there exists a real $\delta < 1$ s.t. $d(F(s_1), F(s_2)) \leq \delta d(s_1, s_2)$,

If for all *discrete* signals s_1, s_2, where $s_1 \neq s_2$.

Causality

Consider a discrete actor $F\colon (\mathbb{R}_+ \rightharpoonup V) \to (\mathbb{R}_+ \rightharpoonup V)$. It is

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- delta causal if there exists a real $\delta < 1$ s.t. $d(F(s_1), F(s_2)) \leq \delta d(s_1, s_2)$,

If for all discrete signals s_1, s_2, where $s_1 \neq s_2$.

Intuition: For two possible inputs,
the outputs differ no earlier than the inputs.

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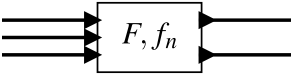
Contracting Functions

A function $F\colon X \to X$ on a metric space X is

- contracting if $d(F(s_1), F(s_2)) \leq d(s_1, s_2)$,
- strictly contracting if $d(F(s_1), F(s_2)) < d(s_1, s_2)$, and
- delta contracting if there exists a real $\delta < 1$ s.t. $d(F(s_1), F(s_2)) \leq \delta d(s_1, s_2)$,

If for all $s_1, s_2 \in X$, where $s_1 \neq s_2$.

Generalize to Multiple Inputs and Outputs



For discrete tuples $\bar{s}_1, \bar{s}_2 \in (\mathbb{R}_+ \rightharpoonup V)^m$, the Cantor metric is

$$d(\bar{s}_1, \bar{s}_2) = \frac{1}{2^\tau}.$$

where τ is the time of the earliest event in any element of $\bar{s}_1$ that is not in the corresponding element of $\bar{s}_2$ or vice versa.

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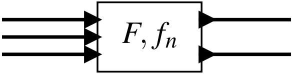
Generalize to Multiple Inputs and Outputs



For tuples $\bar{s}_1, \bar{s}_2 \in (\mathbb{R}_+ \rightharpoonup V)^m$, the Cantor metric is

$$d(\bar{s}_1, \bar{s}_2) = \frac{1}{2^\tau}.$$

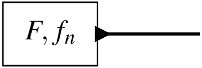
where τ is the time of the earliest event in any element of $\bar{s}_1$ that is not in the corresponding element of $\bar{s}_2$ or vice versa.

What if $m = 0$? Singleton set, signals are identical, so $d(\bar{s}_1, \bar{s}_2) = 0$.

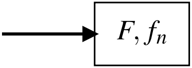
Source and Sink Actors

Quiz:

Is this contracting, strictly contracting, or delta contracting?

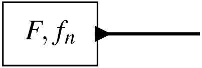


How about this?

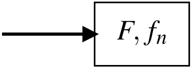


Source and Sink Actors

Quiz:
Is this contracting, strictly contracting, or delta contracting? delta contracting, with $\delta = 0$.



How about this? delta contracting, with $\delta = 0$.



Metric Space Fixed Point Theorem

Let (X, d) be a metric space and $F\colon X \to X$ be strictly contracting. Then F has at most one fixed point.

Proof: It is enough to show

$$x \neq x' \implies F(x) \neq x \text{ or } F(x') \neq x'$$

Suppose to the contrary that $x \neq x'$, but $F(x) = x$ and $F(x') = x'$. But this is not possible because it would imply

$$d(x, x') = d(F(x), F(x')) < d(x, x').$$

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Limits

An element $x \in X$ is the limit of a sequence $\{x_k\}_{k \in \mathbb{N}}$, iff for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $d(x_k, x) < \epsilon$.

The sequence is then said to converge to x, denoted by $x_k \to x$.

Examples

What do the following sequences in $s_0, s_1, \cdots \in (\mathbb{R}_+ \rightharpoonup \mathbb{R})$ converge to?

1. $\mathcal{G}(s_k) = \{(k, 0)\}$
2. $\mathcal{G}(s_k) = \{(n, 0) \mid n \leq k\}$
3. $\mathcal{G}(s_k) = \{(1 - 1/2^n, 0) \mid n \leq k\}$

Examples

What do the following sequences in $s_0, s_1, \cdots \in (\mathbb{R}_+ \rightharpoonup \mathbb{R})$ converge to?

1. $\mathcal{G}(s_k) = \{(k, 0)\} \quad s_k \to \lambda$
2. $\mathcal{G}(s_k) = \{(n, 0) \mid n \leq k\} \quad s_k \to s$ where $\mathcal{G}(s) = \{(n, 0) \mid n \in \mathbb{N}\}$
3. $\mathcal{G}(s_k) = \{(1 - 1/2^n, 0) \mid n \leq k\}$ Nothing. Does not converge.

We might like to view the latter two as computations, constructing a sequence of approximations to a desired signal. But in the latter case, the sequence fails to converge to the desired signal. Maybe there is a better metric space?

Banach Fixed Point Theorem

Let (X, d) be a complete metric space and $F\colon X \to X$ be delta contracting. Then F has a unique fixed point, and the following sequence converges to that fixed point:

$$\begin{aligned} s_0 &= s \\ s_{k+1} &= F(s_k), \quad k \in \mathbb{N}. \end{aligned}$$

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Cauchy Sequences

A sequence $\{x_k\}_{k \in \mathbb{N}}$ is Cauchy iff for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k, l \geq n$, $d(x_k, x_l) < \epsilon$.

A metric space (X, d) is complete iff every Cauchy sequence converges to some $x \in X$.

Intuitively, a Cauchy sequence is a sequence of signals whose earliest time at which they differ goes to infinity, so such a sequence converges.

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Cauchy Sequences

A sequence $\{x_k\}_{k \in \mathbb{N}}$ is Cauchy iff for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k, l \geq n$, $d(x_k, x_l) < \epsilon$.

Consider a sequence in $(\mathbb{R}_+ \rightharpoonup \mathbb{R})$ given by

$$\mathcal{G}(x_k) = \{(k, 0)\}.$$

Is this sequence Cauchy? Yes. $d(x_m, x_n) = 1/2^{\min(m,n)} \to 0$.

Does it converge? To what? Yes, to λ.

Cauchy Sequences

A sequence $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy iff for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for all $k, l \geq n$, $d(x_k, x_l) < \epsilon$.

Consider a sequence in $(\mathbb{R}_+ \rightharpoonup \mathbb{R})$ given by

$$\mathcal{G}(x_k) = \{(i, 0) \mid i \in \mathbb{N}, i \leq k\}.$$

Is this sequence Cauchy? Yes. $d(x_m, x_n) = 1/2^{\min(m,n)} \to 0$.

Does it converge? To what? Yes. To $\{(i, 0) \mid i \in \mathbb{N}\}$.

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Discrete Signals with Cantor Metric is a Complete Metric Space

Theorem

The set of discrete signals (or tuples) with the Cantor metric form a complete metric space.

To show that every Cauchy sequence $\{s_k\}_{k \in \mathbb{N}}$ converges, note that for every $t \in T$, there is a subsequence $\{s_{n(t)}, s_{n(t)+1}, \cdots\}$, for some $n(t) \in \mathbb{N}$, of signals that are identical up to and including t (choose $\epsilon = 1/2^t$). Let $s \in (\mathbb{R}_+ \rightharpoonup V)$ be such that for every $t \in \mathbb{R}_+$, $s(t) = s_{n(t)}(t)$. Then it is easy to show that $\{s_k\}_{k \in \mathbb{N}} \to s$.

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Summary

- Cantor metric provides an ultrametric space for $(\mathbb{R}_+ \rightharpoonup V)$
- Cantor metric equates causality with contracting functions
  (but only for discrete signals)
- Cantor metric provides a metric space for $(\mathbb{R} \rightharpoonup V)$
  (but only for discrete signals) d
- Banach theorem provides constructive convergence
  (but only for delta causal systems).

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